Cut-and-paste operations and exotic 4-manifolds

Jonathan Simone
Brooklyn, New York

Master of Arts, Hunter College, 2011
Bachelor of Arts, New York University, 2009

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
May, 2018

Abstract

We introduce new cut-and-paste operations that can be used to construct exotic 4-manifolds. The idea is to locate a configuration of 2-spheres embedded in an ambient 4-manifold, cut out a neighborhood (called a *plumbing*), and glue in its place a 4-manifold with smaller homology. This serves to generalize the rational blowdown, an operation that has been successfully used in the past to construct new exotic 4-manifolds. In particular, we will develop the notion of k-replaceable plumbings and use a 2-replaceable plumbing to construct a symplectic exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$. Heuristically, a k-replaceable plumbing is a plumbing that can be "symplectically replaced" by a manifold with Euler characteristic k. It will turn out that 1-replaceable plumbings are precisely those that can be rationally blown down.

We then explore the possible existence of non-simply connected plumbings that are 0-replaceable. We first construct examples of plumbings that can be "smoothly replaced" by rational homology $S^1 \times D^3$s and use such replacements to construct 4-manifolds that are homeomorphic to well-known 4-manifolds. For example, we will construct a 4-manifold that is homeomorphic to, but not obviously diffeomorphic to, $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. This would be the first example of an exotic $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. We will also build machinery that can be helpful in determining whether this operation yields exotic 4-manifolds and we will explore the possibility of this being a symplectic operation by classifying the tight contact structures with no Giroux torsion on plumbed 3-manifolds.

Contents

Chapter 1

Introduction

A natural question in the theory of smooth manifolds is: how many smooth structures does a given topological manifold admit?

Definition 1.0.1. *Let X and X' be smooth manifolds. We call (X, X') an **exotic pair** if they are homeomorphic but not diffeomorphic. Furthermore, if X is a well-known smooth manifold, then X' is called an **exotic** X.*

The first examples of exotic pairs were constructed by Milnor [52] in 1956. In this groundbreaking paper, Milnor constructed exotic 7-spheres by considering S^3-bundles over S^4. Since then, much has been discovered. In particular, any closed topological manifold of dimension $n \leq 3$ is known to admit exactly one smooth structure (see [53]) and any closed topological manifold of dimension $n \geq 5$ admits at most finitely many smooth structures (see [43]). Moreover, when $n \neq 4$, simply connected smooth n-manifolds are classified by purely algebraic tools.

When $n = 4$, the story is not as clear. It turns out that a topological 4-manifold can admit infinitely many smooth structures. For example, there are uncountably many exotic $\mathbb{R}^4$s (whereas, for $n \neq 4$, there is a unique smooth structure on $\mathbb{R}^n$). It is not even known whether a 4-manifold can admit only finitely many smooth structures.

Throughout this thesis, we will be concerned with 3- and 4-manifolds. From now on, we assume that all manifolds are smooth and oriented.

1.1 Building exotic 4-manifolds

To show that a closed smooth 4-manifold X' has an exotic smooth structure, one must show: X' is homeomorphic to a well-known smooth 4-manifold X; and X' is not diffeomorphic to X. If X and X' are simply connected, then Freedman's Theorem [22] can be used to show that X and X' are homeomorphic. To show that X and X' are not diffeomorphic, one can compute numerical invariants such as Seiberg-Witten invariants (defined in the early 1990s) or the Ozsváth-Szabó 4-manifold invariant (defined in the early 2000s).

For a closed simply connected 4-manifold X, $H_1(X;\mathbb{Z}) = H_3(X;\mathbb{Z}) = 0$ and $H_0(X;\mathbb{Z}) = H_4(X;\mathbb{Z}) = \mathbb{Z}$. Thus the complexity of X, as measured by homology, is given by $H_2(X;\mathbb{Z})$. Let $b_2(X) = \text{rank}(H_2(X;\mathbb{Z}))$ denote the *second Betti number* of X. For large b_2, there are nice constructions that yield infinitely many exotic versions of given topological 4-manifolds. For example, by performing surgery on a homologically essential torus with self-intersection 0 (see, for example, [33]), it is possible to construct infinitely many exotic 4-manifolds. Unfortunately, these nice constructions break down when b_2 is small. Moreover, the smaller b_2 is, the harder it is to construct exotic manifolds with that prescribed b_2. For example, if $b_2(X) = 0$, then by Freedman's theorem, X is homeomorphic to S^4. However, the existence of an exotic smooth structure on S^4 is the content of the only unknown case of the (smooth) Poincaré conjecture. Moreover, as of early 2018, there are no known examples of exotic 4-manifolds with $b_2 = 1$ or 2.

A rather nice way to construct small (as measured by b_2) exotic 4-manifolds is via the (generalized) rational blowdown, which was introduced by Fintushel and Stern in [18] and generalized by Park in [65]. This is a cut-and-paste operation in which a particular 4-manifold called a *plumbing* is excised from a closed 4-manifold X and a manifold with "smaller" homology is glued in its place, producing a new 4-manifold X'. Further details will be given in Section 2.3. The appeal of this procedure is that the Seiberg-Witten invariants of X' can often be computed from the Seiberg-Witten invariants of X. In many cases, this makes determining that X' is exotic an easy matter. For example, using the rational blowdown: Park constructed an exotic $\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$ in [66] and an exotic $3\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$ in [67]; Stipsicz and Szabó constructed exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$s in [70]; and Yasui constructed an exotic $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ in [77].

In [71], the rational blowdown was extended to a larger class of plumbings and in [51], Michalogiorgaki constructed an exotic $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for $n = 6, 7, 8, 9$ using these plumbings. Moreover, other similar cut-and-paste operations have been defined and used to construct small exotic 4-manifolds. For example, in [42], Karakurt and Starkston defined *star surgery* and used it to construct an exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$ and an exotic $\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$. Moreover, combining star surgery with Fintushel-Stern's knot surgery in a double node neighborhood [19], they constructed infinitely many $\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$s.

In [73], Symington showed that, under favorable conditions on X and P, the rational blowdown is a *symplectic* operation. That is, under these favorable conditions, if X admits a special geometric structure called a *symplectic structure*, then so does X'. Thus, the aforementioned exotic 4-manifolds obtained by rational blowdown admit symplectic structures. It is also known that star surgery is a symplectic operation and so the first two aforementioned 4-manifolds obtained by star surgery admit symplectic structures. The infinitely many exotic $\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$s do not admit symplectic structures, however, because the double node neighborhood construction destroys symplectic structures. We will further explore symplectic cut-and-paste operations in Section 3.2.

There are also examples of exotic $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$s (symplectic and otherwise) when

$n = 2, 3, 4$ (see, for example, [3], [2], [21], [17]). The techniques used to construct these 4-manifolds involve methods that are more complicated than simple cut-and-paste operations like the rational blowdown and star surgery. Moreover, as mentioned above, there is no known example of an exotic $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ (as of early 2018).

1.2 Outline of the main results

This thesis develops new cut-and-paste operations that serve to generalize the rational blowdown and uses these operations to construct exotic and potentially exotic 4-manifolds. Because the cut-and-paste operations introduced here involve wide arrays of plumbings, the hope is that these operations will be able to yield small exotic 4-manifolds that the rational blowdown and other simple cut-and-paste operations have not been able to produce (e.g. exotic $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for $1 \leq n \leq 4$).

In Chapter 2, we will review the basics of certain smooth constructions of exotic 4-manifolds. In particular, we will provide an overview of plumbings (which can be described by weighted graphs), the rational blowdown, and smooth 4-manifold invariants. In Chapter 3, we will bring geometric structures into the mix and give an overview of: the relationship between symplectic, Stein, and contact structures; invariants of contact structures; symplectic cut-and-paste; classifications of contact structures; and Lefschetz fibrations and open book decompositions.

In Chapter 4, we will introduce the notion of *k-replaceable* plumbings, which, heuristically, are plumbings that can be symplectically replaced by 4-manifolds with Euler characteristic k. We will then classify 2-replaceable linear plumbings, construct 2-replaceable plumbing trees, and use one such plumbing tree to build a symplectic exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$.

In Chapters 5 and 6, we will work with a class of plumbings that has not received much attention from the viewpoint of cut-and-paste operations, namely plumbings whose associated graphs are not simply connected. In particular, in Chapter 5 we will develop a method for constructing plumbings whose boundaries also bound manifolds with the rational homology of $S^1 \times D^3$. We will then prove a gluing result that allows one to keep track of the Ozsváth-Szabó 4-manifold invariant when one cuts such a plumbing (satisfying additional conditions) out of an ambient 4-manifold and glues the rational homology $S^1 \times D^3$ in its place.

Using this construction, it will be fairly routine to construct manifolds that are homeomorphic to well-known 4-manifolds. In particular, we will construct a manifold X_n that is homeomorphic to $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$, where $1 \leq n \leq 4$. With the technology developed in this chapter, however, we will not be able to tell if X_n is diffeomorphic to $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$. Different tools need to be developed for this purpose.

In Chapter 6, we will begin exploring whether the cut-and-paste operation introduced in Chapter 5 is a symplectic operation. Namely, we will obtain a classification of tight contact structures with no Giroux torsion on the boundaries of a certain family of plumbings. This result involves proving a generalization of an important result due to Lisca and Matic in [48]. Taking further steps in this direction might help us determine conditions under which this cut-and-paste operation is symplectic, allowing us to bypass the need for a numerical 4-manifold invariant in distinguishing smooth structures, resolving the current shortcomings of the tools developed in Chapter 5.

Chapter 2

Smooth constructions

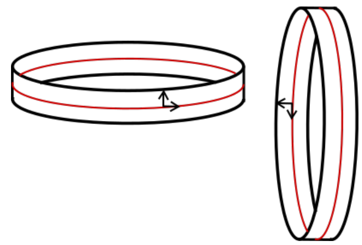
2.1 Notation

We assume the reader is familiar with intersection forms of 4-manifolds and characteristic classes. See [33] for details. Let X be a 4-manifold and let Q_X denote a matrix for the intersection form. Then:

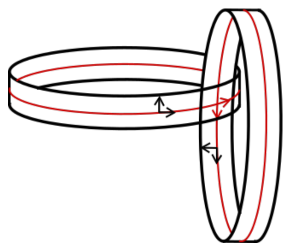
- $\chi(X)$ denotes the *Euler characteristic* of X;
- $b_i(X) = \text{rank}(H_i(X;\mathbb{Z}))$ denotes the i^{th} Betti number;
- $b_2^+(X)$ denotes the number of positive eigenvalues of Q_X;
- $b_2^-(X)$ denotes the number of negative eigenvalues of Q_X;
- $\sigma(X) = b_2^+(X) - b_2^-(X)$ denotes the *signature* of X;
- and $c_1(L)$ denotes the first Chern class of a line bundle L over X;

2.2 Plumbings

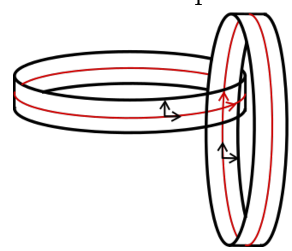
Let $\pi_i : X_i \to S_i^2$ be a D^2-bundle over the 2-sphere for $i = 1, 2$. Fix orientations on each base sphere S_i^2 and on the fibers. These choices induce an orientation on X_i given by the orientation on S_i followed by the orientation on the fibers. Let $D_i \subset S_i^2$ be a disk such that $\pi_i^{-1}(D_i) \cong D_i \times D^2$. Then we *plumb* X_1 and X_2 together by identifying $\pi_1^{-1}(D_1)$ and $\pi_2^{-1}(D_2)$ by a map that preserves the product structure, interchanges the disk factors, and preserves orientation. Since the base spheres and fibers have fixed orientations, this map can be chosen so that the base spheres intersect positively or negatively in the plumbed manifold. Note that these resulting plumbed manifolds are the same oriented manifold; by changing the orientations of S_1 and the fibers of



(a) Two D^1-bundles over S^1 with fixed orientations on the base spheres and fibers



(b) A plumbing in which the base spheres intersect negatively



(c) A plumbing in which the base spheres intersect positively

Figure 2.1: The plumbing operation

π_1, the orientation on X_1 remains unchanged, but the sign of the intersection of S_1 and S_2 changes. See Figure 2.1 for a plumbing of two trivial D^1-bundles over S^1.

We can now iteratively plumb disk bundles over S^2 to form new manifolds. That is, choose one of the disk bundles X_i from the above construction and plumb a new disk bundle $\pi_3 : X_3 \to S^2_3$ to X_i, and continue in this way. A manifold obtained by performing the plumbing operation finitely many times is called a *plumbing*. Notice that a plumbing deformation retracts onto its base spheres, which intersect transversely. Thus, if a configuration of transversely-intersecting spheres $\mathcal{C}$ is embedded in a 4-manifold, then a neighborhood of $\mathcal{C}$ is diffeomorphic to a plumbing.

A plumbing can be described by a weighted graph, called a *plumbing graph*, where each vertex represents a disk bundle and each edge indicates a gluing of the corresponding disk bundle vertices. Each vertex is decorated with an integer, called the *weight*, that denotes the Euler class of the corresponding bundle (and the self-intersection number of the corresponding base sphere). Each edge is decorated with a "+" or "–" to indicate the sign of the intersection of the (oriented) adjacent base spheres.

If a plumbing graph is a tree, then we need not keep track of the signs of the edges, since by simply changing the orientations of the base spheres and fibers of select disk bundles, we may arrange so that all of the intersections of all of the base spheres are positive. Thus, we will leave the edges of plumbing trees undecorated and assume they are all positive. If a plumbing graph is not a tree, then we must take care in decorating the edges of each cycle. For each cycle, there are two distinct possibilities—either all edges of the cycle have sign "+" or all but one edge have sign "+." By changing the orientations of the base spheres and fibers of select disk bundles, it is easy to see that these are the only two possibilities. Undecorated edges are understood to have sign "+." In Chapters 5 and 6, we will work with plumbings whose associated graphs have exactly one cycle. Suppose P is such a plumbing.

Definition 2.2.1. *We say that P is a* ***positive plumbing*** *if, after possible changing the orientations of the base spheres and fibers of select disk bundles, every edge in the cycle can be decorated with "+." Otherwise, we say P is a* ***negative plumbing****.*

Definition 2.2.2. *P is called a* ***cyclic plumbing*** *if its graph is a cycle.*

Given a plumbing graph, there is an obvious handlebody description of the associated 4-manifold. Namely, each vertex corresponds to a 2-handle attachment whose framing equals the weight of the vertex. Moreover, the attaching circles of these 2-handles link according to the edges of the graph. If two vertices share a single edge, then the corresponding attaching circles link once. If there is a cycle, then the sign of the linking numbers of two adjacent (oriented) attaching circles must match the sign of the corresponding edge in the graph. Finally, for every cycle of the plumbing graph, a 1-handle must be introduced in the handlebody diagram that the attaching circles of the cycle "travel through." See Figure 2.2 for an example and see Chapter 6 of [33] for more details.

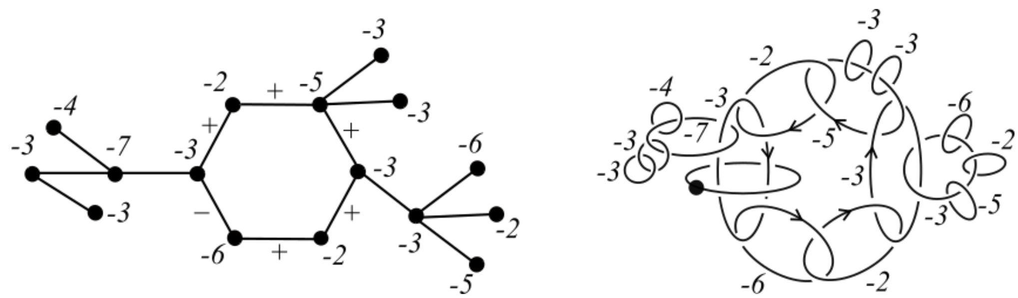


Figure 2.2: A plumbing graph and its associated handlebody diagram. The signs of the linking numbers of adjacent attaching circles in the cycle match the signs of the corresponding edges in the plumbing graph.

It is easy to see from the handlebody diagrams of plumbings that the second homology of any plumbing $H_2(P;\mathbb{Z})$ is a free abelian group generated by the base spheres (i.e. the 2-handles). In this basis, the intersection form of P is simply the intersection matrix of the 2-handles of the handlebody diagram.

Definition 2.2.3. *We say that a plumbing graph is* ***negative definite*** *if the intersection matrix is negative definite.*

Note that since the diagonal entries of the intersection matrix correspond to the weights, all weights of a negative definite plumbing must be negative. We end this section by defining terminology that will appear throughout this thesis and we make an important remark.

Definition 2.2.4. *A vertex of a negative definite plumbing graph is called a* ***bad vertex*** *if its valence is greater than negative its weight.*

Remark 2.2.5. Plumbings can be defined more generally for disk bundles over any orientable or nonorientable surfaces. See Sections 4.6.2 and 6.1 of [33] for more details. Thus, in general, plumbings whose associated graphs are trees might not be simply connected. However, in this thesis, all plumbings are assumed to be built from D^2-bundles over S^2. In Chapters 5 and 6 we will study non-simply connected plumbings, which, in this context, are precisely the plumbings whose associated graphs are non-simply connected.

2.2.1 Boundaries of plumbings

The boundaries of plumbings are called *plumbed 3-manifolds*. Certain plumbed 3-manifolds are well-known. For example, the boundaries of linear plumbings are lens spaces, the boundaries of cyclic plumbings are torus bundles over S^1, and the boundaries of star-shaped plumbings are Seifert fibered spaces. We will mostly be concerned with the first two kinds of plumbings.

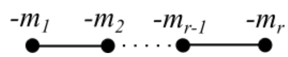
Example 2.2.6 (Lens Spaces)**.** Let P be a linear plumbing with weights $(-m_1, ..., -m_r)$, where $m_i \geq 2$ for all i, as depicted in Figure 2.3a. Moreover, let $p > q > 0$ be relatively prime integers satisfying

$$\frac{p}{q} = [m_1, ..., m_r] = m_1 - \cfrac{1}{m_2 - \cfrac{1}{... - \cfrac{1}{m_r}}}$$

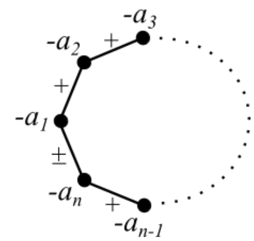
Then P has boundary $L(p, q)$. This can easily be seen by performing the *slam dunk* move to the obvious surgery diagram obtained from the handlebody diagram of the plumbing. After performing this move r times, we obtain $-\frac{p}{q}$-surgery on the unknot (see Section 5.3 in [33] for details).

Example 2.2.7 ((Hyperbolic) Torus Bundles over S^1)**.** Let $P_\pm$ be a positive/negative cyclic plumbing with weights $(-a_1, ..., -a_n)$, as depicted in Figure 2.3b, where $n \geq 2$, $a_i \geq 2$ for all $1 \leq i \leq n$, and $a_1 \geq 3$. Endow $T^2 \times [0, 1] = \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1]$ with the coordinates $(\mathbf{x}, t) = (x, y, t)$. Then by Neumann [54], $\partial P_\pm$ is of the form $T^2 \times [0, 1]/(\mathbf{x}, 1) \sim (\pm B\mathbf{x}, 0)$, where

$$B = B(a_1, ..., a_n) = \begin{bmatrix} p & q \\ -p' & -q' \end{bmatrix}, \frac{p}{q} = [a_1, ..., a_n], \text{ and } \frac{p'}{q'} = [a_1, ..., a_{n-1}].$$



(a) A linear plumbing



(b) A cyclic plumbing

Figure 2.3: Linear and cyclic plumbings

That is, $\partial P_{\pm}$ is a T^2-bundle over S^1. Furthermore, by Lemma 6.6.1 in Section 6.6, $|\mathrm{tr}(B)| = p - q' \geq q + 1 > 2$ and so $\partial P_{\pm}$ is called a *hyperbolic* T^2-bundle over S^1. Finally, since $\mathrm{tr}(B) > 2$, $\partial P_{\pm}$ is a called a *positive/negative* hyperbolic T^2-bundle over S^1.

2.3 Rational blowdown

The *(generalized) rational blowdown*, which was introduced by Fintushel and Stern in [18] and generalized by Park in [65], is the process of cutting a negative definite linear plumbing P out of a 4-manifold and gluing a rational homology ball B with $\partial B = \partial P$ in its place by some diffeomorphism of the boundary. Notice that the rational blowdown lowers b_2 and keeps b_1 the same. Before moving on, we introduce some terminology.

Definition 2.3.1. *If P and B are manifolds such that $\partial B = \partial P$, then we say that P can be* ***(smoothly) replaced*** *by B and we call B a* ***(smooth) replacement*** *of P.*

In [5], Casson and Harer showed that $L(p, q)$ bounds a rational homology 4-ball $B_{m,n}$ if $p = m^2$ and $q = mn - 1$, where $m > n > 0$ are coprime integers. By Example 2.2.6, the linear plumbing $C_{n,m}$ with weights $(-b_1, ..., -b_k)$ satisfying $\frac{p}{q} = [b_1, ..., b_k]$ also has boundary $L(p, q)$. Thus $C_{m,n}$ can be replaced by $B_{m,n}$. In [18] and [65], it was shown that any diffeomorphism of $L(p, q)$ extends over $B_{m,n}$. Thus if $C_{m,n}$ is embedded in a 4-manifold X, then the 4-manifold $X_{m,n} = (X - \mathrm{int}(C_{m,n})) \cup_{L(p,q)} B_{m,n}$ is well-defined (i.e. it does not depend on the diffeomorphism of the boundary).

A cut-and-paste operation is only useful in building new smooth 4-manifolds if one can calculate smooth 4-manifold invariants of the resulting manifold. Fintushel-Stern [18] and Park [65] showed that, under suitable conditions, the Seiberg-Witten invariants of $X_{m,n}$ agree with the Seiberg-Witten invariants of X. Thus, they defined the rational blowdown to be the process of replacing a plumbing $C_{m,n}$ with the rational ball $B_{m,n}$. We will refer to the set of linear plumbings $\{C_{m,n}\}$ as those that can be rationally blown down. In Section 2.4, we will describe this *gluing formula* in detail.

The plumbings $\{C_{m,n}\}$, which arise from the continued fraction expansion of $\frac{m^2}{mn-1}$, can be obtained from a simple inductive procedure. The first such plumbing is $C_{2,1}$, which is the -4-disk bundle over S^2 and whose boundary is $L(4,1)$. By slightly modifying the proof of Lemma 2.1 in [47], it is easy to show that if $[b_1, ..., b_k] = \frac{m^2}{mn-1}$, then $[2, b_1, ..., b_k + 1] = \frac{(2m-n)^2}{(2m-n)m-1}$ and $[b_1 + 1, ..., b_k, 2] = \frac{(2m-n)^2}{(2m-n)(m-n)-1}$. Thus if the plumbing with weights $(-b_1, ..., -b_k)$ can be rationally blown down, then the plumbings with weights $(-2, -b_1, ..., -b_{k-1}, -b_k - 1)$ and $(-b_1 - 1, -b_2, ..., -b_k, -2)$ can also be rationally blown down. This operation will arise many times throughout Chapter 4, so we give it a name.

Definition 2.3.2. *Let P be a linear plumbing with weights $(-b_1, ..., -b_k)$, where $b_i \geq 2$ for all i. The **buddings** of P are the plumbings with weights $(-2, -b_1, ..., -b_{k-1}, -b_k - 1)$ and $(-b_1 - 1, -b_2, ..., -b_k, -2)$.*

A more natural way to see that the iterated buddings of $C_{2,1}$ can be rationally blown down is by realizing these plumbings as complements of rational homology balls in certain blowups of $\overline{\mathbb{C}P^2}$ (see, for example, [65]). Conversely, it can be shown that any plumbing $C_{m,n}$ can be obtained by performing a sequence of buddings to $C_{2,1}$ (see, for example, Corollary 4.5.1 in Section 4.5). Thus, we have the following observation.

Observation 2.3.3. *A linear plumbing can be rationally blown down if and only if it can be obtained by a sequence of buddings of the -4-disk bundle over S^2.*

Remark 2.3.4. By work of Lisca [47], the linear plumbings $\{C_{m,n}\}$ are precisely those that can be *symplectically* rationally blown down. We will explore this notion in Chapter 3. In [71], Stipsicz-Szabó-Wahl went further and classified plumbing trees that can be symplectically rationally blown down.

We end this section by noting that there are other linear plumbings that can be replaced by rational homology balls, but for which the gluing formulas of Fintushel-Stern and Park (c.f. Section 2.4) do not apply. For a complete list of such plumbings, see Lisca's classification of lens spaces that bound rational balls in [46]. We presently highlight one such family. By reversing the orientation of $L(4,1)$, we obtain $L(-4,1)$, which is obtained by 4-surgery on the unknot in S^3. After blowing up and blowing down, as in Figure 2.4, we see that $L(-4,1)$ bounds the linear plumbing with weights $(-2,-2,-2)$, which we denote by $\tilde{C}_{2,1}$. Moreover, $L(-4,1)$ bounds the rational ball $\overline{B}_{2,1}$ (i.e. $B_{2,1}$ with the reversed orientation). Thus $\tilde{C}_{2,1}$ can be replaced by $\overline{B}_{2,1}$.

Similarly, by reversing the orientation of $L(m^2, mn-1)$, we can find a plumbing $\tilde{C}_{m,n}$ that can be replaced by $\overline{B}_{m,n}$. Moreover, it turns out that all such plumbings can be obtained by a sequence of buddings of $\tilde{C}_{2,1}$ (which can be shown by applying Corollary 4.4.4 in Section 4.4).

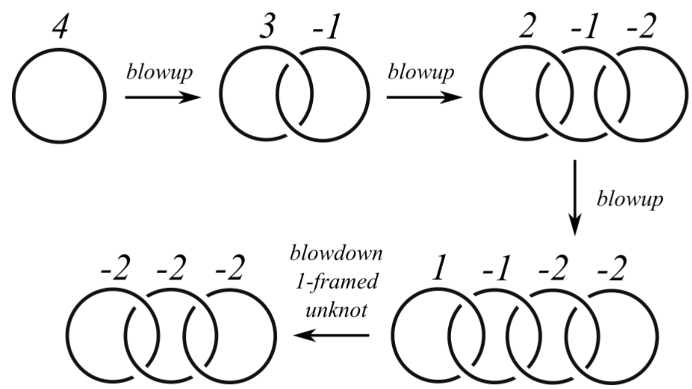


Figure 2.4: Finding another plumbing that can be replaced by a rational ball

Remark 2.3.5. In [69], Roberts proved a gluing formula for the Ozsváth-Szabó 4-manifold invariant analogous to the rational blowdown gluing formula of Fintushel-Stern and Park. In particular, Roberts' result can be used for the plumbings $\{C_{m,n}\}$ as well as the plumbings $\{\tilde{C}_{m,n}\}$.

Remark 2.3.6. Replacing $\tilde{C}_{m,n}$ by $\overline{B}_{m,n}$ can be completed in the smooth category, but by Remark 2.3.4, it cannot be completed in the symplectic category.

2.4 Computing smooth 4-manifold invariants

We assume the reader is familiar with spin^c structures, characteristic classes, and the basics of monopole Floer homology and Heegaard Floer homology. In particular, we assume familiarity with Seiberg-Witten invariants (denoted by SW) and the Ozsváth-Szabó 4-manifold invariant (denoted by Φ). See [58], [64], and [65] for nice expositions. In this short section, we simply highlight the effect that certain cut-and-paste operations have on these 4-manifold invariants. We will refer to the following results as *gluing formulas*.

As mentioned in Section 2.3, the Seiberg-Witten invariants of a 4-manifold obtained by a rational blowdown agree with the Seiberg-Witten invariants of the original manifold. Here is the precise result due to Park, which expands on the original work of Fintushel and Stern in [18].

Theorem 2.4.1 (Park [65])**.** *Suppose X is a smooth 4-manifold containing the plumbing $C_{m,n}$. If L is a characteristic line bundle on X with $SW_X(L) \neq 0$, $c_1(L|_{C_{m,n}}) = -b_2(C_{m,n})$, and $c_1(L|_{L(m^2,mn-1)}) = mp \in \mathbb{Z}_{p^2} \cong H^2(L(m^2,mn-1);\mathbb{Z})$ with $m \equiv (p-1) (\mathrm{mod}\ 2)$, then L induces a characteristic line bundle $\overline{L}$ on $X_{m,n}$ such that $SW_{X_{m,n}}(\overline{L}) = SW_X(L)$.*

Remark 2.4.2. In [71], Stipsicz, Szabó, and Wahl showed that the only linear plumbings for which Theorem 2.4.1 holds are the plumbings in $\{C_{m,n}\}$. This justifies defining the linear plumbings in $\{C_{m,n}\}$ as those that can be rationally blown down.

In [51], Michalogiorgaki extended this result to any pair of negative definite manifolds with "monopole L-space" boundaries by using tools developed in monopole Floer homology. Recall that a monopole L-space is a rational homology 3-sphere with the simplest possible monopole Floer homology. Also recall that for any $\mathfrak{s} \in \mathrm{Spin}^c(X)$, where X is a 4-manifold, $d(\mathfrak{s})$ denotes the formal dimension of the Seiberg-Witten moduli space.

Theorem 2.4.3 (Michalogiorgaki [51])**.** *Suppose Y is a monopole L-space, P and B are negative definite 4-manifolds with $b_1 = 0$, and $X = Z \cup_Y B$, $X' = Z \cup_Y B$ for some 4-manifold Z. If $\mathfrak{s} \in Spin^c(X)$, $\mathfrak{s}' \in Spin^c(X')$, $d(\mathfrak{s}), d(\mathfrak{s}') \geq 0$, and $\mathfrak{s}|_Z = \mathfrak{s}'|_Z$, then $SW_X(\mathfrak{s}) = SW_{X'}(\mathfrak{s}')$. In the case $b_2^+(X) = 1$, $SW_{X,a_1}(\mathfrak{s}) = SW_{X',a_2}(\mathfrak{s}')$, where $a_1 \in H_2(X;\mathbb{Z})$, $a_2 \in H_2(X';\mathbb{Z})$, $a_1|_P = a_2|_B = 0$, and $a_1|_Z = a_2|_Z$.*

Remark 2.4.4. There are similar results for the Ozsváth-Szabó 4-manifold invariant. In particular, as mentioned in Remark 2.3.5, Roberts [69] proved a gluing formula for the rational blowdown.

In Section 4.3, we will apply Michalogiorgaki's result in constructing an exotic 4-manifold. In Section 5.2.1, we will prove an analogous gluing result for the Ozsváth-Szabó 4-manifold invariant for cut-and-paste operations involving certain families of non-simply connected plumbings.

Chapter 3

Symplectic and Stein constructions

3.1 Geometric structures

It is often useful when studying smooth 4-manifolds, to incorporate additional geometric structure. Doing so restricts the class of manifolds that we can consider and allows us to use tools that can help us learn topological characteristics of the underlying 4-manifold. As a simple example, it is known that a *symplectic* 4-manifold has nonvanishing Seiberg-Witten and Ozsváth-Szabó 4-manifold invariants. On the other hand, a 4-manifold of the form $X_1 \# X_2$, where $b_2^+(X_i) \geq 1$ for $i = 1, 2$, has vanishing invariants. Thus if one is studying a 4-manifold that admits a symplectic structure, it cannot be a connected sum of this form.

We begin this chapter by reviewing the notions of contact structures on 3-manifolds, symplectic structures on 4-manifolds, and Stein structures on 4-manifolds, while exploring the interplay between these geometric structures. Thorough expositions can be found in [4], [25], and [58].

3.1.1 Contact 3-manifolds and Legendrian knots

Definition 3.1.1. *A (coorientable)* ***contact form*** *on a 3-manifold Y is a (global) 1-form α such that $\alpha \wedge d\alpha$ is nowhere 0. The 2-plane distribution $\xi = \ker \alpha$ is called a* ***contact structure****. The pair (Y, ξ) is called a* ***contact 3-manifold****.*

We will always assume that contact structures are cooriented and *positive* (i.e. the orientation of Y coincides with the orientation given by $\alpha \wedge d\alpha$). We will be concerned with two equivalences of contact structures—contactomorphism and isotopy.

Definition 3.1.2. *Two contact 3-manifolds (Y, ξ) and (Y', ξ') are* ***contactomorphic*** *if there is a diffeomorphism $f : Y \to Y'$ such that $f_*(\xi) = \xi'$. Two contact structures ν and ν' on Y are called* ***isotopic*** *if there is a contactomorphism $g : (Y, \nu) \to (Y, \nu')$ isotopic to the identity.*

The following class of knots embedded in contact 3-manifolds has been used extensively in constructing *Stein structures* on certain 4-manifolds.

Definition 3.1.3. *A knot L embedded in a contact 3-manifold (Y, ξ) is called **Legendrian** if the tangent vectors of L lie in ξ. The **contact framing** (or **Thurston-Bennequin framing**) of L is defined by pushing L off of itself in a direction orthogonal to ξ.*

If L is null-homologous in (Y, ξ), then it admits a natural 0-framing provided by the Seifert surface framing. In this case, the contact framing can measured with respect to the 0-framing, yielding an integer called the *Thurston-Bennequin number*, denoted by $tb(L)$, which is a numerical invariant of L. The *rotation number* of L, denoted by $r(L)$, is another classical null-homologous (oriented) Legendrian knot invariant. This invariant has a more involved definition which can be found in [25]. We will mainly be concerned with these invariants for Legendrian knots in $\mathbb{R}^3$, equipped with the *standard* contact structure. In this case, these numerical invariants can be easily computed, as we will see in Example 3.1.5 below.

Definition 3.1.4. *A contact 3-manifold (Y, ξ) is **overtwisted** if there exists an embedded disk D such that the contact framing of ∂D coincides with the surface framing given by D. Otherwise, (Y, ξ) is **tight**.*

Example 3.1.5 (Standard (tight) contact structure on $\mathbb{R}^3$)**.** The standard contact structure on $\mathbb{R}^3$ is given by $\xi_{st} = \ker(dz + xdy)$. It can be shown that this contact structure is the unique tight contact structure on $\mathbb{R}^3$. A *front projection* of a Legendrian knot $L \subset (\mathbb{R}^3, \xi_{st})$ is the projection of L onto the yz-plane. Front projections have crossings, cusps, and no vertical tangencies. See [58] for details. Given an oriented front projection $\tilde{L}$ of L, the Thurston-Bennequin and rotation numbers can be computed using the formulas

$$tb(L) = w(\tilde{L}) - \frac{1}{2}c(\tilde{L}) \text{ and } r(L) = \frac{1}{2}(c_d(\tilde{L}) - c_u(\tilde{L})),$$

where $w(\tilde{L})$ denotes the writhe of $\tilde{L}$, $c(\tilde{L})$ denotes the number of cusps of $\tilde{L}$, $c_d(\tilde{L})$ denotes the number of downward cusps of $\tilde{L}$, and $c_u(\tilde{L})$ denotes the number of upward cusps of $\tilde{L}$.

3.1.2 Symplectic and Stein fillings

Definition 3.1.6. *A **symplectic form** on a 4-manifold X is a closed, nondegenerate 2-form ω on X. The pair (X, ω) is called a **symplectic 4-manifold**. (X, ω) is called **minimal** if it contains no symplectically embedded 2-sphere with self-intersection -1.*

In this section we provide an overview of three types of *fillings* of contact structures, namely weak symplectic fillings, strong symplectic fillings, and Stein fillings. For further details, see [6] and [58].

Definition 3.1.7. *A contact 3-manifold* (Y, ξ) *is* ***weakly symplectically fillable*** *if there is a compact symplectic manifold* (X, ω) *such that* $\partial X = Y$ *and* $\omega|_\xi \neq 0$*. We call* (X, ω) *a* ***weak symplectic filling*** *of* (Y, ξ)*.*

Theorem 3.1.8 (Eliashberg [10], Gromov [36])**.** *A weakly symplectically fillable contact structure is tight.*

A *Liouville vector field* on a symplectic manifold (X, ω) is a vector field v such that $\mathcal{L}_v\omega = \omega$ (where $\mathcal{L}$ denotes the Lie derivative). Suppose $Y = \partial X$ and there is a Liouville vector field defined in a neighborhood of Y that is transverse to Y. Then, by work of Weinstein [75], the 1-form $\alpha_\omega = i^*(\iota_v\omega)$ is a contact 1-form on Y, where $i : Y \to X$ is the inclusion map. It can be shown (see [15]) that α does not depend on the choice of Liouville vector field, up to isotopy.

Definition 3.1.9. *A contact 3-manifold* (Y, ξ) *is* ***strongly symplectically fillable*** *if there exists a symplectic manifold* (X, ω) *with* $\partial X = Y$ *and a Liouville vector field* v *defined in a neighborhood of* Y *that is transverse to* Y*, points out of* Y*, and satisfies* $\xi = \ker \alpha_\omega$*. We call* (X, ω) *a* ***strong symplectic filling*** *of* (Y, ξ) *and say that* (X, ω) *admits a* ***strongly convex boundary****.*

Suppose (X, ω) is a strong symplectic filling of (Y, ξ) and let v be a Liouville vector field on (X, ω) satisfying the conditions in the above definition. Then the contact form α_ω satisfies $d\alpha_\omega = i^*\omega$. In particular, $\omega|_\xi \neq 0$ and so (X, ω) is a weak symplectic filling of (Y, ξ).

Definition 3.1.10. *A linear map* $J : TX \to TX$ *such that* $J^2 = -id_{TX}$ *is called an* ***almost complex structure****.* J *is said to be* ***compatible*** *with a symplectic structure* ω *if* $\omega(Ju, Jv) = \omega(u, v)$ *and* $\omega(u, Ju) > 0$ *for all* $u \neq 0$*.*

Remark 3.1.11. It is known that every symplectic form has a compatible almost complex structure and that all such almost complex structures are homotopic. Thus we can define the first Chern class of (X, ω) by $c_1(X, \omega) := c_1(X, J)$, where J is any compatible almost complex structure.

The following formulation of the definition of Stein structures can be found in [72]. Suppose S is a complex surface with almost complex structure J naturally induced by the complex structure on S. Let $\varphi : S \to [0, \infty)$ be a proper, smooth function. Define the 1-form $d^{\mathbb{C}}\varphi$ by $d^{\mathbb{C}}\varphi(v) = d\varphi(Jv)$ and let ω_φ denote the 2-form $\omega_\varphi = -d(d^{\mathbb{C}}\varphi)$.

Definition 3.1.12. *The function* φ *is* J***-convex*** *(or* ***strictly plurisubharmonic****) if the associated 2-form* ω_φ *is a symplectic form on* S*. The triple* (S, J, φ) *is called a* ***Stein surface****. A compact 4-manifold* X *with boundary is called a* ***Stein domain*** *if there is a Stein surface* (S, J, φ) *such that* $X = \varphi^{-1}([0, a])$ *for some regular value* a*. Similarly, a cobordism* W *is a* ***Stein cobordism*** *if there is a Stein surface* (S, J, φ) *such that* $W = \varphi^{-1}([a, b])$ *for some regular values* $0 < a < b$*.*

Remark 3.1.13. It it known (see, for example, Corollary 3.4 in [6]) that a J-convex function has no critical points of index greater than 2. Thus any Stein domain X has a handlebody diagram with no 3- or 4-handles.

Any almost complex structure on a 4-manifold X with boundary $\partial X = Y$ induces a canonical 2-plane field $\xi \subset TY$ made up of the complex tangencies along Y. If X can be equipped with a Stein structure (X, J, φ), this 2-plane field is a contact structure that can be given by the kernel of the 1-form $\alpha_J = -d^{\mathbb{C}}\varphi|_Y$. The *Levi form* on Y is defined to be $L_Y(x, y) = d\alpha(x, Jy)$. The boundary Y is called *J-convex* if L_Y is positive definite.

Definition 3.1.14. (Y, ξ) *is **Stein fillable** if it is the J-convex boundary of a Stein domain* (X, J, φ)*. We call* (X, J, φ) *a **Stein filling** of* (Y, ξ)*.*

By Eliashberg and Gromov (see Proposition 11.22 in [6]), Stein structures only depend on the almost complex structure and not on the choice of J-convex function, up to homotopy. Thus, we can drop φ from the notation and simply write (X, J) for a Stein domain. Moreover, the associated symplectic manifold (X, ω_φ) is independent of φ, up to exact symplectomorphism isotopic to the identity.

Suppose (X, J, φ) is a Stein filling of (Y, ξ). It can be shown (see, for example, Section 8.1 of [58]) that $\nabla\varphi$ is a Liouville vector field that points outward from $Y = \partial X$. Thus (X, ω_φ) is a strong symplectic filling of (Y, ξ). The same computation shows that $\iota_{\nabla\varphi}\omega_\varphi = -d^{\mathbb{C}}\varphi$ and so the the contact 1-form α_J induced by (X, J, φ) is precisely the contact 1-form α_{ω_φ} induced by (X, ω_φ). Since (X, ω_φ) is independent of φ, up to exact symplectomorphism isotopic to the identity, the contact structure induced by (X, J, φ) is independent of J-convex function, up to isotopy. Thus we can rewrite the above definition as:

Definition 3.1.15. (Y, ξ) *is **Stein fillable** if it is the J-convex boundary of a Stein domain* (X, J)*. We call* (X, J) *a **Stein filling** of* (Y, ξ)*.*

To recap, we have the following inclusions of contact structures.

$$\{\text{Stein fillable}\} \subset \{\text{Strongly fillable}\} \subset \{\text{Weakly fillable}\} \subset \{\text{Tight}\}$$

Moreover, we have seen that:

- a symplectic 4-manifold (X, ω) with strongly convex boundary induces a tight contact structure on ∂X that is strongly filled by (X, ω);
- and a Stein domain (X, J) induces a tight contact structure on ∂X that is filled by (X, J).

Example 3.1.16 (Fillings of Lens Spaces)**.** Let ξ_{st} be the canonical tight contact structure on $L(p, q)$ inherited from the unique tight contact structure on S^3 (see, for example, [58]). In [47], Lisca completely classified the minimal weak symplectic fillings of $(L(p,q),\xi_{st})$. In [55], Ohta and Ono showed that any weak symplectic filling of a rational homology sphere can be modified into a strong symplectic filling. Therefore, Lisca's classification is of minimal strong symplectic fillings of $(L(p,q),\xi_{st})$. One such filling is the linear plumbing $C_{m,n}$, as defined in Section 2.3. As a corollary to his classification, Lisca showed that $(L(p,q),\xi_{st})$ has a rational homology ball (strong) symplectic filling if and only if $p = m^2$ and $q = mn-1$, where $m > n > 0$ are coprime integers (c.f. Section 2.3).

3.1.3 Constructing Stein domains

In [75], Weinstein described a way to extend a symplectic structure with strongly convex boundary over 1- and 2-handle attachments. In [11], Eliashberg proved an analogous result when there is a Stein structure present. In both situations, there is a unique way to extend each structure when a 1-handle is attached. If a 2-handle is attached along a Legendrian knot L with framing one less than the framing induced by the contact structure (that is induced by the symplectic/Stein structure), then these structures extend to the resulting 4-manifold. Recall that if L is null-homologous, then there is a well-defined notion of the Thurston-Bennequin number of L, denoted by $tb(L)$. In this case, the smooth framing of the 2-handle attachment must be $tb(L)-1$.

In practice, one can build an explicit Stein structure on a 4-manifold with boundary that has a handlebody diagram without 3-handles by arranging the diagram in a particular way. First arrange the attaching sphere pairs of the 1-handles in rows and ensure that all of the complexity of the 2-handles occurs between the two sets of 1-handle attaching spheres. Then draw each attaching circle of each 2-handle so that it is Legendrian with smooth framing $tb-1$. See Figures 3.1b and 3.2b for some simple examples and see [35] for more details.

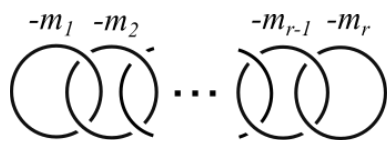
Depending on which Legendrian representatives of the attaching circles are used, we obtain different Stein structures. In particular, two Legendrian representatives of the same smooth knot with framings $tb-1$ and different rotation numbers describe different Stein structures. This is due to the following result of Gompf.

Proposition 3.1.17 (Proposition 2.3 in [35]). *Let (X,J) be a Stein domain. The Chern class $c_1(X,J) \in H_2(X;\mathbb{Z})$ is represented by a cocycle whose value on each 2-handle h_i is $r(K_i)$. That is, $\langle c_1(X,J), h_i\rangle = r(K_i)$.*

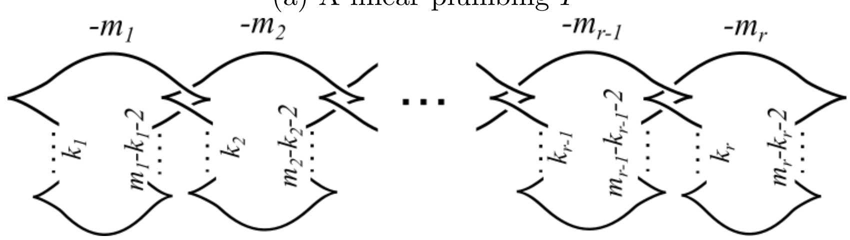
Moreover, by a widely-applied result of Lisca and Matic in [48], different Stein structures on a 4-manifold with boundary induce nonisotopic contact structures. More precisely,

Theorem 3.1.18 (Lisca-Matic [48]). *Let J_1 and J_2 be two Stein structures on a smooth 4-manifold X and let ξ_1 and ξ_2 be the induced (tight) contact structures on ∂X. If $c_1(X,J_1) \neq c_1(X,J_2)$, then ξ_1 and ξ_2 are not isotopic.*

Example 3.1.19 (Linear Plumbings and Lens Spaces)**.** The negative definite linear plumbing P with weights $(-m_1, ..., -m_r)$ has the handlebody diagram depicted in Figure 3.1a. Let L_i denote the attaching circle of the 2-handle with framing $-m_i$. We can equip this 4-manifold with various Stein structures. First make each unknot the standard Legendrian unknot with $tb = -1$ and $r = 0$. Then stabilize L_i $(m_i - 2)$-times so that $tb(L_i) - 1 = -m_i$. By the remarks above, the *standard* Stein structure on D^4 (see, for example, [49]) extends to a Stein structure on P. Notice that there are $m_i - 1$ different ways to stabilize L_i. In particular, as in Figure 3.1b, for each i, we can stabilize L_i k_i-times on the left and $(m_i - k_i - 2)$-times on the right (where $0 \leq k_i \leq m_i - 2$). Orienting each L_i clockwise, we see that $r(L_i) = \frac{1}{2}(1 + 2(m_i - k_i - 2) - (2k_i + 1)) = m_i - 2k_i - 2$. Thus there are $m_i - 1$ Legendrian representatives of L_i with pairwise distinct rotation numbers. So, by Proposition 3.1.17, we obtain $(m_1 - 1) \cdots (m_r - 1)$ distinct Stein structures on P.



(a) A linear plumbing P

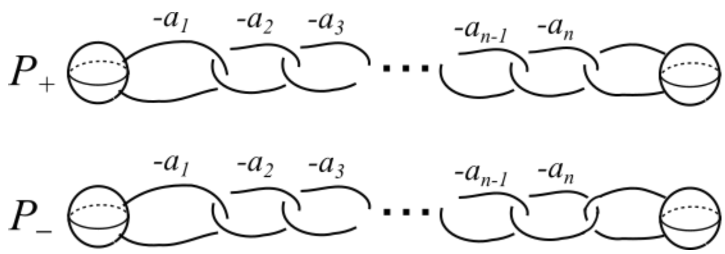


(b) A Stein handlebody diagram for P. The framings are smooth framings.

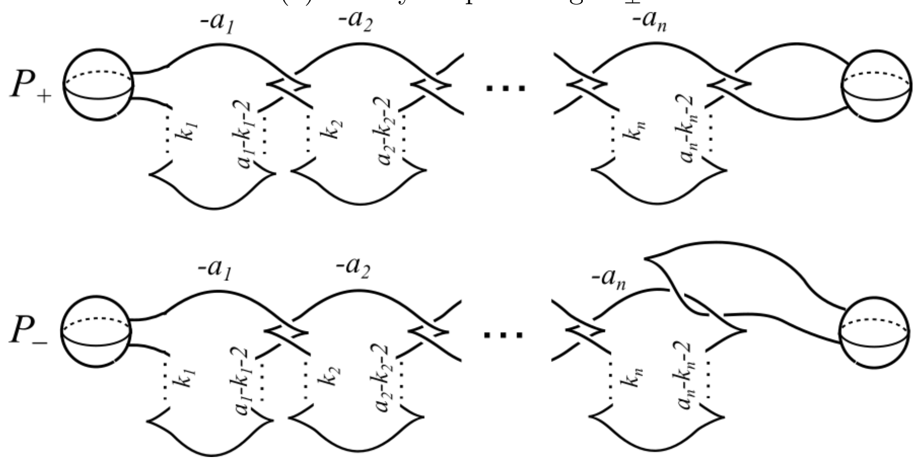
Figure 3.1: Distinct Stein structures on P

By Theorem 3.1.18, these Stein structures induce nonisotopic contact structures on the boundary, which by Example 2.2.6 is the lens space $L(p, q)$, where $\frac{p}{q} = [m_1, ..., m_r]$. Thus $L(p, q)$ admits at least $(m_1 - 1) \cdots (m_r - 1)$ tight contact structures. In [38], Honda showed that there are exactly $(m_1 - 1) \cdots (m_r - 1)$ nonisotopic tight contact structures on $L(p, q)$. Thus all tight contact structures on $L(p, q)$ are Stein fillable. See Section 3.2.1 for more details.

Example 3.1.20 (Cyclic Plumbings and Torus Bundles)**.** Let $P_\pm$ denote the cyclic plumbing with weights $(-a_1, ..., -a_n)$, where $a_1 \geq 3$ and $a_i \geq 2$ for all i. Then by Example 2.2.7, $Y_\pm = \partial P_\pm$ is a hyperbolic torus bundle over S^1. Handlebody diagrams for these two plumbings are depicted in Figure 3.2a. As in the previous example, we can stabilize each unknot $a_i - 2$ times as in Figure 3.2b and obtain distinct Stein structures, which induce $(a_1 - 1) \cdots (a_n - 1)$ nonisotopic tight contact structures on the torus bundle boundary. In [39], Honda showed that ∂P_+ admits



(a) The cyclic plumbings $P_\pm$



(b) Stein handlebody diagrams for $P_\pm$. The framings are smooth framings.

Figure 3.2: Distinct Stein structures on P

exactly $(a_1 - 1) \cdots (a_n - 1)$ tight contact structures with no *Giroux torsion* (see Section 3.2.1 for more details). These are precisely the contact structures induced by the Stein structures depicted in Figure 3.2b. He also showed that ∂P_- admits exactly $(a_1 - 1) \cdots (a_n - 1)$ *virtually overtwisted* contact structures (i.e. tight contact structures that become overtwisted when pulled back to a finite cover) with no Giroux torsion, which are induced by the Stein structures depicted in Figure 3.2b. However, there is a unique *universally tight* contact structure (i.e. a tight contact structure that remains tight when pulled back to the universal cover) on ∂P_- with no Giroux torsion. In [32], Golla and Lisca show that under certain conditions placed on the weights of the plumbing, this contact structure is Stein fillable. Moreover, Ding and Li [9] showed that under other conditions placed on the weights, this universally tight contact structure is not strongly symplectically fillable, although it is always weakly symplectically fillable by [8]. We will further explore these contact structures in Chapter 6.

3.1.4 Invariants of contact structures

In this short section, we highlight two numerical invariants of contact structures, the latter of which will be used in Chapter 6. The first is the d_3-invariant, which is due to Gompf. For details on how to compute $c_1^2(X, J)$, see Chapter 6 of [58].

Theorem 3.1.21 (Gompf [34]). *Let ξ be a contact structure induced by a Stein structure (X, J) on ∂X such that $c_1(\xi)$ is torsion. Then the expression*

$$d_3(\xi) = \frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)) \in \mathbb{Q}$$

is an invariant of ξ up to homotopy.

The second invariant is the Ozsváth-Szabó contact invariant $c(Y, \xi) \in \widehat{HF}(-Y, \mathfrak{t}_\xi)$, where (Y, ξ) is any contact 3-manifold and $\mathfrak{t}_\xi$ is the canonical spin^c structure induced by ξ. See section 14.14.4 of [58] for a nice exposition.

Theorem 3.1.22 (Ozsváth-Szabó [62]). *$c(Y, \xi)$ is an isotopy invariant.*

One of the first uses of this invariant was the following result.

Theorem 3.1.23 (Ozsváth-Szabó [62]). *If (Y, ξ) is overtwisted, then $c(Y, \xi) = 0$.*

By refining the contact invariant with a particular *twisted coefficient system*, Ozsváth and Szabó proved the following, stronger result. We will explore this particular twisted coefficient system and apply the following result in Chapter 6.

Proposition 3.1.24 (Ozsváth-Szabó [60]). *If (Y, ξ) is weakly symplectically fillable, then under this twisted coefficient system the contact invariant does not vanish.*

Remark 3.1.25. Twisted coefficients have been effective in detecting weakly symplectically fillable contact structures that are not strongly symplectically fillable (see, for example, [26]).

3.2 Symplectic cut-and-paste

In [73] and [74], Symington showed that the (generalized) rational blowdown is symplectic. That is, if a plumbing $C_{m,n}$ (as defined in Section 2.3) is embedded in a closed symplectic 4-manifold (X, ω) and the base spheres of the plumbing are symplectic, then the manifold $X_{m,n}$ obtained after the rational blowdown admits a symplectic structure (which is induced by the symplectic structures on $X - \text{int}(C_{m,n})$ and $B_{m,n}$). More generally, suppose (X, ω) is a symplectic 4-manifold and

- (P, ω_1) is a symplectic submanifold of X with strongly convex boundary;

- B is a 4-manifold with $\partial B = \partial P$ and B admits a symplectic structure ω_2 with strongly convex boundary; and
- the induced contact structures on ∂P and ∂B are contactomorphic.

Then, by a result of Etnyre in [15], $Z = (X - \text{int}(P)) \cup B$ inherits a symplectic structure from ω and ω_2. This operation is called *symplectic cut-and-paste*. As a smooth manifold, Z may depend on the choice of contactomorphism. If the induced contact structures happen to be isotopic, however, then by choosing a contactomorphism isotopic to the identity, Z is well-defined as a smooth manifold.

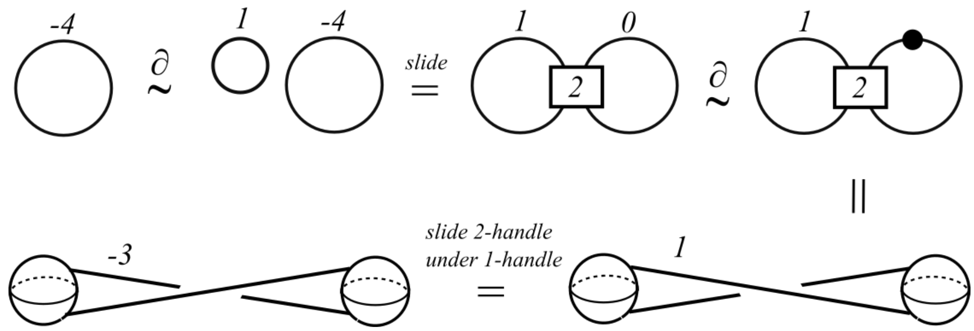
Definition 3.2.1. *Suppose P and B are 4-manifolds with $\partial P = \partial B$. If P and B admit symplectic structures with strongly convex boundary that induce isotopic contact structures, then we say that P can be **symplectically replaced** by B and we call B a **symplectic replacement** of P.*

Coupling Etnyre's result with Lisca's classification of symplectic fillings of lens spaces in Example 3.1.16 and using the terminology introduced in Section 2.3, we can expand on Observation 2.3.3.

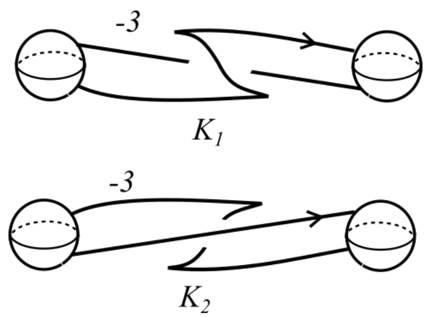
Observation 3.2.2. *A linear plumbing can be symplectically rationally blown down if and only if it can be obtained by a sequence of buddings of the -4-disk bundle over S^2.*

In [23], Gay and Mark showed that under suitable conditions, any negative definite plumbing P with no bad vertices embedded in an ambient symplectic 4-manifold admits a symplectic structure with strongly convex boundary (see Section 3.2.2 for more details). Thus the difficulty in performing symplectic cut-and-paste lies finding a symplectic replacement. To do this from scratch, we can first try to find a smooth 4-manifold B such that $\partial B = \partial P$ using Kirby calculus. Then, if we are lucky, we might be able to realize a handlebody diagram of B as a Stein diagram, implying that B admits a symplectic structure with strongly convex boundary. Finally, we would have to decide whether the contact structure induced on ∂B is contactomorphic to the contact structure induced on ∂P. This might be accomplished by first deciding how many fillable contact structures ∂P admits and then applying contact structure invariants.

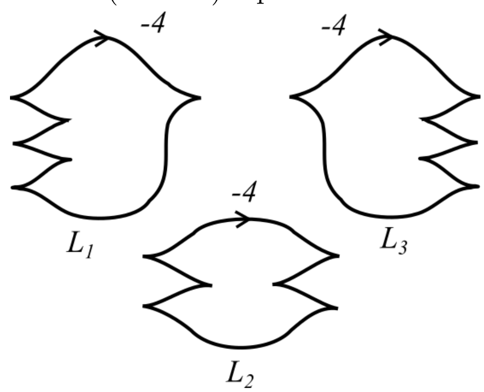
Example 3.2.3 (Symplectic rational blowdown)**.** Following the ideas above, we will show that rationally blowing down the plumbing $P = C_{2,1}$ can be done symplectically (without relying on Symington's result). We first find a rational ball (smooth) replacement B of P using Kirby calculus, as shown in Figure 3.3a. The top-left diagram is the obvious handlebody diagram for P and the top-right diagram is the rational ball B. By changing the dotted circle into two 3-balls, we obtain the bottom-right diagram. After sliding the 2-handle under the 1-handle, we obtain the bottom-left diagram.



(a) Using Kirby calculus to find the rational ball (smooth) replacement B.



(b) Two Stein structures on B.



(c) Three Stein structures on P.

Figure 3.3: All framings are smooth framings

In [35], Gompf showed that for non null-homologous knots such as K_i in Figure 3.3b, $tb(K_i)$ and $r(K_i)$ can be defined and computed using the usual formulas given in Example 3.1.5. Thus the two diagrams in Figure 3.3b describe Stein structures on B, since $tb(K_i) - 1 = -2 - 1 = -3$. Moreover, since $r(K_1) = -1$ and $r(K_2) = 1$, the Stein structures are different, by Proposition 3.1.17. Moreover, by Theorem 3.1.18, these induce nonisotopic contact structures on $L(4,1)$. Similarly, there are three Stein structures on P, as depicted in Figure 3.3c, which induce nonisotopic contact structures on $L(4,1)$, since $r(L_1) = -2$, $r(L_2) = 0$, and $r(L_3) = 2$. As mentioned in Example 3.1.19, by Honda's classification [38], $L(4,1)$ admits exactly three nonisotopic tight contact structures.

Recall, if Σ is a symplectic surface embedded in a symplectic 4-manifold (X,ω), then the following *adjunction equality* holds:

$$\langle c_1(X,J), [\Sigma] \rangle = [\Sigma]^2 + \chi(\Sigma),$$

where J is an ω-compatible almost complex structure. Thus, if a -4-sphere S is symplectically embedded in a symplectic 4-manifold (X,ω), we necessarily have that $\langle c_1(X,J), [S] \rangle = -2$. Moreover, by Proposition 3.1.17, $\langle c_1(X,J), [S] \rangle = r(K)$, where

K is the attaching circle of the 2-handle making up S. Thus, the Stein structure on P must be the one obtained by attaching a 2-handle to L_1.

Let ξ_{L_i} denote the contact structure induced on $L(4,1)$ by the Stein structure obtained by attaching a 2-handle along L_i. Then $d_3(\xi_{L_1}) = d_3(\xi_{L_3}) = \frac{1}{4}(-1-3(-1)-2(2)) = -\frac{1}{2}$ and $d_3(\xi_{L_2}) = \frac{1}{4}(0-3(-1)-2(2)) = -\frac{1}{4}$. Similarly, if ξ_{K_i} denotes the contact structure induced on $L(4,1)$ by the Stein structure obtained by attaching a 2-handle along K_i, then $d_3(\xi_{K_1}) = d_3(\xi_{K_2}) = \frac{1}{4}(0-3(0)-2(1)) = -\frac{1}{2}$. Thus, $\{\xi_{L_1}, \xi_{L_3}\} = \{\xi_{K_1}, \xi_{K_2}\}$ up to isotopy and so there exists a Stein structure on B such that the induced contact structure on ∂B is isotopic to ξ_{L_1}. Therefore, B is a symplectic replacement of P and so P can be symplectically rationally blown down.

Remark 3.2.4. As mentioned in Remark 2.3.6, by Lisca's classification (Example 3.1.16) the plumbing $\tilde{C}_{2,1}$ cannot be symplectically replaced by a rational ball. Notice that, in particular, we cannot obtain a Stein structure on the rational ball $\overline{B}_{2,1}$ in the same way that we obtained a Stein structure on $B_{2,1}$ in Example 3.2.3.

3.2.1 Classifications of contact structures

A crucial step in the calculation of Example 3.2.3 was knowing how many fillable contact structures the boundary admitted. There are many classification results for (strongly symplectically) fillable contact structures on plumbed 3-manifolds. As we saw in Example 3.1.19, Honda classified the number of fillable contact structures on lens spaces, which are the boundaries of linear plumbings. Fillable contact structures on small Seifert fibered spaces have also been classified in [27], [28], and [76]. These Seifert fibered spaces bound plumbings whose associated graphs are "star-shaped."

The typical approach to the proofs of these results are broken into two parts. First an upper bound k for the number of tight contact structures is given using *convex surface theory* (c.f. Chapter 6) and applications of Honda's classifications of tight contact structures on the "building blocks" $S^1 \times D^2$, $T^2 \times I$, and $S^1 \times \Sigma$, where Σ is a pair of pants ([38], [39]). Then k is shown to be a lower bound by exhibiting k distinct Stein diagrams, which induce k nonisotopic contact structures, by Theorem 3.1.18.

By a deep theorem of Colin, Giroux, and Honda [7], a closed, oriented, atoroidal 3-manifold admits finitely many tight contact structures up to isotopy. In particular, the boundary of any plumbing tree admits finitely many tight contact structures. On the other hand, Honda, Kazez, and Matic showed in [40] that if a 3-manifold contains an incompressible torus T, then it admits infinitely many tight contact structures. These contact structures can be constructed by starting with a tight contact structure and adding *Giroux torsion* in a neighborhood of T. We say a contact 3-manifold contains *Giroux n-torsion*, for $n \geq 1$, if there exists an embedding of $(T^2 \times I, \xi_n = \ker(\sin(2n\pi z)dx + \cos(2n\pi z)dy))$. Luckily, in [24], Gay showed that tight contact structures that are strongly symplectically fillable have no Giroux torsion (i.e. there does not exist such an embedding).

In Example 3.1.20, we mentioned that Honda classified the number of tight contact structures on the boundaries of cyclic plumbings (i.e. on torus bundles over S^1). These contain incompressible tori and thus admit infinitely many tight contact structures, giving an unfortunate upper bound on the number of strongly symplectically fillable contact structures. However, Honda's classification shows that these torus bundles admit only finitely many tight contact structures with no Giroux torsion. Thus, in light of Gay's result, the set of possible strongly symplectically fillable contact structures on such 3-manifolds is finite. This set turns out to be the set of contact structures mentioned in Example 3.1.20. In Chapter 6, by using the above techniques and a generalization of Lisca-Matic's Theorem 3.1.18, we will classify the number of tight contact structures with no Giroux torsion on the boundaries of plumbings with a single cycle and an "arm" emanating from the cycle.

3.2.2 Lefschetz fibrations and open books

In this section, we will explore how Lefschetz fibrations and open book decompositions can be used to perform symplectic cut-and-paste. We assume the reader is familiar with the basics of Lefschetz fibrations and open book decompositions (see [16] and [58] for nice expositions). We first recall an important result of Giroux.

Theorem 3.2.5 (Giroux's Correspondence [31]). *Let Y be a fixed 3-manifold. Then there is a one-to-one correspondence between contact structures on Y, up to isotopy, and open book decompositions of Y, up to common positive stabilization. In particular, given a contact structure ξ, there is a unique open book decomposition (up to common stabilization) that supports ξ. Conversely, given an open book decomposition, there is a unique contact structure (up to isotopy) that it supports.*

Let $\mathcal{C} = C_1 \cup \cdots \cup C_n$ denote a configuration of symplectic spheres embedded in a symplectic 4-manifold (X, ω). Then any neighborhood of $\mathcal{C}$ is a plumbing. Suppose the associated plumbing graph is negative definite and has no bad vertices. Further assume that these spheres intersect ω-orthogonally and positively. Let v_i denote the valence and let $-m_i$ denote the weight of the vertex corresponding to C_i. Let Σ denote the surface with boundary obtained by connect-summing $m_i - v_i$ copies of D^2 to each C_i and then connect summing these surfaces according the plumbing graph. Let $\{t_1, ..., t_k\}$ be the collection of simple closed curves on Σ consisting of one curve around each connected sum neck. Let τ denote the product of right Dehn twists along these curves. See Figure 3.4 for an example. In [23], Gay and Mark showed the following.

Theorem 3.2.6 (Gay-Mark [23]). *There exists a symplectic neighborhood (Z, ν) of $\mathcal{C}$ with strongly convex boundary that admits a symplectic Lefschetz fibration $\pi : Z \to D^2$ having regular fiber Σ and exactly one singular fiber $\Sigma_0 = \pi^{-1}(0)$. The vanishing cycles* *are* $t_1, ..., t_k$ *and* $\mathcal{C}$ *is the union of the closed components of* Σ_0*. The induced contact structure* ξ *on* ∂Z *is supported by the open book induced by* (Σ, τ)*.*

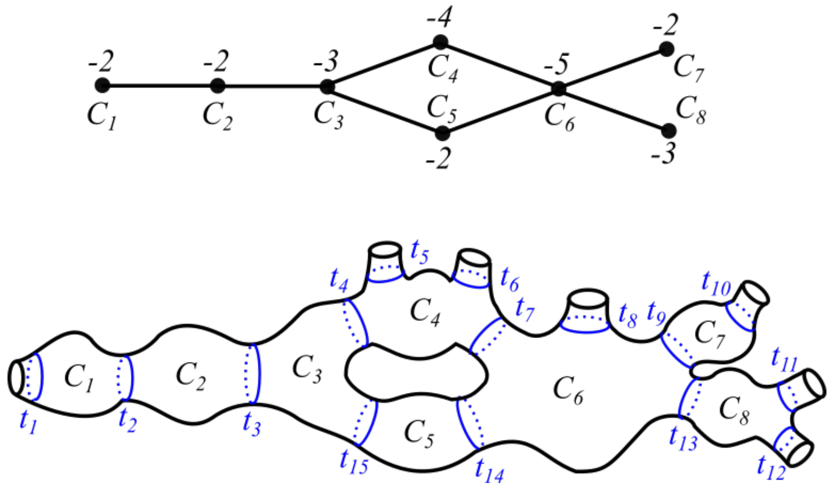


Figure 3.4: A (positive) plumbing and the associated surface Σ with vanishing cycles $t_1, ..., t_{15}$ corresponding to the Lefschetz fibration of Theorem 3.2.6.

Remark 3.2.7. This theorem is also true for plumbings whose base surfaces C_i have arbitrary genus. We focus on spheres here because it is the only case with which we will be concerned.

By Theorem 3.2.5, the open book decomposition (Σ, τ) of ∂Z induced by the Lefschetz fibration of Theorem 3.2.6 supports a unique contact structure ξ, up to isotopy. Now suppose we can re-factor the monodromy τ into $\tau' = \tau_{t'_1} \cdots \tau_{t'_{k'}}$, where $t'_1, ..., t'_{k'}$ are homologically essential curves on Σ. Then (Σ, τ') is another open book decomposition for ∂Z that supports ξ. This monodromy also describes a *positive allowable* Lefschetz fibration (PALF) $\pi' : Z' \to D^2$ (i.e. a Lefschetz fibration whose monodromy is composed of right Dehn twists about homologically essential curves), where $\partial Z' = Y = \partial Z$. By Loi and Peirgallini [50], Z' naturally admits a Stein structure and by Plamanevskaya [68], this structure fills (Y, ξ). Thus Z' is a symplectic replacement of the plumbing Z. This is summed up in a corollary of Gay and Mark.

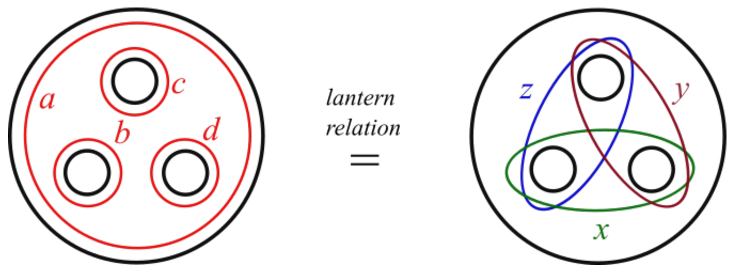
Corollary 3.2.8 (Gay-Mark [23])**.** *In the setting of Theorem 3.2.6, suppose that* $t'_1, ..., t'_{k'}$ *is a sequence of homologically essential simple closed curves in* Σ *such that the product* τ' *of Dehn twists along these curves is isotopic to* τ *. Let* Z' *be the 4-manifold with smooth Lefschetz fibration over* D^2 *having regular fiber* Σ *and vanishing cycles* $t'_1, ..., t'_{k'}$ *(on disjoint fibers). Then* $(X - Z) \cup Z'$ *supports a symplectic form* ν *inherited from* ω *on* X *and* ω' *on* Z'*.*

Finally, notice that it is possible to draw a handlebody diagram for Z using a regular fiber Σ and vanishing cycles $t_1, ..., t_k$. We will only focus on the case when Σ is a planar surface, which will be used in Section 4.2. Suppose Σ is a disk with n holes. Then a neighborhood $\Sigma \times D^2$ is diffeomorphic to $(D^2 \times D^2) - [(D_1^2 \times D^2) \sqcup \cdots \sqcup (D_n^2 \times D^2)]$, where D_i is a small disk in the first factor of $D^2 \times D^2$ and $D_i \cap D_j$ is empty for $i \neq j$. Thus $\Sigma \times D^2$ is diffeomorphic to a 1-handlebody with n 1-handles and so it has a handlebody diagram consisting of an n-component unlink decorated with dots. Considering Σ as a round disk in the xy plane with n holes, the components of this unlink can be thought of as vertical line segments that pass through the holes (and close up into unknots in the obvious, trivial way). To finish building Z, we must attach a 2-handle to each vanishing cycle with framing -1 in parallel copies of Σ, in the order (from bottom to top) in which they appear in the monodromy factorization. See [13] for more details.

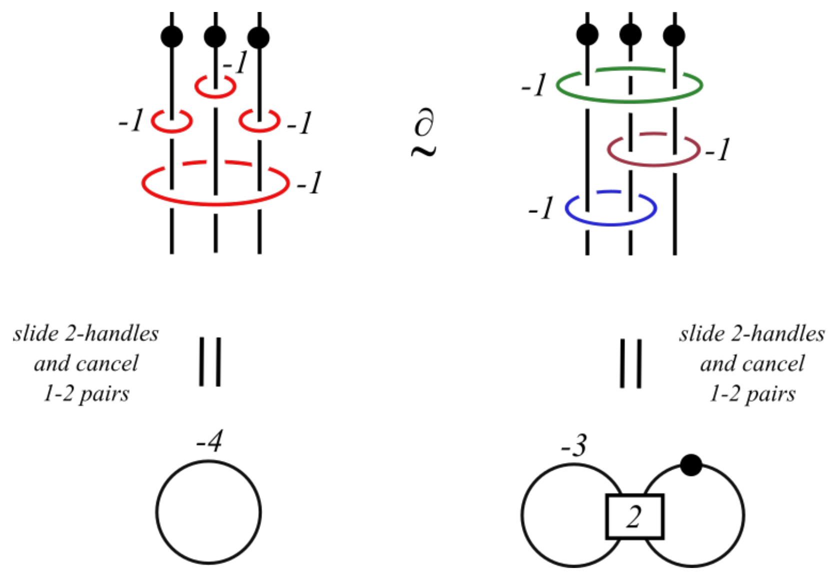
Example 3.2.9 (Symplectic rational blowdown, revisited)**.** We revisit Example 3.2.3 and show that rationally blowing down the plumbing $P = C_{2,1}$ can be done symplectically by using Lefschetz fibrations and Corollary 3.2.8. First, by Theorem 3.2.6, we can view P as a Lefschetz fibration that has the mondromy depicted on the left of Figure 3.5a. The associated handlebody diagram is shown on the top-left of Figure 3.5b. Note that by performing handle slides and cancellations, we can easily obtain the obvious handlebody diagram for P shown in the bottom-left of Figure 3.5b.

By using the famous *lantern relation* (see [41]), we obtain the monodromy factorization shown in the right of Figure 3.5a. Let B' be the total space of the Lefschetz fibration described by this latter factorization. By drawing the obvious handlebody diagram for B', as in the top-right of Figure 3.5b, an easy computation shows that B' is a rational homology 4-ball. By Corollary 3.2.8, B' is a symplectic replacement of P. Thus, P can be symplectically rationally blown down.

Finally, consider the following moves applied to the handlebody diagram of B' depicted in the top-right of Figure 3.5b: slide the red 2-handle over the green 2-handle; cancel the resulting 1-2 pair; slide the blue 2-handle over the green 2-handle; and cancel the resulting 1-2 pair. The result of this Kirby calculus is the handlebody diagram at the bottom-right of Figure 3.5b. Notice that B is diffeomorphic to the rational homology 4-ball replacement of P constructed in Example 3.2.3 and depicted in Figure 3.3a. Thus the symplectic replacement constructed in this example is the same as the symplectic replacement constructed in Example 3.2.3.



(a) The lantern relation $abcd = xyz$



(b) Handlebody diagrams of the total spaces of the Lefschetz fibrations given by the monodromy factorizations $abcd = xyz$ given in Figure 3.5a. The tops and bottoms of the vertical dotted lines are identified trivially to dotted circles.

Figure 3.5: Using Lefschetz fibrations for symplectic cut-and-paste

Chapter 4

k-Replaceability

Suppose P is a negative definite plumbing embedded in a symplectic 4-manifold (X, ω) that has no bad vertices and whose base spheres are symplectic and intersect ω-orthogonally and positively. By Theorem 3.2.6, P admits a symplectic structure with strongly convex boundary. Moreover, this symplectic structure induces a canonical contact structure on ∂P that does not depend on the ambient symplectic manifold, up to isotopy.

Definition 4.0.1. *P is called **k-replaceable** if it can be symplectically replaced by a negative definite, minimal symplectic 4-manifold B satisfying $\chi(B) = k$ and $b_3(B) = 0$. We say that P can be **k-replaced** by B and we call B a **k-replacement** of P.*

Our goal is to use k-replaceable plumbings to construct closed, simply connected, symplectic, exotic 4-manifolds that are "small," as measured by b_2. Thus we would like B to be an Euler characteristic k manifold with the smallest possible second Betti number. This is why we require that B is minimal and that $b_3(B) = 0$. We further require B to be negative definite so that Michalogiorgaki's gluing formula (Theorem 2.4.3) is applicable. Moreover, by considering the long exact sequences of the pairs $(B, \partial B)$ and $(P, \partial P)$, since B and P are negative definite and $b_3(P) = b_3(B) = 0$, it follows that $b_1(P) = b_1(B)$. With this terminology in place, we can slightly modify Observation 3.2.2.

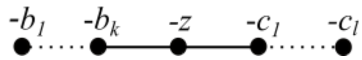
Observation 4.0.2. *A linear plumbing is 1-replaceable if and only if it can be (symplectically) rationally blown down if and only if it can be obtained by a sequence of buddings of the -4-disk bundle over S^2.*

In this chapter, we will focus on 2-replaceable plumbings. In particular, we will classify 2-replaceable linear plumbings, construct 2-replaceable plumbing trees, and construct a symplectic exotic $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$. This chapter is organized as follows. In Section 4.1, we will state the main results in detail. In Section 4.2, we prove the result regarding 2-replaceable trees. In Section 4.3, we use one such plumbing tree to construct the symplectic exotic $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$. Finally the proof of the classification of 2-replaceable linear plumbings can be found in Sections 4.4 and 4.5.

4.1 2-replaceable plumbings

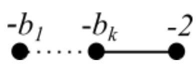
Since 1-replaceable plumbings trees were classified by Lisca in [47] and Stipsicz-Szabó-Wahl in [71], we focus on 2-replaceable plumbings. In particular, we will prove the following theorems.

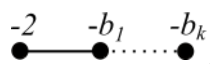
Theorem 4.1.1. *Let $(-b_1, ..., -b_k)$ and $(-c_1, ..., -c_l)$ be obtained by sequences of buddings of -4 and let $z \geq 2$ be any integer. Then a minimal linear plumbing is 2-replaceable if and only if it is either of the form:*

(a) 

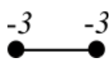
for $k, l \geq 0$

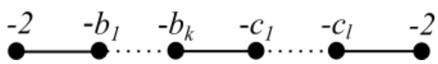
or can be obtained by a sequence of buddings of one of the linear plumbings of the form:

(b) 

(or 

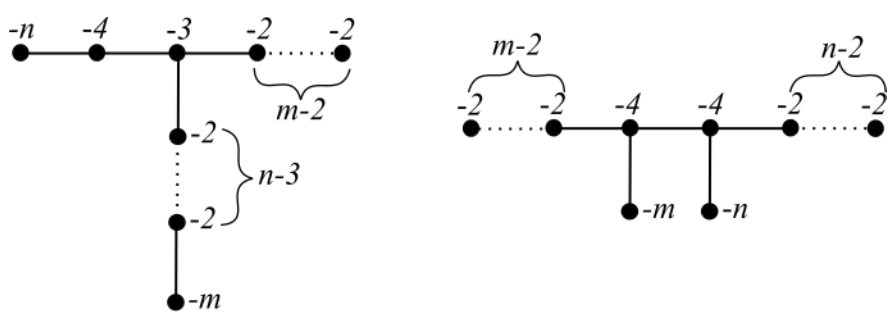
) *for* $k \geq 0$.

(c) 

(d) 

for $k, l \geq 1$

Theorem 4.1.2. *For any integers $n, m \geq 3$, the following plumbing trees are 2-replaceable.*



Remark 4.1.3. The proof of Theorem 4.1.1 relies on Lisca's classification of symplectic fillings of lens spaces equipped with the canonical contact structure inherited from the unique tight contact structure on S^3 (c.f. Example 3.1.16). The proof, which can be found in Section 4.5, mainly consists of computations involving continued fractions. Similar computations can be completed to find (and classify) families of k-replaceable linear plumbings for $k \geq 3$.

Remark 4.1.4. The families of plumbing trees in Theorem 4.1.2 will be constructed from the (2-replaceable) linear plumbing with weights $(-2,-4,-4,-2)$. In the proof of the theorem, we will show that this linear plumbing is indeed 2-replaceable without relying on Theorem 4.1.1. Instead, we will apply the theory of Lefschetz fibrations (c.f. Section 3.2.2). It turns out that the families of plumbing trees of Theorem 4.1.2 are interesting in the sense that they cannot all be built trivially by plumbing the 1-replaceable trees of [71] to a disk bundle over S^2 (c.f. the plumbings in Theorem 4.1.1a). Moreover, the technique used in the proof of Theorem 4.1.2 can be applied to obtain more families of 2-replaceable trees. For example, instead of starting with the linear plumbing with weights $(-2,-4,-4,-2)$, one could start with a different 2-replaceable linear plumbing.

Theorem 4.1.5. *The 2-replaceable tree of Theorem 4.1.2(a) with $n = 9$ and $m = 3$ can be embedded in $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$. Call this tree P and let B denote its Euler characteristic 2 replacement. Then $X = (\mathbb{C}P^2\#16\overline{\mathbb{C}P^2} - \mathrm{int}(P)) \cup_{\partial P} B$ is homeomorphic but not diffeomorphic to $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$. Furthermore, X admits a symplectic structure.*

4.2 Lefschetz fibrations and the Key Lemma

We first highlight the strategy used to prove Theorem 4.1.2, relying on the background developed in Section 3.2.2. Let P be a symplectic negative definite plumbing with strongly convex boundary that admits a symplectic Lefschetz fibration over D^2 with monodromy τ that can be written down in an explicit factorization. This monodromy naturally describes an open book decomposition of Y that supports the contact structure ξ induced by the symplectic structure. Suppose there is a different factorization of τ into right Dehn twists about homologically essential curves such that the total space B of the corresponding Lefschetz fibration has Euler characteristic 2. Then, as discussed in Section 3.2.2, B is a symplectic replacement of P. Since the obvious handlebody diagram of B obtained from the monodromy (c.f. Section 3.2.2) has no 3-handles, we have that $b_3(B) = 0$. Finally, in [14], Etnyre showed that any strong symplectic filling of a contact manifold supported by a planar open book is negative definite. Thus B is a 2-replacement of P.

We will apply the following *Key Lemma* due to Endo, Mark, and Van Horn-Morris in [13] to the monodromy factorizations associated to P and B.

Lemma 4.2.1. *(Key Lemma [13]) Let F be a planar surface containing as a subsurface a pair of pants, S_3. Let z and d be the boundary parallel curves marked in Figure 4.1 and let the boundary component of S_3 corresponding to z coincide with a component of ∂F. Let F' be the planar surface obtained from F by gluing a disk with two holes into the hole enclosed by z. Suppose that in the planar mapping class group $Mod(F, \partial F)$, the relation $w_1 z w_2 = w_1' d w_2'$ holds for some*

$w_1, w_2, w_1', w_2' \in Mod(F, \partial F)$. If a commutes with either w_1 and w_1' or w_2 and w_2', then in $Mod(F', \partial F')$ we have the relation $w_1 abc w_2 = w_1' xy w_2'$.

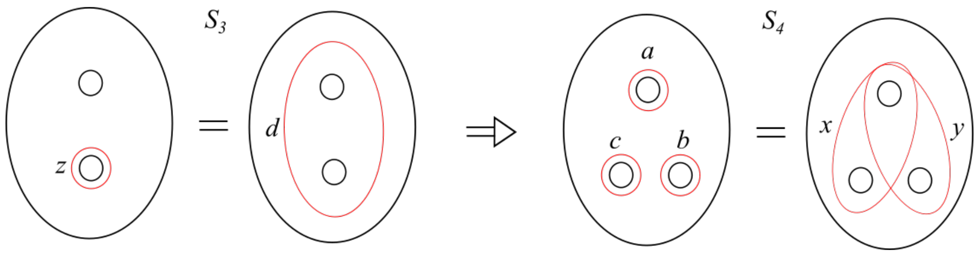


Figure 4.1: The Key Lemma

Assume the Key Lemma applies to the monodromies of P and B and suppose P contains the curve z and B contains the curve d, as depicted in Figure 4.1. Let P' and B' denote the total spaces the the Lefschetz fibrations associated to the two new equivalent monodromy factorizations obtained from the Key Lemma. Then $Y' = \partial P' = \partial B'$ has an open book decomposition that can be described by these two factorizations. By Giroux's correspondence (Theorem 3.2.5), $Y = \partial P' = \partial B'$ admits a contact structure ξ' that is supported by both open books. By [23] and [68], P' and B' both admit symplectic structures that are (strong) symplectic fillings of (Y', ξ'). Thus B' is a symplectic replacement of P'.

We now claim that $\chi(B') = 2$, $b_3(B') = 0$, and that B' is negative definite. Since the obvious handlebody diagram of B' obtained from its monodromy (c.f. Section 3.2.2) has one more 1-handle and one more 2-handle than the obvious handlebody diagram of B, it follows that $\chi(B') = \chi(B) = 2$. Since there are no 3-handles in this diagram, we have that $b_3(B') = 0$. Once again, by a result of Etnyre in [14], since B' is a strong symplectic filling of a contact manifold supported by a planar open book, B' is negative definite.

Finally, for B' to be a 2-replacement of P', B' must be minimal. For this, we restrict our attention to the plumbings of Theorem 4.1.2. Using the above arguments, we will construct these plumbings and Euler characteristic 2 symplectic replacements in the next section. Suppose P' is such a plumbing and let B' be its Euler characteristic 2 symplectic replacement. If B' is not minimal, then we can symplectically blow down a symplectic -1-sphere to obtain a 1-replacement of P'. In other words, P' can be symplectically rationally blown down. All such plumbing trees are classified by Stipsicz-Szabó-Wahl in [71]. Since P' is not among those trees, B' must be minimal and so P' is 2-replaceable.

4.2.1 Proof of Theorem 4.1.2

To construct the families of plumbing trees of Theorem 4.1.2, we will iteratively apply the Key Lemma. By the remarks above, these trees will automatically be

2-replaceable. All monodromy factorizations will be products of right Dehn twists around simple closed curves. For simplicity, a curve and a right Dehn twist about the curve will have the same label.

Let P be the linear plumbing with framings $(-2,-4,-4,-2)$. By Theorem 3.2.6, P can be viewed as a Lefschetz fibration with the monodromy factorization drawn on the left side of Figure 4.2. It is given by $x_0^2x_1x_2x_3yx_4x_5^2$. Using a lantern relation applied to $x_3yx_4x_5$, we obtain the middle factorization in Figure 4.2, $x_0^2x_1x_2zefx_5 = x_0^2x_1x_2zx_5ef$. Finally, using the more general *daisy relation* (defined in [13]), applied to $x_0^2x_1x_2zx_5$, we obtain the factorization $abcdef$ pictured on the right side of Figure 4.2. By drawing a handlebody diagram of the total space of the Lefschetz fibration described by the monodromy factorization $abcdef$ (as discussed in Section 3.2.2), easy homology calculations show that this 4-manifold is a 2-replacement of P.

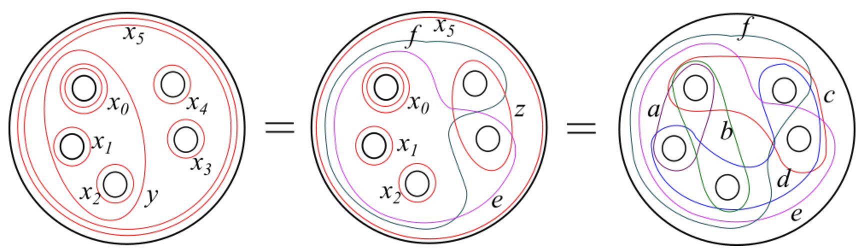
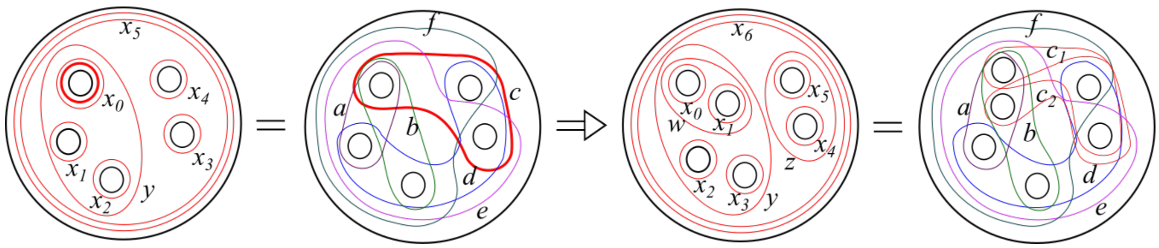


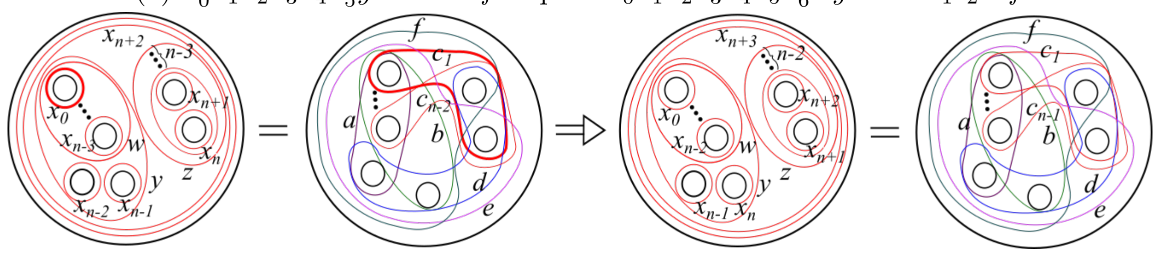
Figure 4.2: $x_0^2x_1x_2x_3x_4x_5^2y = x_0^2x_1x_2zx_5ef = abcdef$

Now we will repeatedly apply the Key Lemma to the relation $x_0^2x_1x_2x_3x_4x_5^2y = abcdef$ to build a family of 2-replaceable trees. Notice that the curve labelled z in the right side of Figure 4.3a commutes with the curves labelled a and b in the left side of Figure 4.3a. Thus the Dehn twist z commutes with the Dehn twist ab and so we can apply the Key Lemma to x_0 and c, which are shown in bold in the left of Figure 4.3a. Thus the hole encircled by x_0 splits and we obtain the relation $zx_1x_0wx_2x_3x_4x_5x_6^2y = abc_1c_2def$, or $x_0x_1x_2x_3x_4x_5x_6^2wyz = abc_1c_2def$, depicted on the right side of Figure 4.3a. Notice that in the new relation, we relabelled the boundary parallel curves for convenience and one of the curves that was labelled x_0 is now labelled w. This relabelling will be done throughout. Once again it is easy to see that the total space of the Lefschetz fibration described by the monodromy abc_1c_2def is a 2-replacement of the plumbing tree associated to the monodromy $x_0x_1x_2x_3x_4x_5x_6^2wyz$, which is depicted in Figure 4.4a. Now, inductively assume that the relation $x_0\cdots x_{n+3}^2wyz^{n-3} = abc_1\cdots c_{n-1}def$ holds, as in the left side of Figure 4.3b. Then since z commutes with ab, we can apply the Key Lemma to x_0 and c_1 to obtain the relation $x_0\cdots x_{n+3}^2wyz^{n-2} = abc_1\cdots c_{n-1}def$. As before, the total space of the Leftschetz fibration described by the monodromy $abc_1\cdots c_{n-1}def$ is a 2-replacement of the plumbing tree shown in Figure 4.4b.

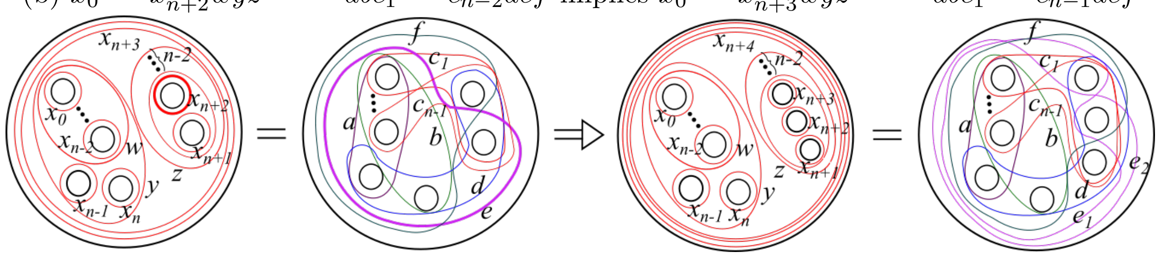
Next, we apply the Key Lemma to x_{n+2} and e. To do this, view the $n{+}3$ punctured disk as an $n+4$ punctured sphere so that the outermost boundary of the disk is just



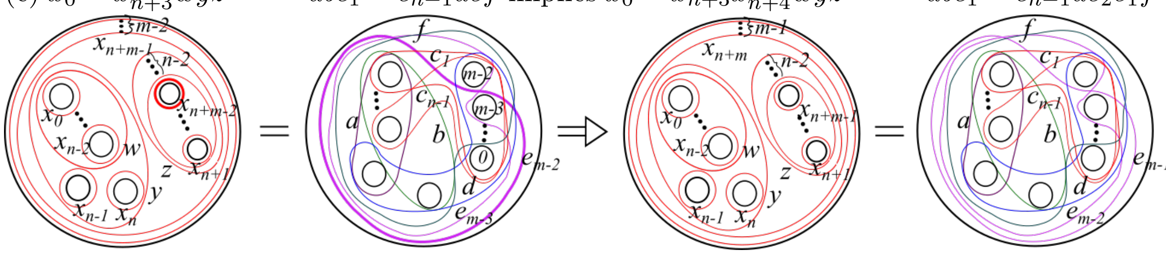
(a) $x_0^2 x_1 x_2 x_3 x_4 x_5^2 y = abcdef$ implies $x_0 x_1 x_2 x_3 x_4 x_5 x_6^2 wyz = abc_1c_2def$



(b) $x_0 \cdots x_{n+2}^2 wyz^{n-3} = abc_1 \cdots c_{n-2}def$ implies $x_0 \cdots x_{n+3}^2 wyz^{n-2} = abc_1 \cdots c_{n-1}def$



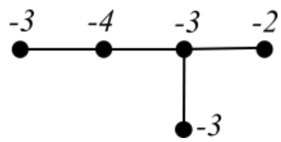
(c) $x_0 \cdots x_{n+3}^2 wyz^{n-2} = abc_1 \cdots c_{n-1}def$ implies $x_0 \cdots x_{n+3} x_{n+4}^3 wyz^{n-2} = abc_1 \cdots c_{n-1}de_2e_1f$



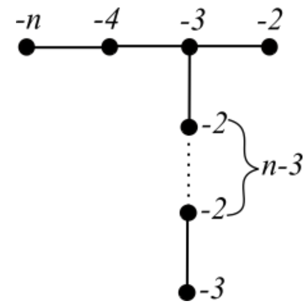
(d) $x_0 \cdots x_{n+m-2} x_{n+m-1}^{m-2} wyz^{n-2} = abc_1 \cdots c_{n-1}de_{m-2} \cdots e_1 f$ implies
$x_0 \cdots x_{n+m-1} x_{n+m}^{m-1} wyz^{n-2} = abc_1 \cdots c_{n-1}de_{m-1} \cdots e_1 f$

Figure 4.3: Repeated applications of the Key Lemma to the bold circles

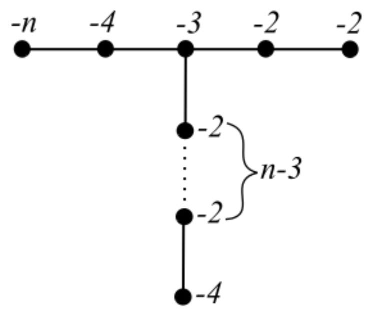
another puncture. In this way, we can view x_{n+3} as a curve around a puncture and e as a curve around the two punctures with boundary parallel curves x_{n+2} and x_{n+3}. These are shown in bold on the left side of Figure 4.3c. Since x_{n+4} (as labelled on the right side of Figure 4.3c) commutes with everything, the Key Lemma applies, yielding the relation $x_0 \cdots x_{n+1} x_{n+4} x_{n+2} x_{n+3} x_{n+4}^2 wyz^{n-2} = abc_1 \cdots c_{n-1}de_2e_1f$, or $x_0 \cdots x_{n+3} x_{n+4}^3 wyz^{n-2} = abc_1 \cdots c_{n-1}de_2e_1f$. This relation proves that the linear plumbing depicted in Figure 4.4c is 2-replaceable. Now, inductively assume that the



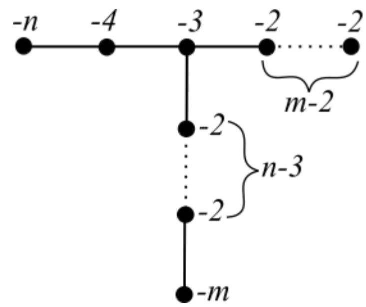
(a) Plumbing associated to Figure 4.3a



(b) Plumbing associated to Figure 4.3b



(c) Plumbing associated to Figure 4.3c

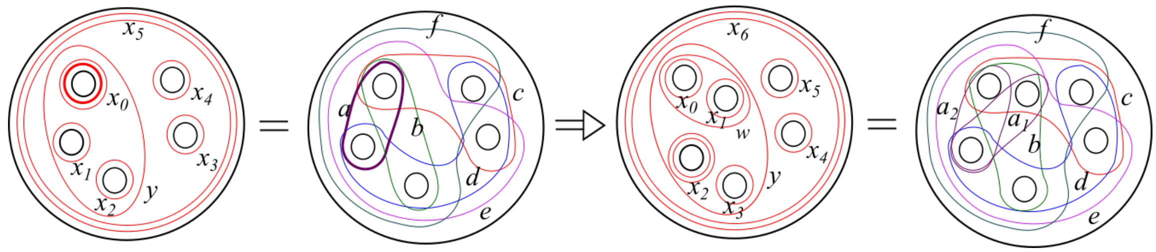


(d) Plumbing associated to Figure 4.3d

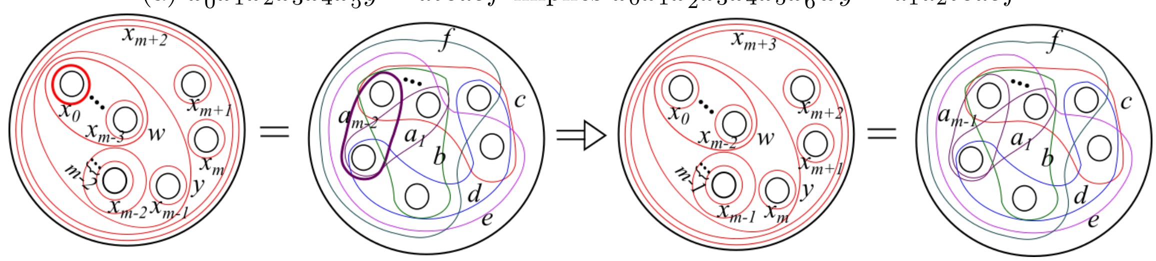
Figure 4.4: 2-replaceable plumbings associated to the monodromies in Figure 4.3

relation $x_0 \cdots x_{n+m-2} x_{n+m-1}^{m-2} wyz^{n-2} = abc_1 \cdots c_{n-1} de_{m-2} \cdots e_1 f$ holds, as in the left side of Figure 4.3d. Notice that each curve e_i encircles all the punctures except the one labelled i for $1 \leq i \leq m-2$, as depicted in Figure 4.3d. Also note that, with this labelling, we can write $f = e_0$. We now apply the Key Lemma as we did previously to the bold curves labelled x_{n+m-2} and e_{m-2} to obtain the relation $x_0 \cdots x_{n+m-1} x_{n+m}^{m-1} wyz^{n-2} = abc_1 \cdots c_{n-1} de_{m-1} \cdots e_1 f$. This relation proves that the plumbing tree depicted in Figure 4.4d is 2-replaceable for all $m \geq 3$. Thus we have proved that the plumbing trees of Theorem 4.1.2a are indeed 2-replaceable.

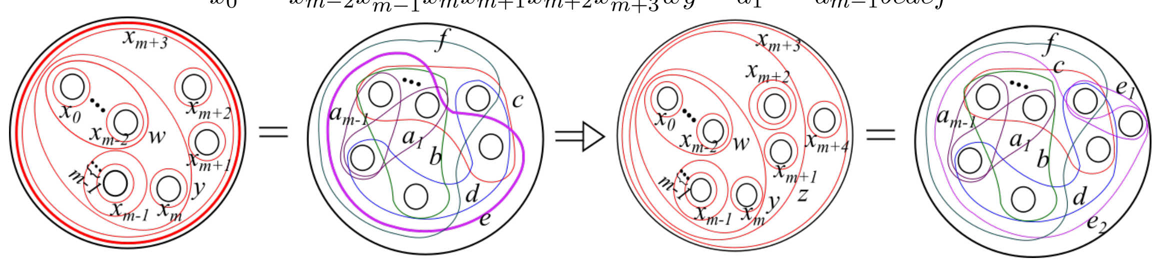
To obtain the family in Theorem 4.1.2b, we go back to the the relation depicted in Figure 4.2, namely $x_0^2 x_1 x_2 x_3 x_4 x_5^2 y = abcdef$. We will apply the Key Lemma to the bold circles x_0 and a shown in Figure 4.5a. Since x_2, as labelled in the third surface in Figure 4.5a commutes with everything, the Key Lemma applies and we obtain the relation $wx_2 x_0 x_1 x_2 x_3 x_4 x_5 x_6^2 y = a_1 a_2 bcdef$, or $x_0 x_1 x_2^2 x_3 x_4 x_5 x_6^2 wy = a_1 a_2 bcdef$, as shown in Figure 4.5a. Thus, the plumbing tree shown in Figure 4.6a is 2-replaceable. Inductively assume that $x_0 \cdots x_{m-3} x_{m-2}^{m-2} x_{m-1} x_m x_{m+1} x_{m+2}^2 wy = a_1 \cdots a_{m-2} bcdef$, as in Figure 4.5b. Again, since x_{m-1}, as labelled in the third monodromy in Figure 4.5b, commutes with everything, we can apply the Key Lemma to obtain the relation $x_0 \cdots x_{m-2} x_{m-1}^{m-1} x_m x_{m+1} x_{m+2} x_{m+3}^2 wy = a_1 \cdots a_{m-1} bcdef$. Thus, the plumbing tree in Figure 4.6b is 2-replaceable.



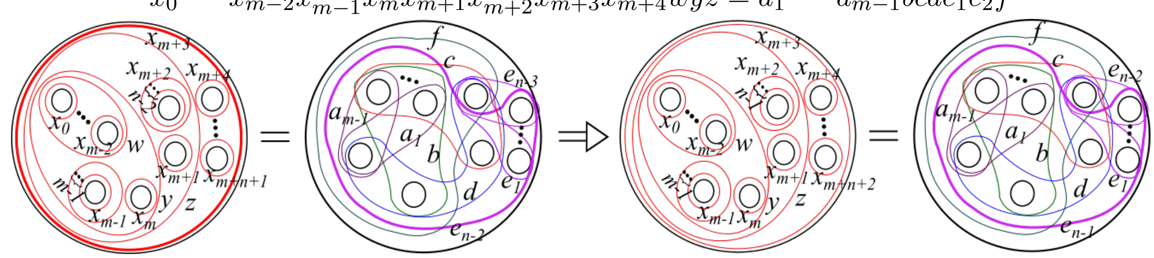
(a) $x_0^2 x_1 x_2 x_3 x_4 x_5^2 y = abcdef$ implies $x_0 x_1 x_2^2 x_3 x_4 x_5 x_6^2 wy = a_1 a_2 bcdef$



(b) $x_0 \cdots x_{m-3} x_{m-2}^{m-2} x_{m-1} x_m x_{m+1} x_{m+2}^2 wy = a_1 \cdots a_{m-2} bcdef$ implies
$x_0 \cdots x_{m-2} x_{m-1}^{m-1} x_m x_{m+1} x_{m+2} x_{m+3}^2 wy = a_1 \cdots a_{m-1} bcdef$



(c) $x_0 \cdots x_{m-2} x_{m-1}^{m-1} x_m x_{m+1} x_{m+2} x_{m+3}^2 wy = a_1 \cdots a_{m-1} bcdef$ implies
$x_0 \cdots x_{m-2} x_{m-1}^{m-1} x_m x_{m+1} x_{m+2}^2 x_{m+3} x_{m+4} wyz = a_1 \cdots a_{m-1} bcde_1 e_2 f$

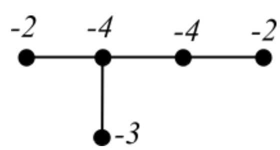


(d) $x_0 \cdots x_{m-2} x_{m-1}^{m-1} x_m x_{m+1} x_{m+2}^{n-2} x_{m+3} x_{m+4} wyz = a_1 \cdots a_{m-1} bcde_1 e_2 f$ implies
$x_0 \cdots x_{m-2} x_{m-1}^{m-1} x_m x_{m+1} x_{m+2}^{n-1} x_{m+3} \cdots x_{m+n+2} wyz = a_1 \cdots a_{m-1} bcde_1 \cdots e_{n-1} f$

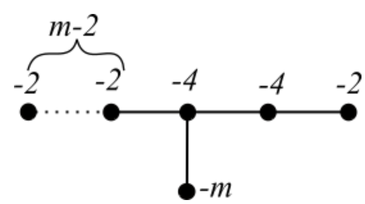
Figure 4.5: Repeated applications of the Key Lemma to the bold circles

Now view the leftmost punctured disk in Figure 4.2 as a sphere with six punctures. Then we can arrange the sphere so that the curve labelled y is the equator and the northern and southern hemispheres both have 3 punctures, two of which have one parallel curve and one of which has two parallel curves. In the previous paragraph, we repeatedly applied the Key Lemma to curves in only one of the hemispheres (without involving the equator y). Thus we can also apply it to the other hemisphere in the exact same way. We now do this explicitly. In the relation $x_0 \cdots x_{m-2}x_{m-1}^{m-1}x_m x_{m+1}x_{m+2}x_{m+3}^2 wy = a_1 \cdots a_{m-1}bcdef$, consider the bold curves x_{m+3} and a_{m-1} shown in Figure 4.5b . We view the latter as a curve containing the two punctures with boundary parallel curves x_{m+2} and x_{m+3}.

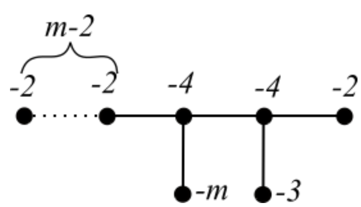
Since x_{m+2} commutes with all other Dehn twists, we can apply the Key Lemma to obtain $x_0 \cdots x_{m-2}x_{m-1}^{m-1}x_m x_{m+1}x_{m+2}x_{m+2}x_{m+3}x_{m+4}zwy = a_1 \cdots a_{m-1}bcde_1e_2f$, or $x_0 \cdots x_{m-2}x_{m-1}^{m-1}x_m x_{m+1}x_{m+2}^2 x_{m+3}x_{m+4}zwy = a_1 \cdots a_{m-1}bcde_1e_2f$. Thus the plumbing tree in Figure 4.6c is 2-replaceable. Inductively assume the relation $x_0 \cdots x_{m-2}x_{m-1}^{m-1}x_m x_{m+1}x_{m+2}^{n-2}x_{m+3}\cdots x_{m+n+1}zwy = a_1 \cdots a_{m-1}bcde_1 \cdots e_{n-2}f$, as in Figure 4.5d, holds. Again, since x_{m+2} commutes with everything, we can apply the Key Lemma to obtain the relation $x_0 \cdots x_{m-2}x_{m-1}^{m-1}x_m x_{m+1}x_{m+2}^{n-1}x_{m+3} \cdots x_{m+n+2}zwy = a_1 \cdots a_{m-1}bcde_1 \cdots e_1 \cdots e_{n-1}f$. Thus the plumbing tree in Figure 4.6d is 2-replaceable and so the family of trees in Theorem 4.1.2b are indeed 2-replaceable.



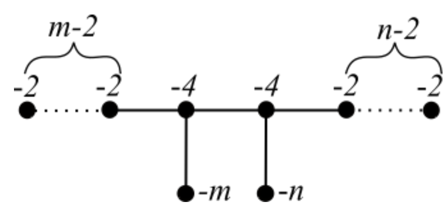
(a) Plumbing associated to Figure 4.5a



(b) Plumbing associated to Figure 4.5b



(c) Plumbing associated to Figure 4.5c



(d) Plumbing associated to Figure 4.5d

Figure 4.6: 2-replaceable plumbings associated to the monodromies in Figure 4.5

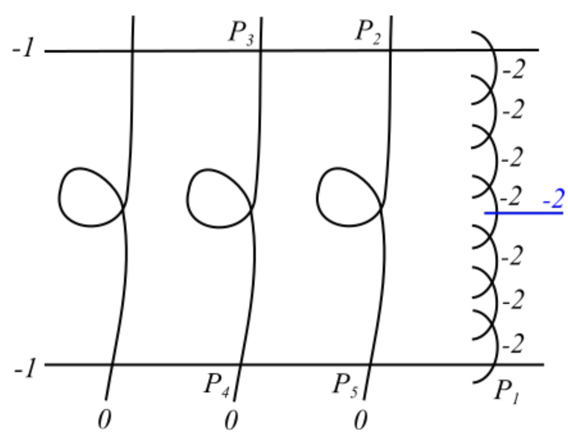
4.3 A symplectic exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$

In this section we will prove Theorem 4.1.5. In particular, we will find the 2-replaceable plumbing tree of Theorem 4.1.2(a) with $n = 9$ and $m = 3$ embedded in $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$, excise it, and replace it with the 2-replacement constructed in the proof of Theorem 4.1.2. We will then show that the resulting 4-manifold X is a symplectic exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$. We assume the reader is familiar with elliptic fibrations and blowups. See [33] for details.

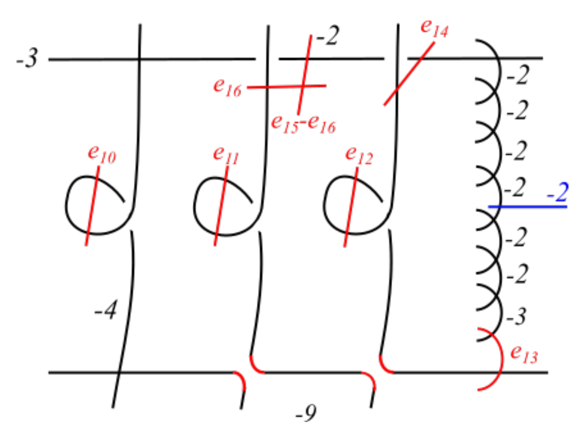
In [70], Stipsicz and Szabó showed that there is an elliptic fibration $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$, called $E(1)$, with three fishtail fibers, two sections, and a singular fiber of type III* (i.e. an $\tilde{E}_7$ singular fiber) which intersect as in Figure 4.7a. Starting with this configuration, perform the following moves:

- blow up the three double points in the fishtail fibers and call the exceptional spheres e_{10}, e_{11}, and e_{12};
- blow up the points P_1, P_2, and P_3 and call the exceptional spheres e_{13}, e_{14}, and e_{15} respectively;
- blow up the intersection between e_{15} and image of the adjacent fishtail fiber and call the new exceptional sphere e_{16};
- and smooth the intersection points P_4 and P_5.

The resulting configuration is shown in Figure 4.7b.



(a) A configuration in $E(1)$



(b) Configuration after seven blowups

Figure 4.7: Blowing up $E(1)$ seven times

Since we performed seven blowups, this configuration lives in $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$. Furthermore, notice that the the plumbing P depicted in Figure 4.8 is embedded in this configuration. By Theorem 4.1.2a, this plumbing is 2-replaceable. Let B denote the 2-replacement of P constructed in the proof of Theorem 4.1.2. Let $Z = \mathbb{C}P^2\#16\overline{\mathbb{C}P^2} - \text{int}(P)$ and $X = Z \cup_{\partial P} B$, where the gluing is by a contactomorphism isotopic to the identity.

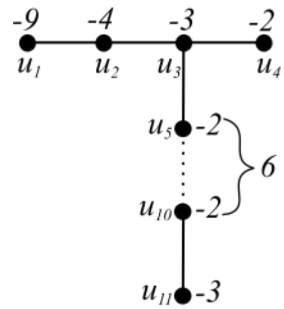


Figure 4.8: The configuration P

Proposition 4.3.1. *X is homeomorphic to $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$.*

Proof. We first prove that X is simply connected. Since $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$ is simply connected, the inclusion $\partial P \hookrightarrow Z$ induces a surjection $\pi_1(\partial P) \to \pi_1(Z)$. Furthermore, since B is built out of 0-, 1-, and 2-handles, the inclusion $\partial B = \partial P \hookrightarrow B$ also induces a surjection $\pi_1(\partial P) \to \pi_1(B)$. By the Seifert Van-Kampen theorem, we have $\pi_1(X) = \pi_1(Z) *_{\pi_1(\partial P)} \pi_1(B)$. Thus, in the amalgamation, the generators of $\pi_1(Z)$ can be expressed in terms of the generators of $\pi_1(B)$. Therefore, if the generators of $\pi_1(B)$ bound disks in X, then $\pi_1(X)$ is trivial. We first prove that $\pi_1(B)$ is cyclic of order 17 and then show that a particular generator of $\pi_1(B)$ bounds a disk in X.

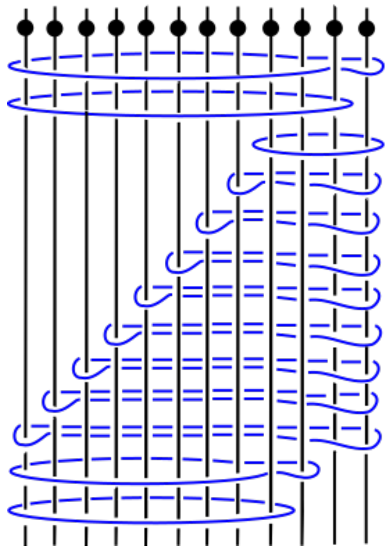
In the proof of Theorem 4.1.2a, we explicitly described the monodromy of the Lefschetz fibration associated to B (see Figure 4.3b). Figure 4.9a depicts a handlebody diagram of B obtained from this monodromy (c.f. Section 3.2.2). Each blue unknot has framing -1 and, from bottom to top, these unknots correspond to the curves $a, b, c_1, ..., c_8, d, e, f$ in the monodromy factorization shown in Figure 4.3b. Let m_i be a meridian around the i^{th} 1-handle of the handlebody diagram (Figure 4.9a), counting left to right. Then $\pi_1(B)$ is generated by $\{m_i\}$, which are subject to the relations (given by the 2-handles)

$$m_1 \cdots m_9 = 1, \quad m_1 \cdots m_8 m_{10} = 1, \quad m_1 \cdots m_{11} = 1, \quad m_1 \cdots m_{10} m_{12} = 1,$$

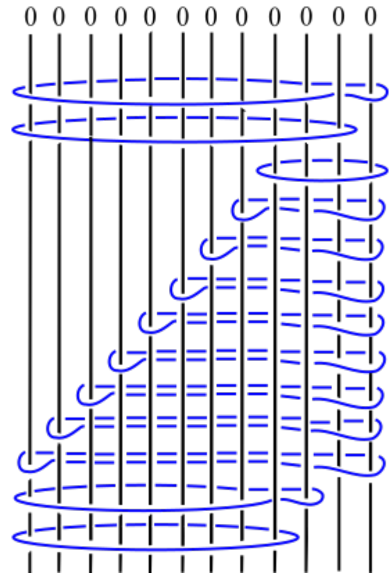
$$m_9 \cdots m_{12} = 1, \text{ and } m_i m_{11} m_{12} = 1 \text{ for } i = 1, ..., 8.$$

The last relation shows that $m_1 = \cdots = m_8$. Call this element m. Furthermore, we have $m_9 = m_{10} = m^{-8}$ and $m_{11} = m_{12} = m^8$. Thus, $1 = m_1 m_{11} m_{12} = m^{17}$ and so $\pi_1(B) \cong \mathbb{Z}_{17}$.

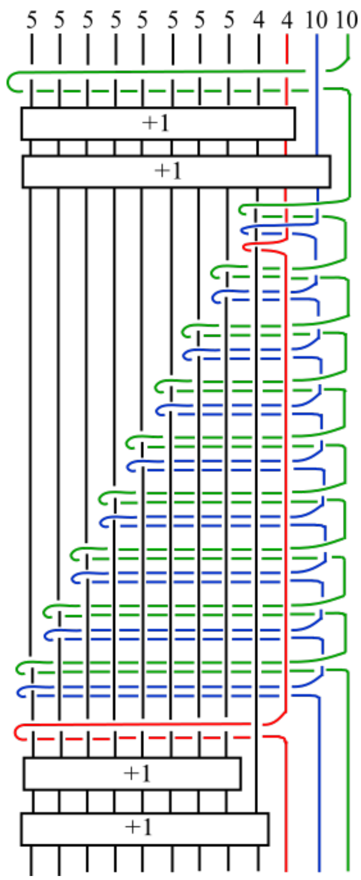
We now use Kirby calculus to move from the handlebody diagram of B depicted in Figure 4.9a to the handlebody diagram of P depicted in Figure 4.9h. Start with the handlebody diagram for B depicted in Figure 4.9a and change the dotted circles to 0-framed unknots to obtain the surgery diagram for ∂B depicted in Figure 4.9b.



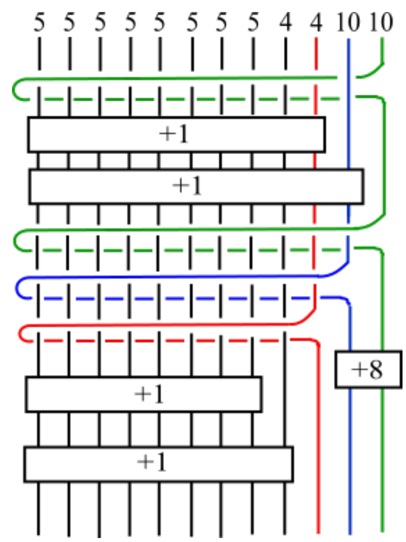
(a) A handlebody diagram for B



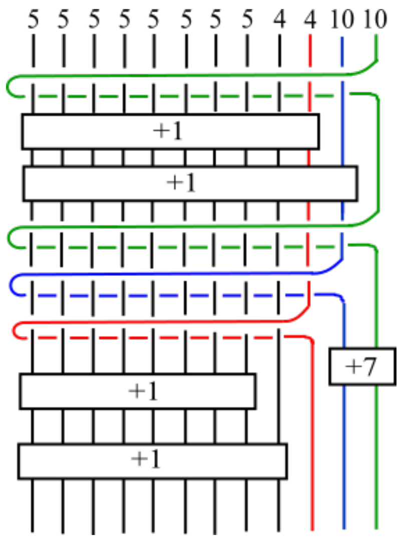
(b) A surgery description for ∂B



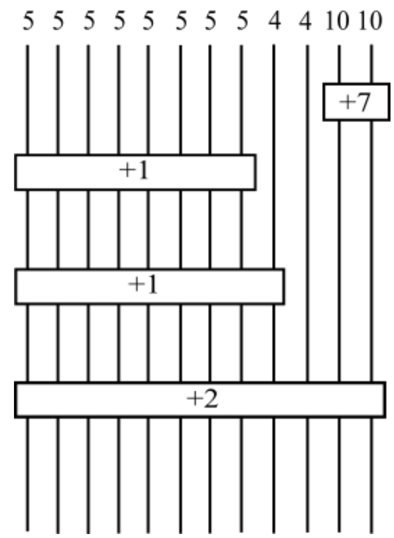
(c) Blow down the blue unknots in (b)



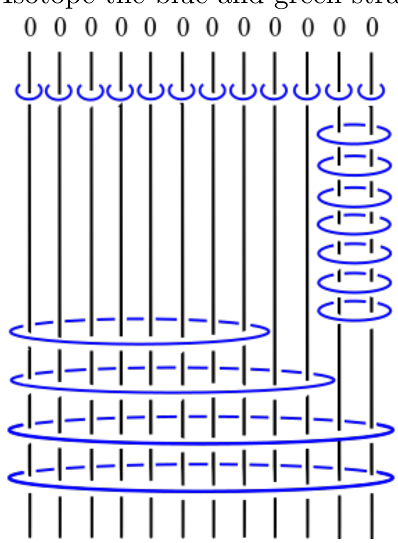
(d) Result after isotoping the red, green, and blue strands



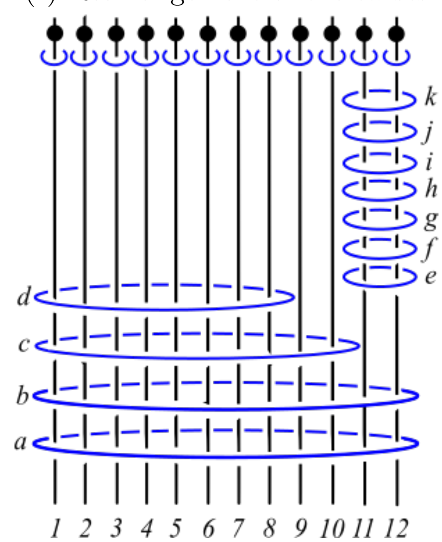
(e) Isotope the blue and green strands



(f) Rearrangement of the twists



(g) Perform blowups



(h) A handlebody diagram for P

Figure 4.9: Using Kirby calculus to show B and P have the same boundary

Now perform the following moves:

- blow down all of the blue -1 framed unknots to obtain Figure 4.9c;
- isotope the vertical red strand under the strand immediately to its left and pull it leftward;
- pull the blue and green strands leftward to obtain Figure 4.9d;
- introduce a positive twist at the top of the blue and green strands and a negative twist at the bottom of the same strands (these twists undo each other) to obtain Figure 4.9e;
- rearrange the strands to appear as in Figure 4.9f;
- and perform 23 blowups to obtain Figure 4.9g.

Finally, change the 0-framed unknots to dotted circles to obtain the handlebody diagram depicted in Figure 4.9h. Notice that this is a handlebody diagram for P, namely the diagram obtained from the monodromy associated to P in Figure 4.3b.

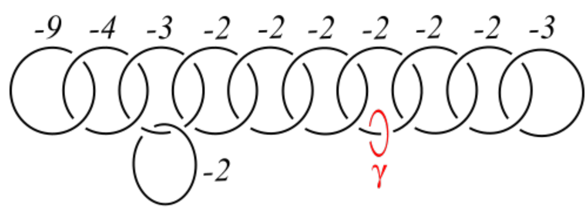
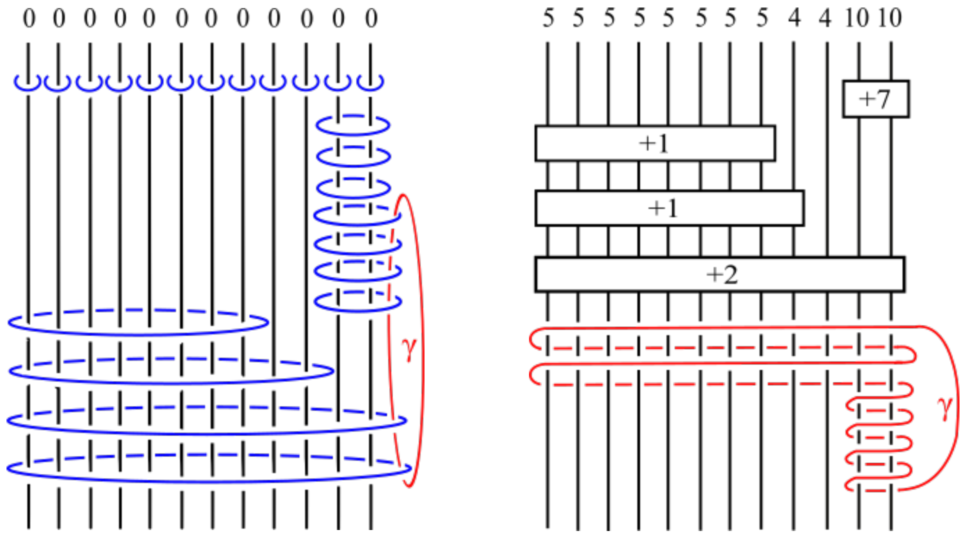


Figure 4.10: Handlebody diagram of P with a meridian γ

Now consider the obvious handlebody diagram for P depicted in Figure 4.10 (without the red meridian labelled γ). We can explicitly show that the handlebody diagram in Figure 4.9h is indeed a handlebody diagram for P via the following moves.

- In Figure 4.9h, slide a over b, followed by b over c, and followed by c over d.
- Slide d over each of the 8 blue unknots at the top of the 1-handles labelled 1-8.
- Slide c over the blue unknots at the top of the 1-handles labelled 9 and 10.
- Slide b over e, e over f, f over g, g over h, h over i, i over j, and j over k.
- Slide k over the blue unknots at the top of the 1-handles labelled 11 and 12.
- Cancel the 1-2 handle pairs to obtain the handlebody diagram in Figure 4.10.



(a) γ in ∂P (b) γ after blowing down

Figure 4.11: Keeping track of γ

Now consider the red meridian labelled γ in Figure 4.10. By reversing the moves outlined above, in the handlebody diagram in Figure 4.9h, γ links each of the curves labelled $a, b, e, f, g,$ and h exactly once. Changing the dotted circles to 0-framed unknots, we can see γ in a surgery diagram of ∂P as in Figure 4.11a (imagining that the base point is above the diagram). After blowing down all of the -1-framed blue unknots and isotoping γ, we obtain Figure 4.11b. Tracing through the Kirby calculus to obtain the handlebody diagram of B depicted in Figure 4.9a, it is easy to see that γ remains at the bottom of the diagram in the same position as in Figure 4.11b. Thus, in $\pi_1(B)$, $\gamma = m_1^2 \cdots m_{10}^2 m_{11}^6 m_{12}^6 = m^{12}$, which is a generator of $\pi_1(B) \cong \mathbb{Z}_{17}$.

Notice, in the original configuration of spheres found in $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$ (Figure 4.7b), γ can be identified with the equator of the -2-sphere colored in blue that is "dangling off" the singular fiber of type III*. Thus this meridian bounds a disk (a hemisphere of the blue -2-sphere) in Z and thus bounds a disk in X. Since γ generates $\pi_1(B)$, X is simply connected.

Next, notice that $\chi(X) = \chi(\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}) - \chi(P) + \chi(B) = 19 - 12 + 2 = 9 = \chi(\mathbb{C}P^2\#6\overline{\mathbb{C}P^2})$. Since $\chi(B) = 2$ and $b_1(B) = b_3(B) = b_4(B) = 0$, we must have $b_2(B) = 1$. Since B is negative definite, the signature of B is $-b_2(B) = -1$ and so $\sigma(X) = \sigma(\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}) - \sigma(P) + \sigma(B) = -15-(-11)+(-1) = -5 = \sigma(\mathbb{C}P^2\#6\overline{\mathbb{C}P^2})$. Finally, since -5 is not divisible by 16, the intersection forms of X and $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$ are both odd. Thus, by Freedman's theorem, X is homeomorphic to $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$. □

Proposition 4.3.2. *X is admits a symplectic structure.*

Proof. Since all of the spheres in the configuration P are complex submanifolds of $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$, they are symplectic and intersect positively (c.f. [70]). By [34], these spheres can be made ω-orthogonal by an isotopy through symplectic spheres. Thus, by Theorem 3.2.6, P admits a symplectic structure with strongly convex boundary and by Corollary 3.2.8, X admits a symplectic structure. $\square$

Proposition 4.3.3. *X is not diffeomorphic to $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$.*

Proof. Let h denote the canonical generator of $H_2(\mathbb{C}P^2;\mathbb{Z})$ in $H_2(\mathbb{C}P^2\#16\overline{\mathbb{C}P^2};\mathbb{Z}) = H_2(\mathbb{C}P^2;\mathbb{Z})\bigoplus 16H_2(\overline{\mathbb{C}P^2};\mathbb{Z})$ and, with abuse of notation, let e_i, for $1 \leq i \leq 16$, denote the homology class of the i^{th} exceptional sphere of the i^{th} blowup, which generates the i^{th} copy of $H_2(\overline{\mathbb{C}P^2};\mathbb{Z})$. Consider the configuration of spheres depicted in Figure 4.7a. Then, as shown in [70], the bottom horizontal section has homology class e_1, the top horizontal section has homology class e_9, each fishtail fiber has homology $3h-\sum_{i=1}^{9} e_i$, the vertical chain of -2-spheres have homology classes (working bottom to top) $h-e_1-e_2-e_3, e_3-e_4, e_4-e_5, e_5-e_6, e_6-e_7, e_7-e_8, e_8-e_9$, and the blue -2-sphere has homology class $h-e_3-e_4-e_5$. After performing the seven blowups to obtain the configuration of Figure 4.7b described earlier, the spheres in our configuration P, labelled as in Figure 4.8, have homology classes

$$\begin{aligned}
&u_1 = 6h - e_1 - \sum_{i=2}^{9} 2e_i - 2e_{11} - 2e_{12} - \sum_{i=13}^{16} e_i, \quad u_2 = 3h - \sum_{i=1}^{9} e_i - 2e_{10},\\
&u_3 = e_9 - e_{14} - e_{15} - e_{16}, \quad u_4 = e_{15} - e_{16}, \quad u_5 = e_8 - e_9,\\
&u_6 = e_7 - e_8, \quad u_7 = e_6 - e_7, \quad u_8 = e_5 - e_6, \quad u_9 = e_4 - e_5\\
&u_{10} = e_3 - e_4, \text{ and } u_{11} = h - e_1 - e_2 - e_3 - e_{13}.
\end{aligned}$$

For quick expositions of Seiberg-Witten invariants when $b^+ = 1$, see [20] and [70]. It is known that the small perturbation Seiberg-Witten invariant $SW^{\circ}_{\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}}$ is identically 0 because $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$ admits a metric of positive scalar curvature. Thus, we must find $\tilde{K} \in H^2(X;\mathbb{Z})$ such that $SW^{\circ}_X(\tilde{K}) \neq 0$. Let K be the canonical class of $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$ associated to the canonical symplectic form ω on $\mathbb{C}P^2\#16\overline{\mathbb{C}P^2}$. Then K is of the form $K = PD(-3h + \sum_{i=1}^{16} e_i)$ Let $\tilde{K}$ be the canonical class of X, induced by the symplectic structure $\tilde{\omega}$ on X. Since, by construction, $\omega|_Z = \tilde{\omega}|_Z$, we necessarily have that $K|_Z = \tilde{K}|_Z$. Furthermore, the dimensions of the Seiberg-Witten moduli spaces associated to K and $\tilde{K}$ are both 0.

By the proof of Corollary 9.4 in [45], ∂P is an L-space. Since P and B are both negative definite, by Michalogiorgaki's gluing formula (Theorem 2.4.3), $SW^{\circ}_{X,PD(a_2)}(\tilde{K}) = SW^{\circ}_{\mathbb{C}P^2\#16\overline{\mathbb{C}P^2},PD(a_1)}(K)$, where $a_1 \in H_2(\mathbb{C}P^2\#16\overline{\mathbb{C}P^2};\mathbb{Z})$ and $a_2 \in H_2(X;\mathbb{Z})$ such that $a_1|_Z = a_2|_Z$ and $a_1|_P = a_2|_B = 0$. Let

$$a = 10h - 3e_1 - 2e_2 - \sum_{i=3}^{9} 3e_i - 2e_{10} - e_{11} - 2e_{12} - 2e_{13} - 3e_{14}.$$

Then $a \cdot u_i = 0$ for all $1 \leq i \leq 11$ and so $a|_P = 0$. Thus a is represented in Z and can also be thought of as a homology class in $H_2(X;\mathbb{Z})$ such that $a|_B = 0$. Thus we have

$$SW^{\circ}_{X,PD(a)}(\tilde{K}) = SW^{\circ}_{\mathbb{C}P^2\#16\overline{\mathbb{C}P^2},PD(a)}(K).$$

Since the cohomology class $PD(h)$ gives the chamber that contains the point of positive scalar curvature, $SW^{\circ}_{\mathbb{C}P^2\#16\overline{\mathbb{C}P^2},PD(h)} = 0$ (see, e.g. [20]). Since $a \cdot a > 0$, $h \cdot h > 0$, $K \cdot PD(h) = -3 < 0$, $K \cdot PD(a) = 6 > 0$ and $h \cdot a = 10 > 0$, by the wall crossing formula, we have

$$SW^{\circ}_{\mathbb{C}P^2\#16\overline{\mathbb{C}P^2},PD(h)}(K) - SW^{\circ}_{\mathbb{C}P^2\#16\overline{\mathbb{C}P^2},\alpha}(K) = (-1)^{1+d(k)/2}$$

and so $SW^{\circ}_{X,\alpha}(\tilde{K}) = SW^{\circ}_{\mathbb{C}P^2\#16\overline{\mathbb{C}P^2},\alpha}(K) \neq 0$. □

4.4 Continued fractions

In this section we outline and prove useful facts about Hirzebruch-Jung continued fractions that will be needed for the proof of Theorem 4.1.1. Given a sequence of integers $(a_1, ..., a_n)$ the (Hirzebruch-Jung) continued fraction expansion is given by

$$[a_1, ..., a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}}$$

If $a_i \geq 2$ for all $1 \leq i \leq n$, then this fraction is well-defined and the numerator is greater than the denominator. In fact, for coprime $p > q > 0 \in \mathbb{Z}$, there exists a unique continued fraction expansion $[a_1, ..., a_n] = \frac{p}{q}$, where $a_i \geq 2$ for all $1 \leq i \leq n$.

We call the continued fraction expansions of $\frac{p}{q}$ and $\frac{p}{p-q}$ *dual* to each other. The following relationship between these two continued fractions is well-known (see, for example, Theorem 7.1 and Lemma 7.2 of [54]).

Theorem 4.4.1. *If*

$$\frac{p}{q} = [\underbrace{2, ..., 2}_{n_0}, m_1 + 3, \underbrace{2, ..., 2}_{n_1}, m_2 + 3, ..., m_k + 3, \underbrace{2, ..., 2}_{n_k}]$$

then

$$\frac{p}{p-q} = [n_0 + 2, \underbrace{2, ..., 2}_{m_1}, n_1 + 3, \underbrace{2, ..., 2}_{m_2}, ..., n_{s-1} + 3, \underbrace{2, ..., 2}_{m_s}, n_s + 2]$$

The following corollary follows from Theorem 4.4.1. It will be used throughout the proof of Theorem 4.1.1.

Corollary 4.4.2. *If* $[m_1, ..., m_k]$ *has dual* $[a_1, ..., a_n]$ *and* $[s_1, ..., s_l]$ *has dual* $[b_1, ...b_t]$*, then* $[m_1, ..., m_k, s_1, ..., s_l]$ *has dual* $[a_1, ...a_{n-1}, a_n + b_1 - 1, b_2, ..., b_t]$*. Conversely, suppose* $[m_1, ..., m_k, s_1, ..., s_l]$ *has dual* $[a_1, ..., a_n]$*. Then* $[m_1, ..., m_k]$ *and* $[s_1, ..., s_l]$ *have duals of the form* $[a_1, ..., a_{i-1}, a'_i]$ *and* $[a''_i, a_{i+1}, ..., a_n]$*, where* $a'_i + a''_i - 1 = a_i$ *and* $1 \leq i \leq n$*.*

Definition 4.4.3. *The* ***buddings*** *of the fraction* $[a_1, ..., a_n]$ *are the fractions* $[a_1 + 1, a_2, ..., a_n, 2]$ *and* $[2, a_1, ..., a_{n-1}, a_n + 1]$*. The* ***debudding*** *of* $[a_1, ..., a_n]$ *is the reverse operation. (Note: to be able to perform a debudding, we must have either* $a_1 = 2$ *and* $a_n > 2$ *or* $a_1 > 2$ *and* $a_n = 2$*. For example, the debudding of* $[2, a_2, ..., a_n]$*, where* $a_n > 2$*, is* $[a_2, ..., a_n - 1]$*.) Furthermore, by saying* $[a_1, ..., a_n]$ *is a budding of* $[a'_1, ..., a'_l]$*, we mean that* $[a_1, ..., a_n]$ *can be obtained by a finite sequence of buddings of* $[a'_1, ..., a'_l]$ *and by saying* $[a_1, ..., a_n]$ *is a debudding of* $[a'_1, ..., a'_l]$*, we mean that* $[a_1, ..., a_n]$ *can be obtained by a finite sequence of debuddings of* $[a'_1, ..., a'_l]$*.*

Equipped with this definition, the following corollary is a direct consequence of Theorem 4.4.1.

Corollary 4.4.4. *If* $[a_1, ..., a_n]$ *has dual* $[m_1, ..., m_k]$*, then the dual of a budding of* $[a_1, ..., a_n]$ *is a budding of* $[m_1, ..., m_k]$*. For example,* $[2, a_1, ..., a_n + 1]$ *has dual* $[1 + m_1, m_2, ..., m_k, 2]$*.*

4.4.1 Admissible fractions

In this section, we will consider continued fractions in which all entries are positive and in which each denominator appearing in the fraction is nonzero. Such a fraction is called *admissible*. Note that admissible fractions yield defined rational numbers (see, for example, [57]). In this section, we will consider fractions with entries greater than or equal to 1 and so requiring admissibility is important; for example, $[2, 1, 1]$ is not admissible and is undefined. Moreover, we will consider admissible fractions $[a_1, ..., a_n]$ that are equal to 0. In this case, there must exist an i such that $a_i = 1$ (see, for example, [57]).

Definition 4.4.5. *Let* $[a_1, ..., a_n] = 0$ *be admissible. Then the* ***blowup before*** a_i *is the fraction* $[a_1, ..., a_{i-1} + 1, 1, a_i + 1, ..., a_n]$ *and the* ***blowup after*** a_i *is the fraction* $[a_1, ..., a_i + 1, 1, a_{i+1} + 1, ..., a_n]$*. If* $a_i = 1$*, then the* ***blowdown at*** a_i *is* $[a_1, ..., a_{i-1} - 1, a_{i+1} - 1, ..., a_n]$*. By saying* $[a_1, ..., a_n]$ *is a blowup of* $[a'_1, ..., a'_l]$*, we mean that* $[a_1, ..., a_n]$ *can be obtained by a finite sequence of blowups of* $[a'_1, ..., a'_l]$*. Similarly,* $[a'_1, ..., a'_l]$ *is a blowdown of* $[a_1, ..., a_n]$ *if it can be obtained by a finite sequence of blowdowns of* $[a_1, ..., a_n]$*.*

The facts collected in the following proposition are well-known. See, for example, the Appendix of [57] and Section 2 of [47].

Proposition 4.4.6. *If* $[a_1, ..., a_n]$ *is admissible, then:*

1. *any blowup or blowdown of* $[a_1, ..., a_n]$ *is also admissible;*
2. $[a_n, ..., a_1]$ *is admissible;*
3. $[a_i, a_{i+1}, ..., a_j]$ *is admissible for all* $1 \leq i \leq j \leq n$*;*
4. *if* $[a_1, ..., a_n] = 0$*, then* $[a_n, ..., a_1] = 0$*;*
5. *if* $[a_1, ..., a_n] = 0$*, then any blowup or blowdown is also equal to 0; and*
6. *if* $[a_1, ..., a_n] = 0$*, then it can be obtained by a sequence of blowups of* $[0]$.

Note that the only blowup of $[0]$ is $[1, 1]$ and the only two blowups of $[1, 1]$ are $[1, 2, 1]$ and $[2, 1, 2]$. We will consider fractions obtained by sequences of blowups of these two fractions.

Lemma 4.4.7. *If* $n \geq 3$ *and* $[a_1, ..., a_n] = 0$ *is admissible and* $a_1 = 1$ *or* $a_n = 1$*, then it is a blowup of* $[1, 2, 1]$.

Proof. Suppose $[a_1, ..., a_n] = 0$ and without loss of generality assume $a_1 = 1$. Notice that there must be an $i \neq 1$ such that $a_i = 1$. If $a_n = 1$, then the fraction is necessarily $[1, 2, ..., 2, 1]$, which is a blowup of $[1, 2, 1]$. If $a_2 = 1$, then since $[1, 1, a_3, ..., a_n]$ is admissible, so is $[a_n, ..., a_3, 1, 1]$, by Proposition 4.4.6. But $[1, 1] = 0$, which contradicts admissibility. Thus we may assume $2 < i < n$. We proceed by induction. Let $n = 4$. Then the only fraction satisfying $[1, a_2, a_3, a_4] = 0$ is $[1, 3, 1, 2]$. By blowing down at the third entry, we obtain $[1, 2, 1]$. Now suppose all length $n-1$ (where $n > 5$) fractions with $a_1 = 1$ are blowups of $[1, 2, 1]$. Let $[1, a_2, ..., a_n] = 0$ have an entry $a_i = 1$, where $2 < i < n$. Blowing down at a_i, we obtain $[1, a_2, ..., a_{i-1} - 1, a_{i+1} - 1, ..., a_n] = 0$. By the inductive hypothesis, this fraction is a blowup of $[1, 2, 1]$. Thus $[1, a_2, ..., a_n]$ is a blowup of $[1, 2, 1]$. □

Lemma 4.4.8. *Let* $[a_1, ..., a_n] = 0$ *be a blowup of* $[2, 1, 2]$ *that is not a blowup of* $[1, 2, 1]$*. Then the buddings of* $[a_1, ..., a_n]$ *are also blowups of* $[2, 1, 2]$ *and not of* $[1, 2, 1]$*. By Proposition 4.4.6, the buddings are admissible and equal to 0.*

Proof. Let $[a_1, ..., a_n]$ be as in the statement of the lemma. Then there is a sequence of blowdowns that obtains $[2, 1, 2]$. Performing this sequence to the budding $[2, a_1, ..., a_n + 1]$, we obtain $[2, 2, 1, 3]$, which is a blowup of $[2, 1, 2]$. Similarly, $[a_1 + 1, ..., a_n, 2]$ is a blowup of $[3, 1, 2, 2]$, which is a blowup of $[2, 1, 2]$. □

By Proposition 4.4.6 and Lemma 4.4.8, we will not have to check admissibility of any fractions throughout the remainder of this chapter. The following is a partial converse to Lemma 4.4.8.

Lemma 4.4.9. *Let $[a_1, ..., a_n] = 0$ be a blowup of $[2, 1, 2]$ with exactly one entry that is 1. Then there is one possible debudding of $[a_1, ..., a_n]$ and it is a blowup of $[2, 1, 2]$. By Proposition 4.4.6, this debudding is admissible and equal to 0.*

Proof. We proceed by induction on n. First notice that the only blowups of $[2, 1, 2]$ with exactly one entry equal to 1 are $[2, 2, 1, 3]$ and $[3, 1, 2, 2]$. These have one possible debudding each, namely $[2, 1, 2]$. Inductively assume that the lemma is true for all length $n - 1$ fractions satisfying the hypotheses. Let $[a_1, ..., a_n] = 0$ be a blowup of $[2, 1, 2]$ with exactly one entry, a_i, that is 1, where $1 < i < n$. Then either $a_1 = 2$ and $a_n > 2$ or $a_1 > 2$ and $a_n = 2$. Thus there is one possible debudding, namely $[a_2, ..., a_n - 1]$ or $[a_1 - 1, ..., a_{n-1}]$. Note that in the former case we have $2 < i < n$ and in the latter case we have $1 < i < n - 1$. By blowing down at a_i, we obtain a length $n - 1$ fraction $[a_1, ..., a_{i-1} - 1, a_{i+1} - 1, ..., a_n] = 0$ with exactly one entry that is 1, namely either $a_{i-1} = 1$ or $a_{i+1} = 1$. By the inductive hypothesis, $[a_1, ..., a_{i-1} - 1, a_{i+1} - 1, ..., a_n]$ has one debudding and it is a blowup of $[2, 1, 2]$. Without loss of generality, suppose $a_1 = 2$ and $a_n > 2$ so that the debudding is $[a_2, ..., a_{i-1} - 1, a_{i+1} - 1, ..., a_n - 1]$. Now, perform a blowup before $a_{i+1} - 1$ to obtain $[a_2, ..., a_n - 1]$, which is the debudding of $[a_1, ..., a_n]$ and a blowup of $[2, 1, 2]$. □

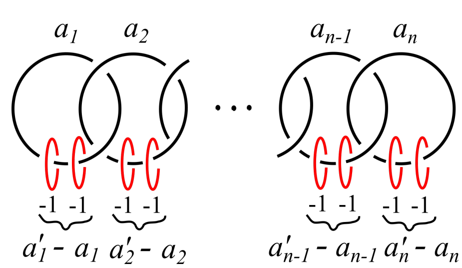
Corollary 4.4.10. *A fraction $[a_1, ..., a_n]$ is a budding of $[2, 1, 2]$ if and only if it is a blowup of $[2, 1, 2]$ that has exactly one entry that is 1.*

Proof. This follows from Lemmas 4.4.8 and 4.4.9. □

4.5 Proof of Theorem 4.1.1

4.5.1 Lisca's classification of symplectic fillings of $(L(p, q), \xi_{st})$

Let $p > q > 0 \in \mathbb{Z}$ be coprime. Recall from Example 3.1.16 that Lisca classified all minimal (strong) symplectic fillings of $(L(p, q), \xi_{st})$, where ξ_{st} is the standard contact structure on $L(p, q)$. These fillings correspond to the continued fraction expansion of $\frac{p}{p-q} = [a'_1, ..., a'_n]$. In particular, Lisca proved that any minimal symplectic filling of $(L(p, q), \xi_{st})$ is orientation preserving diffeomorphic to the manifold described by:



where the red -1-framed unknots are the attaching circles of 2-handles attached to $S^1 \times D^3$, whose boundary $S^1 \times S^2$ is given by surgery along the horizontal chain of unlinks, where $[a_1, ..., a_n] = 0$ is an admissible fraction with $a_i \leq a'_i$ for all $1 \leq i \leq n$. Note that since this filling is obtained from $S^1 \times D^3$ by attaching $\sum_{i=1}^{k}(a'_i - a_i)$ 2-handles, it has Euler characteristic $\sum_{i=1}^{k}(a'_i - a_i)$. To obtain a handlebody diagram of this 4-manifold, we must iteratively blow down the 1-framed unknots in the horizontal chain of unknots with framings $(a_1, ..., a_n)$ until we obtain a single unknot with framing 0, which we can then change to a dotted circle. The images of the -1-framed red unknots become a complicated link with negative framings. From this description, it is easy to see that the filling is negative definite and its first and third Betti numbers are 0. Thus this filling is a $(\sum_{i=1}^{k}(a'_i - a_i))$-replacement of the linear plumbing with weights arising from the continued fraction expansion of $-\frac{p}{q}$.

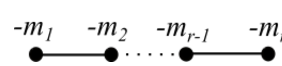
Recall, from Observation 4.0.2, that 1-replaceable linear plumbings are precisely those that can be obtained by sequences of buddings the -4-disk bundle over S^2. Reframing this in terms of Lisca's classification and dual fractions, we have the following.

Corollary 4.5.1. *$[b_1, ..., b_k]$ is a budding of $[4]$ if and only if $[b_1, ..., b_k]$ has dual of the form $[a_1, ..., a_i + 1, ..., a_n]$, where $a_i = 1$, $1 < i < n$, and $[a_1, ..., a_i, ..., a_n]$ is a blowup of $[2, 1, 2]$ with exactly one entry that is 1.*

Proof. Since $[4]$ has dual $[2, 2, 2]$, this result follows from Corollaries 4.4.4 and 4.4.10. This also follows from Lisca's work in [47]. □

Throughout this section any continued fraction $[a_1, ..., a_n] = 0$ is assumed to be admissible with at most two entries that are 1. Suppose $a_i = a_j = 1$ are the entries that are equal to 1. If $i \neq j$, then $[a_1, ...a_i + 1, ..., a_j + 1, ..., a_n] = \frac{p}{p-q}$ describes an Euler characteristic 2 symplectic filling of $(L(p, q), \xi_{st})$ (as above) and if $i = j$, then $[a_1 + 1, ...a_i + 1, ..., a_n] = \frac{p_1}{p_1-q_1}, [a_1, a_2 + 1, ..., a_i + 1, ..., a_n] = \frac{p_2}{p_2-q_2}, ..., [a_1, ...a_i + 2, ..., a_n] = \frac{p_i}{p_i-q_i}, ..., [a_1, ...a_i + 1, ..., a_n + 1] = \frac{p_n}{p_n-q_n}$ describe Euler characteristic 2 symplectic fillings of $(L(p_1, q_1), \xi_{st}), ..., (L(p_n, q_n), \xi_{st})$, respectively.

On the other hand, given the continued fraction expansion of $\frac{p}{p-q}$, the dual fraction expansion of $\frac{p}{q}$ corresponds to a linear plumbing that is also a symplectic filling of $(L(p, q), \xi_{st})$. That is, if $\frac{p}{q} = [m_1, ..., m_r]$, then the plumbing described by the graph



admits a symplectic structure that makes it a symplectic filling of $(L(p, q), \xi_{st})$.

Putting these facts together, the linear plumbing corresponding to $\frac{p}{q}$ is 2-replaceable if and only if it has dual $\frac{p}{p-q}$ of one of the following forms:

1. $[a_1, ..., a_i+1, ..., a_j+1, ..., a_n]$, where $a_i = 1, a_j \neq 1$ and $[a_1, ..., a_i, ..., a_j, ..., a_n] = 0$ has exactly one entry that is 1.

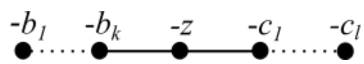
2. $[a_1, ..., a_i+2, ..., a_n]$, where $a_i = 1$ and $[a_1, ..., a_i, ..., a_n] = 0$ has exactly one entry that is 1.

3. $[a_1, ..., a_i + 1, ..., a_j + 1, ..., a_n]$, where $a_i = a_j = 1$ and $[a_1, ..., a_i, ..., a_j, ..., a_n] = 0$ has exactly two entries that are 1.

To prove Theorem 4.1.1, we will consider these three cases.

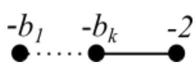
4.5.2 Proof of Theorem 4.1.1

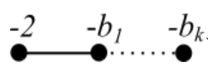
For convenience, we recall Theorem 4.1.1:
Theorem 4.1.1 *Let $(-b_1, ..., -b_k)$ and $(-c_1, ..., -c_l)$ be obtained by sequences of buddings of -4 and let $z \geq 2$ be any integer. Then a linear plumbing is 2-replaceable if and only if it is either of the form:*

(a) 

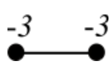
for $k, l \geq 0$

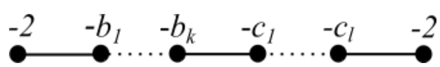
or is obtained by a sequence of buddings of one of the linear plumbings of the form:

(b) 

(or 

) for $k \geq 0$.

(c) 

(d) 

for $k, l \geq 1$

We first show that all plumbings listed in the theorem are indeed 2-replaceable. This is clear for the linear plumbings of type (a), since the "subplumbings" on either side of the $-z$-disk bundle can be symplectically rationally blown down, revealing an Euler characteristic 2 symplectic 4-manifold. Buddings of plumbings of type (b), (c), and (d) can also be seen to be 2-replaceable via rational blowdowns. Given the usual handlebody diagram of (a budding of) a plumbing of type (b) or (c), it can be shown that a simple handle slide (or no handle slide) reveals a 1-replaceable linear plumbing of length one less than the length of the plumbing in question. This plumbing can then be rationally blown down, leaving an Euler characteristic 2 manifold. For (a budding of) a plumbing of type (d), one could blowup the intersection between the $-b_k$- and $-c_1$-spheres multiple times in such a way that reveals two 1-replaceable linear plumbings that are plumbed to a -1-sphere. After rationally blowing down these plumbings, we are left with an Euler characteristic 2 manifold. Instead of working through these details, we will instead use Lisca's classification and work case by case.

Type (b): Suppose $[b_1, ..., b_k]$ is a budding of $[4]$. By Corollary 4.5.1, $[b_1, ..., b_k]$ has dual of the form $[a_1, ..., a_i + 1, ..., a_n]$, where $[a_1, ..., a_i, ..., a_n] = 0$ has exactly one entry that is equal to 1, namely a_i, where $i \neq 1, n$. By Theorem 4.4.1, $[b_1, ..., b_k, 2]$ has dual $[a_1, ..., a_i + 1, ..., a_n + 1]$ and so the plumbing corresponding to $[b_1, ..., b_k, 2]$ is 2-replaceable. Now let $[m_1, ..., m_r]$ be a budding of $[b_1, ..., b_k, 2]$. Then, by Corollary 4.4.4, it has dual that is a budding of $[a_1, ..., a_i + 1, ..., a_n + 1]$. Call this dual $[a_{-j}, ..., a_0, a_1 + 1 + t, a_2, ..., a_i + 1, ..., a_n + s, a_{n+1}, ..., a_{n+m}]$, for some $j, m, s, t \geq 0$. By Corollary 4.4.10, $[a_{-j}, ..., a_0, a_1 + t, a_2, ..., a_i, ..., a_n + s, a_{n+1}, ..., a_{n+m}]$ is a blowup of $[2, 1, 2]$ with exactly one entry equal to 1. Thus, the plumbing corresponding to $[m_1, ..., m_r]$ is 2-replaceable. □

Type (c): By a similar argument, it is clear that any budding of $[3, 3]$ is 2-replaceable. □

Type (d): Let $[b_1, ..., b_k]$ have dual $[a_1, ..., a_i + 1, ..., a_n]$, where $a_i = 1$ and $[a_1, ..., a_i, ..., a_n] = 0$ is a blowup of $[2, 1, 2]$ with exactly one entry that is 1, and let $[c_1, ..., c_l]$ have dual $[a'_1, ..., a'_j + 1, ..., a'_m]$, where $a'_j = 1$ and $[a'_1, ..., a'_j, ..., a'_m] = 0$ is a blowup of $[2, 1, 2]$ with exactly one entry that is 1. By Corollary 4.4.2, $[b_1, ..., b_k, c_1, ..., c_l]$ has dual $[a_1, ..., a_i + 1, ..., a_{n-1}, a_n + a'_1 - 1, a'_2, ..., a'_j + 1, ..., a'_m]$ and so, by Theorem 4.4.1, $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$ has dual $[a_1 + 1, ..., a_i + 1, ..., a_{n-1}, a_n + a'_1 - 1, a'_2, ..., a'_j + 1, ..., a'_m + 1]$.

We now claim that $[a_1 + 1, ..., a_i, ..., a_{n-1}, a_n + a'_1 - 1, a'_2, ..., a'_j, ..., a'_m + 1] = 0$. First note that by Lemma 4.4.8, $[2, a'_1, ..., a'_j, ..., a'_m + 1] = 0$ and so $[a'_1, ..., a'_j, ..., a'_m + 1] = \frac{1}{2}$. Thus $[a_n + a'_1 - 1, a'_2, ..., a'_j, ..., a'_m + 1] = a_n - 1 + [a'_1, ..., a'_j, ..., a'_m] = a_n - \frac{1}{2}$. Once again, by Lemma 4.4.8, $[a_1 + 1, ..., a_i, ..., a_{n-1}, a_n - \frac{1}{2}] = [a_1 + 1, ..., a_i, ..., a_{n-1}, a_n, 2] = 0$. Thus $[a_1 + 1, ..., a_i, ..., a_{n-1}, a_n + a'_1 - 1, a'_2, ..., a'_j, ..., a'_m + 1] = [a_1 + 1, ..., a_i, ..., a_{n-1}, a_n - \frac{1}{2}] = 0$. Thus, the plumbing corresponding to $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$ is 2-replaceable.

Now, suppose $[m_1, ..., m_r]$ is a budding of $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$. Then, by Corollary 4.4.4, $[m_1, ..., m_r]$ has dual that is a budding of $[a_1 + 1, ..., a_i + 1, ..., a_{n-1}, a_n + a'_1 - 1, a'_2, ..., a'_j + 1, ..., a'_m + 1]$. Then by applying Lemma 4.4.8 as in the proof of "Type (b)," the plumbing corresponding to $[m_1, ..., m_r]$ is 2-replaceable. □

We have shown that all linear plumbings of Theorem 4.1.1 are indeed 2-replaceable. Next we show that these are the only 2-replaceable linear plumbings. To do this we consider the continued fractions that are of the three forms listed at the end of Section 4.5.1 and show that the linear plumbings corresponding to their dual fractions are of one of the forms listed in Theorem 4.1.1.

We start with the trivial cases. If $n = 1$, then the only admissible fraction that is equal to 0 is $[0]$. Adding 2 gives the fraction $[2] = \frac{2}{1}$, which has dual fraction $[2] = \frac{2}{1}$. This corresponds to the -2-disk bundle over S^2. If $n = 2$, then the only such fraction is $[1, 1]$. Adding 1 to each entry gives $[2, 2] = \frac{3}{2}$, which has dual fraction 3. This corresponds to the -3-disk bundle over S^2. More generally, consider $[1, \overbrace{2, 2, ..., 2, 2}^{p}, 1] = 0$, where $p \geq 0$. Adding 1 to the first and last entry gives $[2, 2, ..., 2, 2] = \frac{p+3}{p+2}$, which has dual fraction $p + 3$. This corresponds to the $-(p + 3)$-disk bundle over S^2. These plumbings already have Euler characteristic 2, thus they are trivially 2-replaceable.

We now assume that $[a_1, ..., a_n] = 0$ has at most two entries that are equal to 1, $n \geq 3$, and a_1, a_n are not both 1. Since the only admissible fractions equal to 0 of length 3 are $[1, 2, 1]$ and $[2, 1, 2]$, the fraction $[a_1, ..., a_n]$ must be a blowup of one of these two fractions. Thus we need to consider the following cases.

1. Blowups of $[1, 2, 1]$ with exactly two entries that are 1. (Note that no blowup of $[1, 2, 1]$ can contain exactly one entry that is 1)
   - Blowups with $a_1 = 1$ and $a_n \neq 1$ (equivalently, with $a_n = 1$ and $a_1 \neq 1$)
   - Blowups with $a_1, a_n \neq 1$
2. Blowups of $[2, 1, 2]$ that are not blowups of $[1, 2, 1]$. By Lemma 4.4.7, such blowups have first and last entry not equal to 1.
   - Blowups with exactly one entry that is 1
   - Blowups with exactly two entries that are 1

Lemma 4.5.2. *Suppose* $[a_1, ..., a_n] = 0$ *is a blowup of* $[1, 2, 1]$ *with* $a_1 = 1$ *and* $a_i = 1$, *where* $1 < i < n$. *Then* $[a_1 + 1, ..., a_i + 1, ..., a_n]$ *has dual of the form* $[z, b_1, ..., b_k]$, *where* $[b_1, ..., b_k]$ *is a budding of* $[4]$ *and* $z \geq 3$. *Thus, any linear plumbing gotten this way is of type (a).*

Proof. Let $[a_1, ..., a_i, ..., a_n] = 0$ be as in the statement of the lemma. First notice that since $n \geq 3$, $i \neq 2$. Otherwise, $[a_1, ..., a_n] = [1, 1, a_3, ..., a_n] = 0$ and so $[a_n, ..., a_3, 1, 1] = 0$. But, $[1, 1] = 0$ and so $[a_n, ..., a_3, 1, 1]$ is undefined. Next, we claim that there exists $2 \leq j < i$ such that $a_j \geq 3$. Assume otherwise, so that for all $2 \leq j < i$, $a_j = 2$. Then the fraction is $[1, 2, ..., 2, 1, a_{i+1}, ..., a_n] = 0$ and so $[a_n, ..., a_{i+1}, 1, 2, ..., 2, 1] = 0$. But $[1, 2, ..., 2, 1] = 0$ and so $[a_n, ..., a_{i+1}, 1, 2, ..., 2, 1]$ is undefined. Let a_j be the first element such that $a_j \geq 3$. Then $a_2 = a_3 = ... = a_{j-1} = 2$ and $j < i$. Now blow down repeatedly at the first entry until we obtain $[a_j - 1, ..., a_i, ..., a_n] = 0$. This fraction has exactly one entry that is 1, namely a_i. Thus it is a blowup of $[2, 1, 2]$ and so by Corollary 4.5.1, $[a_j - 1, ..., a_i + 1, ..., a_n] = 0$ has dual, say $[b_1, ..., b_k]$, that is a budding of $[4]$. By Theorem 4.4.1, $[a_j, ..., a_i + 1, ..., a_n]$ has dual $[2, b_1, ..., b_k]$ and so $[a_1 + 1, ..., a_j, ..., a_i + 1, ..., a_n] = [\underbrace{2, ..., 2}_{j-1}, a_j, ..., a_i, ..., a_n]$ has dual $[j + 1, b_1, ..., b_k]$, where $j + 1 \geq 3$. □

Corollary 4.5.3. *Suppose* $[a_1, ..., a_n] = 0$ *is a blowup of* $[1, 2, 1]$ *with* $a_n = 1$ *and* $a_i = 1$*, where* $1 < i < n$*. Then* $[a_1, ..., a_i + 1, ..., a_n + 1]$ *has dual of the form* $[b_1, ..., b_k, z]$*, where* $[b_1, ..., b_k]$ *is a budding of* $[4]$*. Thus, any linear plumbing gotten this way is of type (a).*

Proof. Let $[a_1, ..., a_n] = 0$ be of the form described in the corollary. Then $[a_n, ..., a_1] = 0$. By Lemma 4.5.2, we have that $[a_n + 1, ..., a_i + 1, ..., a_1]$ has dual $[z, b_k, ..., b_1]$, where $[b_k, ..., b_1]$ is a budding of $[4]$. Thus, by Theorem 4.4.1, the dual of $[a_1, ..., a_i + 1, ..., a_n + 1]$ is $[b_1, ..., b_k, z]$, where $[b_1, ..., b_k]$ is again a blowup of 4 and $z \geq 3$. □

Lemma 4.5.4. *Let* $[a_1, ..., a_n] = 0$ *be a blowup of* $[1, 2, 1]$ *such that* $a_i = a_j = 1$*, where* $1 < i < j < n$*. Then* $[a_1, ..., a_i + 1, ..., a_j + 1, ..., a_n]$ *has dual* $[c_1, ..., c_l, z, b_1, ..., b_k]$*, where* $[c_1, ..., c_l]$ *and* $[b_1, ..., b_k]$ *are buddings of* $[4]$ *and* $z \geq 2$*. Thus, any linear plumbing gotten this way is of type (a).*

Proof. Let $[a_1, ..., a_n] = 0$ be as in the statement of the lemma. Then it can be repeatedly blown down at the first occurrence of 1 until it is of the form $[1, a_r - m, ..., a_n] = 0$, for some $r < j$ and $m \leq a_r - 2$. Note that such values for r and m can be realized since, otherwise, if $r = j$ or $a_r - m = 1$, we obtain contradictions similar to the contradictions reached in the proof of Lemma 4.5.2. Assume that this sequence of blowdowns is minimal in the sense that once we have a fraction beginning with 1, we stop blowing down (for example, if $a_r - m = 2$, one could blowdown at the first entry to obtain another fraction of this form). By Lemma 4.5.2, we know that $[2, a_r - m, ..., a_j + 1, ..., a_n]$ has dual $[z, b_1, ..., b_k]$, where $z \geq 3$ and $[b_1, ..., b_k]$ is a budding of $[4]$. To recover the original fraction, we now perform blowups. Note that the first blowup must be after the first entry, since otherwise, we obtain $[1, 2, a_r - m, ..., a_n]$, which contradicts the minimality assumption. Thus, the first blowup yields $[2, 1, a_r - m + 1, ...a_n] = 0$. By Theorem 4.4.1, $[a_r - m, ..., a_j + 1, ..., a_n]$ has dual $[z-1, b_1, ..., b_k]$ and so $[a_r - m + 1, ..., a_j + 1, ..., a_n]$ has dual $[2, z-1, b_1, ..., b_k]$ and so $[2, 2, a_r - m + 1, ..., a_j + 1, ..., a_n]$ has dual $[4, z-1, b_1, ..., b_k]$. Writing $[2, 2, a_r - m + 1, ..., a_j + 1, ..., a_n]$ as $[2, 2, 2 + (a_r - m - 1), ..., a_j + 1, ..., a_n]$, we can think of $[2, 2, 2]$ as a "subfraction" of $[2, 2, a_r - m + 1, ..., a_j + 1, ..., a_n]$. Furthermore, note that $[2, 2, 2]$ has dual $[4]$.

Now, blow up $[2, 1, a_r - m + 1, ...a_n] = 0$ repeatedly before or after the first occurrence of 1 to recover the original fraction $[a_1, ..., a_i, ..., a_r, ..., a_j, ..., a_n] = [a_1, ..., a_i, ..., m + 1 + (a_r - m - 1), ..., a_j, ..., a_n] = 0$. Note that by doing this, we also end up blowing up $[2, 1, 2]$, considered a subfraction, before or after the only entry that is 1 to obtain $[a_1, ..., a_i, ..., a_{r-1}, m + 1]$, which we can consider as a subfraction of $[a_1, ..., a_i, ..., a_r, ..., a_j, ..., a_n]$. Thus $[a_1, ..., a_i + 1, ..., a_{r-1}, m + 1]$ has dual that is a budding of $[4]$; call it $[c_1, ..., c_l]$. Since $[a_r - m, ..., a_j + 1, ..., a_n]$ has dual $[z-1, b_1, ..., b_k]$, by Lemma 4.4.2, $[a_1, ..., a_i + 1, ..., a_r, ..., a_j + 1, ..., a_n]$ has dual $[c_1, ..., c_l, z-1, b_1, ..., b_k]$, where $z - 1 \geq 2$. □

Lemma 4.5.5. *Let* $[a_1, ..., a_n] = 0$ *be a blowup of* $[2, 1, 2]$ *with exactly one entry that is 1, called* a_i *(where* $1 < i < n$*). If* $j \neq i$*, then* $[a_1, ..., a_i + 1, ..., a_j + 1, ..., a_n]$ *has dual that is a budding of either* $[2, b_1, ..., b_k]$ *or* $[b_1, ..., b_k, 2]$*, where* $[b_1, ..., b_k]$ *is a budding of* $[4]$*. If* $i = j$*, then* $[a_1, ..., a_i + 2, ..., a_n]$ *has dual that is a budding of* $[3, 3]$*. Thus any linear plumbing gotten this way is of type (b) or (c).*

Proof. Suppose $j < i$. Since $[a_1, ..., a_n] = 0$ is a blowup of $[2, 1, 2]$ and a_i is the only entry that is 1, we can also view it as a budding of $[2, 1, 2]$, by Corollary 4.4.10. Furthermore, we can view a_i as the image of 1 after performing the sequence of buddings of $[2, 1, 2]$ to obtain $[a_1, ..., a_n]$. Thus we can perform debuddings until we obtain $[a_j, ..., a_i, ..., a'_l]$ (which is still a blowup of $[2, 1, 2]$ by Corollary 4.4.10), where $i < l \leq n$ and $a'_l \leq a_l$. By Corollary 4.5.1, $[a_j, ..., a_i+1, ...a'_l]$ has dual that is a budding of $[4]$; call it $[b_1, ..., b_k]$. By Theorem 4.4.1, $[a_j + 1, ..., a_i + 1, ...a'_l]$ has dual $[2, b_1, ..., b_k]$. By performing buddings to recover the original fraction, by Corollary 4.4.4, we have that $[a_1, ..., a_j+1, ..., a_i+1, ..., a_n]$ has dual that is a budding of $[2, b_1, ..., b_k]$. Similarly, if $j > i$, then $[a_1, ..., a_i + 1, ..., a_j + 1, ..., a_n]$ has dual that is a budding of $[c_1, ..., c_l, 2]$, where $[c_1, ..., c_l]$ is a budding of $[4]$.

Now suppose $j = i$. The only such fraction of length 3 is $[2, 3, 2]$, which has dual $[3, 3]$. Now let $[a_1, ..., a_n] = 0$ be a blowup of $[2, 1, 2]$ with $a_i = 1$ and $i \neq 1, n$. By Corollary 4.4.10, $[a_1, ..., a_n]$ is a budding of $[2, 1, 2]$. Thus $[a_1, ..., a_i + 2, ..., a_n]$ is a budding of $[2, 3, 2]$. By Corollary 4.4.4, the dual of $[a_1, ..., a_i + 2, ..., a_n]$ must be a budding of the dual of $[2, 3, 2]$; that is, the dual is a budding of $[3, 3]$. □

Lemma 4.5.6. *Let* $[a_1, ..., a_n] = 0$ *be a blowup of* $[2, 1, 2]$ *and not a blowup of* $[1, 2, 1]$ *with exactly two entries that are 1. Call them* $a_i = a_j = 1$ *with* $i < j$ *(and* $i, j \neq 1, n$*). Then* $[a_1, ..., a_i+1, ..., a_j+1, ..., a_n]$ *has dual that is a budding of* $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$*, where* $[b_1, ..., b_k]$ *and* $[c_1, ...c_l]$ *are buddings of* $[4]$*. Thus linear plumbings gotten this way are of type (d).*

Proof. Note that the minimal length of such a continued fraction is 5 and the only such fraction of length 5 is $[3, 1, 3, 1, 3] = 0$. Clearly $[3, 2, 3, 2, 3]$ has dual $[2, 4, 4, 2]$. Also notice that $[2, 4, 4, 2]$ is the minimal length dual fraction of the desired form. We start by considering blowups of $[3, 1, 3, 1, 3]$. Let $[a_1, ..., a_i, ..., a_{n-2}, 1, 3] = 0$ with $a_i = 1$ and $1 < i < n - 2$ be obtained by a sequence of blowups of $[3, 1, 3, 1, 3]$ before or after the first instance of 1. Then $a_1, a_{n-2} \geq 3$. Blow down $[a_1, ..., a_i, ..., a_{n-2}, 1, 3]$ at the second occurrence of 1 to obtain $[a_1, ..., a_i, ..., a_{n-2} - 1, 2] = 0$. Next, perform a debudding to obtain $[a_1 - 1, ..., a_i, ..., a_{n-2} - 1]$, which is 0 by Lemma 4.4.10. Since $a_1, a_{n-2} \geq 3$, this fraction contains exactly one 1, namely a_i, and so by Corollary 4.5.1, $[a_1 - 1, ..., a_i + 1, ..., a_{n-2} - 1]$ has dual that is a budding of $[4]$, which we call $[b_1, ..., b_k]$. Now by Theorem 4.4.1 and Corollary 4.4.2, $[a_1, ..., a_i + 1, ..., a_{n-2}, 2, 3]$ has dual $[2, b_1, ..., b_k, 4, 2]$. Thus, any fraction obtained by a sequence of blowups of $[3, 1, 3, 1, 3]$ before or after the first instance of 1 is of the form $[2, b_1, ..., b_k, 4, 2]$, where $[b_1, ..., b_k]$ is a budding of $[4]$.

Now let $[a_1, ..., a_i, ..., a_j, ..., a_n] = 0$ be a blowup of $[3, 1, 3, 1, 3]$ with precisely two entries that are 1, namely a_i and a_j. Notice that we can order the blowups so that we first perform all blowups before or after the first occurrence of 1 and then all blowups before or after the second occurrence of 1 to obtain $[a_1, ..., a_i, ..., a_j, ..., a_n] = 0$. After performing all blowups of the former type, we obtain a fraction of the form $[a_1, ..., a_i, ..., a'_m, 1, 3] = 0$, where $a_i = 1$, $1 < i < m$, and $a'_m \leq a_m$. Thus, by the previous paragraph, $[a_1, ..., a_i + 1, ..., a'_m, 2, 3]$ has dual $[2, b_1, ..., b_k, 4, 2]$, where $[b_1, ..., b_k]$ is a budding of $[4]$. Furthermore, $[a_1, ..., a_i + 1, ..., a'_m - 1]$ has dual $[2, b_1, ..., b_k]$. Now, blow up $[a_1, ..., a_i, ..., a'_m, 1, 3] = 0$ repeatedly before or after the second occurrence of 1 to obtain the original fraction $[a_1, ..., a_i, ...a_{m-1}, a_m, ..., a_j, ..., a_n] = 0$. Since $a_n \geq 3$, by Theorem 4.4.1, $[a_m - a'_m + 2, a_{m+1}, ..., a_j + 1, ..., a_n]$ has dual of the form $[c_1, ..., c_l, 2]$ for some sequence $(c_1, ..., c_l)$. By Corollary 4.4.2, since $[a_1, ..., a_i + 1, ..., a'_m - 1]$ has dual $[2, b_1, ..., b_k]$, we have that $[a_1, ..., a_i + 1, ...a_{m-1}, a_m, a_{m+1} ..., a_j + 1, ..., a_n] = [a_1, ..., a_i + 1, ...a_{m-1}, (a'_m - 1) + (a_m - a'_m + 2) - 1, a_{m+1}, ..., a_j + 1, ..., a_n]$ has dual $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$. It remains to show that $[c_1, ..., c_l]$ is a budding of $[4]$. To do this, we show that its dual $[a_m - a'_m + 2, a_{m+1}, ..., a_j + 1, ..., a_n - 1]$, which has exactly one entry equal to 1, is a blowup of $[2, 1, 2]$. First recall that $[a_1, ..., a_i + 1, ..., a'_m, 2, 3]$ has dual $[2, b_1, ..., b_k, 4, 2]$. By writing $[a_1, ..., a_i + 1, ..., a'_m, 2, 3]$ as $[a_1, ..., a_i + 1, ..., (a'_m - 2) + 2, 2, 3]$, we can view $[2, 2, 3]$ as a subfraction of $[a_1, ..., a_i + 1, ..., a'_m, 2, 3]$. Recall that $[a_1, ..., a_i + 1, ..., a'_m - 1]$ has dual $[2, b_1, ..., b_k]$ and $[2, 2, 3]$ has dual $[4, 2]$. To obtain $[a_1, ..., a_i, ..., a_m, ..., a_j, ..., a_n] = 0$, we must perform a sequence of blowups before or after the second occurrence of 1 in $[a_1, ..., a_i, ..., a'_m, 1, 3] = 0$. Note that by doing this, we also end up blowing up $[2, 1, 3]$, considered as a subfraction, before or after the only entry that is 1, to obtain $[a_m - a'_m + 2, a_{m+1}, ..., a_j, ..., a_n]$. Thus, $[a_m - a'_m + 2, a_{m+1}, ..., a_j, ..., a_n - 1]$ is a blowup of $[2, 1, 2]$.

Now suppose $[a_1, ..., a_n]$ is as in the statement of the lemma. Since $[a_1, ..., a_n]$ is not a blowup of $[1, 2, 1]$, the only way to obtain $[a_1, ..., a_n]$ through blowups of $[2, 1, 2]$ is to first blowup $[2, 1, 2]$ before or after the middle entry. This gives $[3, 1, 2, 2]$ or $[2, 2, 1, 3]$. Furthermore, at each step, we cannot blowup at the beginning or end of the fraction (otherwise, we obtain a fraction with first or last entry 1 and thus it is a blowup of $[1, 2, 1]$, by Lemma 4.4.7). Thus either the first or last entry of $[a_1, ..., a_n]$ must be at least 3.

We now claim that if $a_1, a_n > 2$, then $[a_1, ..., a_n] = 0$ is a blowup of $[3, 1, 3, 1, 3]$. To see this, first blowdown $[a_1, ..., a_n]$ until it is of minimal length and still has exactly two entries that are 1 (and with first and last entry not equal to 1, since it is not a blowup of $[1, 2, 1]$). This is possible, since after sufficiently many blowdowns, we must obtain $[2, 1, 2]$. With abuse of notation, call this blowdown $[a_1, ..., a_n]$ with $a_i = a_j = 1$ for $i, j \neq 1, n$ and $i \neq j$. If we blowdown at a_i or at a_j, then we must have exactly one entry that is 1 (since the resulting fraction is still a blowup of $[2, 1, 2]$). Thus $a_{i-1}, a_{i+1}, a_{j-1}, a_{j+1} \neq 2$. If $j - 1 \neq i + 1$, then after blowing down both a_i and a_j, we obtain a fraction with no entries that are 1, a contradiction. Thus we must have $j - 1 = i + 1$. Moreover, we must have $a_{j+1} = 3$. Otherwise, once again after blowing down a_i and a_j there would be no entries that are 1. Thus the fraction must be of the form $[a_1, ..., a_i, 3, a_j, ..., a_n]$. Now, blowing down a_i gives $[a_1, ..., a_{i-1} - 1, 2, 1, ..., a_n] = 0$, which has exactly one entry that is 1. By Corollary 4.5.1, $[a_1, ..., a_{i-1} - 1, 2, 2, ..., a_n]$ has dual $[d_1, ..., d_k]$, which is a budding of [4]. If $i - 1 \neq 1$, then since $a_1, a_n > 2$, we have $d_1 = d_k = 2$, by Theorem 4.4.1. But no such budding of [4] exists. Thus $i - 1 = 1$. Similarly, $j + 1 = n$. Thus our fraction is of the form $[a_1, 1, 3, 1, a_5]$. The only such continued fraction that is a blowup of $[2, 1, 2]$ is $[3, 1, 3, 1, 3]$. Thus, if $[a_1, ..., a_i, ..., a_j, ..., a_n] = 0$ has $a_1, a_n > 2$, then it is a blowup of $[3, 1, 3, 1, 3]$ and thus $[a_1, ..., a_i + 1, ..., a_j + 1, ..., a_n]$ has dual of the form $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$, where $[b_1, ..., b_k], [c_1, ..., c_l]$ are buddings of [4].

Finally, suppose $[a_1, ..., a_n] = 0$, with $a_i = a_j = 1$, $i \neq j$, $a_1 = 2$, and $a_n > 2$ (or, similarly $a_1 > 2$ and $a_n = 2$). We claim that by a sequence of debuddings, we can obtain a fraction that is a blowup of $[3, 1, 3, 1, 3]$. The first debudding yields $[a_2, ..., a_n - 1]$. If $a_2, a_n - 1 > 2$, then by the previous paragraph, we are done. If $a_2 = 2$ and $a_n - 1 > 2$ (or vice versa), then perform another debudding. Since the fraction has finite length, this process terminates, yielding a fraction with first and last entry greater than 2 (they cannot both be 2 by the remarks above). So the result is a blowup $[3, 1, 3, 1, 3]$. Call this fraction $[a'_1, ..., a'_m]$, where $a'_i = a'_j = 1$, $i \neq j$, and $i, j \neq 1, n$. Then $[a'_1, ..., a'_i + 1, ..., a'_j + 1, ..., a'_m]$ has dual of the form $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$, where $[b_1, ..., b_k]$ and $[c_1, ..., c_l]$ are buddings of [4]. Now we can perform buddings to $[a'_1, ..., a'_i + 1, ..., a'_j + 1, ..., a'_m]$ to obtain the original fraction $[a_1, ..., a_i + 1, ..., a_j + 1, ..., a_n]$. By Corollary 4.4.4, its dual fraction is obtained by a sequence of buddings of $[2, b_1, ..., b_k, c_1, ..., c_l, 2]$. □

We have exhausted all possibilities and thus have proved Theorem 4.1.1.

Chapter 5

Non-simply connected plumbings

Simple cut-and-paste operations such as the rational blowdown, star surgery, and the cut-and-paste operations with k-replaceable plumbings introduced in Section 4 all involve simply connected plumbing trees. A natural followup question is: can we do the same with non-simply connected plumbings? In this chapter, we will focus on plumbings whose associated graphs have a single cycle. For such a plumbing, the best case scenario in terms of lowering b_2 is that it can be replaced by a 4-manifold with the rational homology of $S^1 \times D^3$, which we denote by $\mathbb{Q}S^1 \times D^3$. We will also refer to any 3-manifold with the rational homology of $S^1 \times S^2$ as a $\mathbb{Q}S^1 \times S^2$.

Question 5.0.1. *Do there exist plumbings with a single cycle that can be (smoothly) replaced by a* $\mathbb{Q}S^1 \times D^3$*?*

If there do exist such plumbings, then the next natural question is:

Question 5.0.2. *Do there exist 0-replaceable plumbings?*

Finally, in the context of constructing exotic 4-manifolds, we may ask:

Question 5.0.3. *Can such plumbings be used to construct exotic 4-manifolds, symplectic or otherwise?*

In this chapter, we will explore Questions 5.0.1 and 5.0.3. In particular, in Section 5.1, we will describe a way to construct such plumbings, leading to an affirmative answer to Question 5.0.1. In Section 5.2, we will prove the following gluing formula (c.f. Section 2.4) for the Ozsváth-Szabó 4-manifold invariant, which may help in deciding whether a cut-and-paste operation involving one of these plumbings yields an exotic manifold.

Theorem 5.2.1 *Let Y be a $\mathbb{Q}S^1 \times S^2$. Suppose that $HF^-_{red}(Y, \mathfrak{s}) = 0$ in degree $-\frac{3}{2}$ for all torsion $\mathfrak{s} \in Spin^c(Y)$. Let P be negative definite manifold with $b_2 > 0$ and no 3-handles and let B be a $\mathbb{Q}S^1 \times D^3$ with no 3-handles such that $\partial P = \partial B = Y$. Suppose P is embedded in a 4-manifold X_1 with $b_2^+(X_1) \geq 2$. Let $Z = X_1 - P$*

and $X_2 = Z \cup B$. *Let* $\mathfrak{t}_1 \in Spin^c(X_1)$ *such that* $c_1^2(\mathfrak{t}_1|_P) = -b_2(P)$. *Then for any* $\mathfrak{t}_2 \in Spin^c(X_2)$ *satisfying* $\mathfrak{t}_2|_Z = \mathfrak{t}_1|_Z$, *we have* $\Phi_{X_1,\mathfrak{t}_1} = \pm\Phi_{X_2,\mathfrak{t}_2}$.

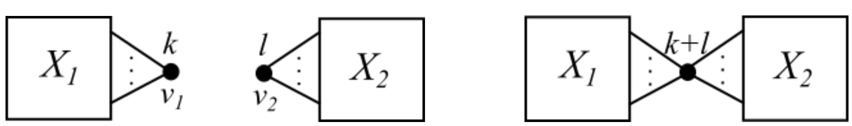
In Section 5.2, we will also construct families of plumbings with a single cycle whose boundaries satisfy the vanishing condition in the theorem. Finally, in Section 5.3, we will construct 4-manifolds that are homeomorphic to, but not obviously diffeomorphic to, $(2n-1)\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ for $n = 1,2$ and $1 \leq k \leq 9n-3$, and $(2n-1)\mathbb{C}P^2\#(\lfloor\frac{11n+4}{4}\rfloor)\overline{\mathbb{C}P^2}$, where $2n-1-\lfloor\frac{11n+4}{4}\rfloor$ is not divisible by 16. Unfortunately, Theorem 5.2.1 cannot be used to determine whether these manifolds are diffeomorphic. New tools need to be developed for this purpose.

In Chapter 6, we will take a nascent step in answering Question 5.0.2 by classifying contact structures with no Giroux torsion on a certain family of plumbed 3-manifolds. Recall, by the discussion in Section 3.2, that classifying such contact structures is a useful step towards answering Question 5.0.2.

5.1 Constructing $\mathbb{Q}S^1 \times D^3$s

In this section, we will construct non-simply connected plumbings with a single cycle that can be replaced by 4-manifolds with the rational homology of $S^1 \times D^3$. Throughout this section, we will use the same notation to denote a plumbing and its associated graph. We begin by reviewing a useful construction by Aceto [1].

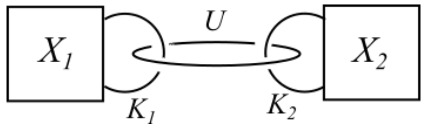
Definition 5.1.1 (Aceto [1])**.** *Let* X_i *be a plumbing tree with a distinguished vertex* v_i, *for* $i = 1,2$. *Let* X *be the plumbing tree obtained from* X_1 *and* X_2 *by identifying the two distinguished vertices and taking the sum of the corresponding weights to be the new weight (See Figure 5.1). We say that* X *is obtained by* ***joining*** *together* X_1 *and* X_2 *along* v_1 *and* v_2 *and we write* $X = X_1 \bigvee_{v_1=v_2} X_2$. *We call this operation the* ***join operation****.*



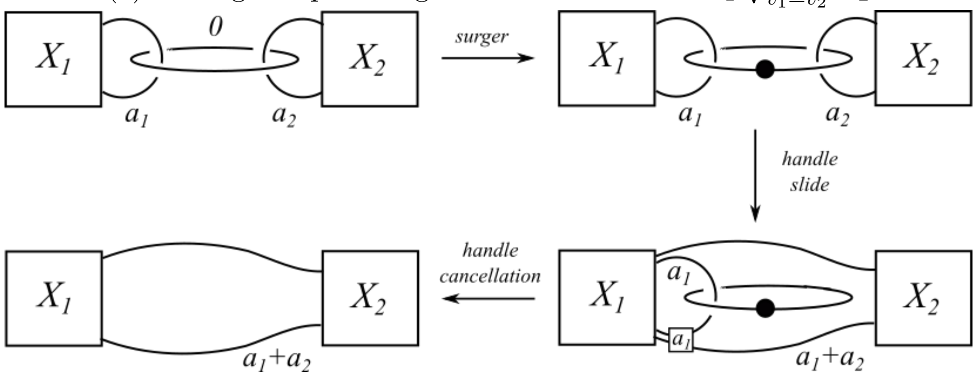
(a) Two plumbing trees (X_i, v_i) (b) The plumbing $X_1 \bigvee_{v_1=v_2} X_2$

Figure 5.1: Applying the join operation to the vertices v_1 and v_2

On the 4-manifold level, consider the obvious handlebody diagram of $X_1 \natural X_2$. Let K_i denote the unknot to which the 2-handle associated with the vertex v_i is attached. Let U be an unknot such that $lk(K_i, U) = 1$ for $i = 1,2$ and such that there exists a sphere surrounding U that intersects the handlebody diagram of $X_1 \natural X_2$ in exactly four points, namely two points on K_1 and two points on K_2 (See Figure 5.2a). Now attach a 0-framed 2-handle along U. By sliding K_2 over K_1, surgering U into a dotted circle, and performing a handle cancellation, we obtain the obvious handlebody diagram of X. This process is depicted in Figure 5.2b. The following result prescribes a way to construct plumbing trees that can be replaced by $\mathbb{Q}S^1 \times D^3$s.



(a) Joining two plumbing trees to obtain $X = X_1 \bigvee_{v_1=v_2} X_2$



(b) Using Kirby calculus to obtain the obvious handlebody diagram of X.

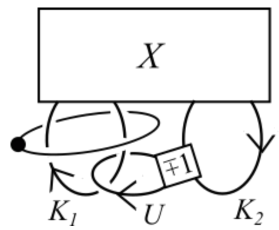
Figure 5.2: Obtaining handlebody descriptions of the join operation

Proposition 5.1.2 (Aceto [1]). *Let (X, v) be a plumbing tree with distinguished vertex v such that $\partial X = S^1 \times S^2$ and $\partial(X \backslash v)$ is a $\mathbb{Q}S^3$. Let (X', v') be a plumbing tree with distinguished vertex v' such that $\partial X'$ is a $\mathbb{Q}S^1 \times S^2$. Then $\partial X'$ and $\partial(X \bigvee_{v=v'} X')$ are rational homology cobordant. In particular, if $\partial X'$ bounds a $\mathbb{Q}S^1 \times D^3$, then so does $\partial(X \bigvee_{v=v'} X')$.*

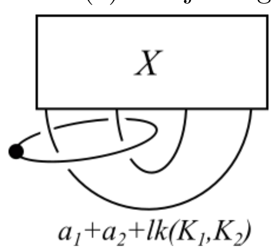
We now define a "self-join" operation that will lead to a similar result detailing a way to construct non-simply connected plumbings that can be replaced by $\mathbb{Q}S^1 \times D^3$s.

Definition 5.1.3. *Let X be a plumbing tree with distinguished vertices v_i for $i = 1, 2$. Then $X_{v_1=v_2}$ is the plumbing graph obtained by joining X to itself along v_1 and v_2 in a way that yields a positive cycle. Similarly, $X_{v_1=-v_2}$ is the plumbing graph obtained by joining X to itself along v_1 and v_2 in a way that yields a negative cycle.*

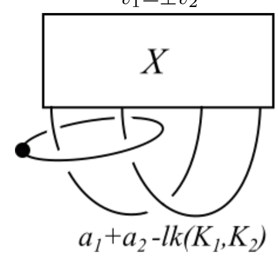
On the 4-manifold level, we can once again consider the obvious handlebody diagram of X. Orient the attaching circles of the 2-handles so that all linking numbers of all adjacent unknots are $+1$. Let K_i denote the unknot to which the 2-handle associated with the vertex v_i is attached. Consider the obvious handlebody diagram for $X \natural (S^1 \times D^3)$ obtained by adding a 1-handle to X. Now, as above, we can obtain $X_{v_1=\pm v_2}$ from $X \natural (S^1 \times D^3)$ by adding a particular 0-framed 2-handle. Let $U_\mp$ be an unknot such that: $lk(K_1, U_\mp) = 1$ and $lk(K_2, U_\mp) = \mp 1$; there exists a sphere surrounding $U_\mp$ that intersects the handlebody diagram of $X \natural (S^1 \times D^3)$ in precisely four points, namely two points on K_1 and two points on K_2; and $U_\mp$ "passes through" the 1-handle (see Figure 5.3a). Now attach a 0-framed 2-handle along $U_\mp$. As in the case with trees, by sliding K_2 over K_1, surgering $U_\mp$ into a dotted circle, and performing a handle cancellation, we obtain the obvious handlebody diagram of $X_{v_1=\pm v_2}$ (see Figures 5.3b and 5.3c). The following proposition gives us a way to construct plumbings with a single cycle that can be replaced by $\mathbb{Q}S^1 \times D^3$s.



(a) Self-joining a plumbing tree to obtain $X_{v_1=\pm v_2}$



(b) Handlebody diagram of $X_{v_1=v_2}$



(c) Handlebody diagram of $X_{v_1=-v_2}$

Figure 5.3: Obtaining handlebody descriptions of the self-join operation

Proposition 5.1.4. *Let X be a plumbing tree such that $Y = \partial X$ bounds a $\mathbb{Q}S^1 \times D^3$ W and $H_1(Y;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is a surjection. Let v_1 and v_2 be vertices of X and let μ_1 and μ_2 denote the elements of $H_1(Y;\mathbb{Z})$ represented by the meridians of v_1 and v_2, respectively. Further suppose that μ_1 has infinite order in $H_1(Y;\mathbb{Z})$. If $\mu_1 \neq \mu_2$, then $\partial(X_{v_1=v_2})$ bounds a $\mathbb{Q}S^1 \times D^3$. If $\mu_1 \neq -\mu_2$, then $\partial(X_{v_1=-v_2})$ bounds a $\mathbb{Q}S^1 \times D^3$.*

Proof. Let X, Y, W, v_i, and μ_i be as in the statement of the Proposition. Choose a basis for $H_1(Y;\mathbb{Z})$ and let ν be the free generator. Then $\mu_1 = a\nu + \sum_i t_i$, where $a \neq 0$ and t_i are torsion elements of $H_1(Y;\mathbb{Z})$. Let $\tilde{\nu}$ denote the image of ν in $H_1(W;\mathbb{Z})$. Since $H_1(Y;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is a surjection, we may choose a basis for $H_1(W;\mathbb{Z})$ such that $\tilde{\nu}$ is the free generator of $H_1(W;\mathbb{Z})$.

Let $X' = X \natural (S^1 \times D^3)$. Then $Y' = \partial X' = Y \# (S^1 \times S^2)$ bounds a rational homology $(S^1 \times D^3) \natural (S^1 \times D^3)$, namely $W' = W \natural (S^1 \times D^3)$. Consider the obvious surgery diagram of Y' obtained as the boundary of the handlebody diagram of X'. By attaching a 0-framed 2-handle to W' along an unknot $U_\mp$, as described in the paragraph preceding the statement of this proposition, we will obtain a 4-manifold $W''_{\pm}$ with boundary $\partial(X_{v_1=\pm v_2})$. Let μ denote the image of ν in $H_1(Y';\mathbb{Z})$ and let $\tilde{\mu}$ denote the image of $\tilde{\nu}$ in $H_1(W';\mathbb{Z})$. Note that μ maps to $\tilde{\mu}$ and, under suitable bases, μ and $\tilde{\mu}$ are free generators of there respective homology groups.

With abuse of notation, let μ_i denote the image of μ_i in $H_1(Y';\mathbb{Z})$. Thus $[U_{\mp}] = \mu_1 \mp \mu_2 \in H_1(Y';\mathbb{Z})$. If $\mu_1 \neq \mu_2$, then $[U_-]$ is nonzero and if $\mu_1 \neq -\mu_2$, then $[U_+]$ is nonzero. Moreover, by the assumption on μ_1, the homology class of $U_{\mp}$ is of the form $[U_{\mp}] = a\mu + \sum \lambda_i$ in $H_1(Y';\mathbb{Z})$, where λ_i are torsion elements of $H_1(Y';\mathbb{Z})$. Thus in $H_1(W';\mathbb{Z})$, $[U_{\mp}] = a\tilde{\mu} + \sum \delta_i$, where δ_i are torsion elements of $H_1(W';\mathbb{Z})$. Now, since $U_{\mp}$ bounds a disk in $W''_{\pm}$, we have that $[U_{\mp}] = a\tilde{\mu} + \sum \delta_i = 0$ in $H_1(W''_{\pm};\mathbb{Z})$. Thus the image of $\tilde{\mu}$ in $H_1(W''_{\pm};\mathbb{Z})$ is torsion and so $H_1(W''_{\pm};\mathbb{Q}) = \mathbb{Q}$.

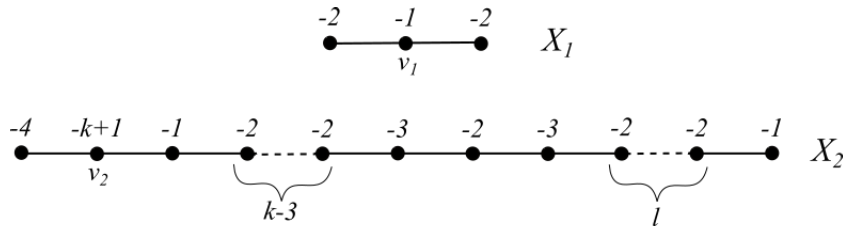
Now by Mayer-Vietoris, we have the sequence $H_2(W';\mathbb{Z}) \oplus H_2(D^4;\mathbb{Z}) \xrightarrow{p_* - q_*} H_2(W''_{\pm};\mathbb{Z}) \xrightarrow{h_*} H_1(S^1 \times D^2;\mathbb{Z}) \xrightarrow{(i_*, j_*)} H_1(W';\mathbb{Z}) \oplus H_1(D^4;\mathbb{Z})$ corresponding to the 2-handle attachment along $U_{\mp}$. Since $[U_{\mp}]$ is not a torsion element of $H_1(W';\mathbb{Z})$, (i_*, j_*) is injective. Thus $p_* - q_*$ is surjective and so, since $H_2(W';\mathbb{Z})$ is torsion, so is $H_2(W''_{\pm};\mathbb{Z})$, which implies that $H_2(W''_{\pm};\mathbb{Q}) = \mathbb{Q}$. □

Remark 5.1.5. It is easy to obtain plumbings that satisfy the hypothesis of Proposition 5.1.4. In particular, Aceto's construction (Proposition 5.1.2) gives many examples of plumbing trees X such that ∂X bounds a $\mathbb{Q}S^1 \times D^3$. Moreover, by his construction, these $\mathbb{Q}S^1 \times D^3s$ have no 3-handles, implying that $H_1(Y;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is a surjection.

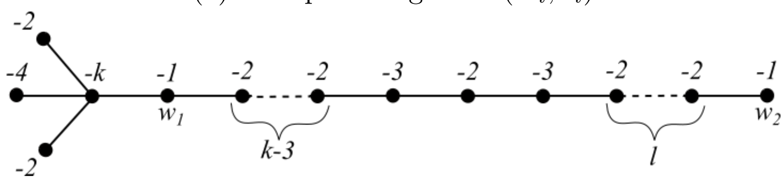
Example 5.1.6. Consider the two plumbings X_1 and X_2 with distinguished vertices v_1 and v_2 depicted in Figure 5.4a. Notice that: $\partial X_1 = S^1 \times S^2$; $\partial(X_1 \backslash v_1) = L(2,1)\#L(2,1)$ is a $\mathbb{Q}S^3$; and ∂X_2 bounds $S^1 \times D^3$, since $\partial X_2 = S^1 \times S^2$. Thus by Proposition 5.1.2, $\partial X = \partial(X_1 \bigvee_{v_1=v_2} X_2)$ bounds a $\mathbb{Q}S^1 \times D^3$. Now consider the distinguished vertices w_1 and w_2 of X depicted in Figure 5.4b. Homology calculations show that if m_i is the meridian of w_i, then $[m_2]$ has infinite order in $H_1(\partial X;\mathbb{Z})$ and $[m_1] = (4k-5)[m_2]$. Thus by Proposition 5.1.4, $\partial(X_{w_1=-w_2})$ (shown in Figure 5.4c) also bounds a $\mathbb{Q}S^1 \times D^3$. Moreover notice that by changing the orientations of the attaching circles of the 2-handles in the obvious handlebody diagram of the plumbing cycle, we can move the negative intersection and obtain Figure 5.4d. (c.f. Section 2.2). This particular family of plumbings will be used in Section 5.3.2.

5.2 Heegaard Floer homology calculations

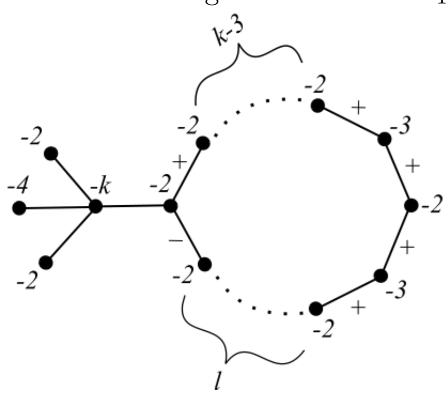
Now that we have an affirmative answer to Question 5.0.1, we can think about Question 5.0.3. Namely, after locating such a non-simply connected plumbing with a single cycle in an ambient 4-manifold and replacing it with a $\mathbb{Q}S^1 \times D^3$, do we end up with an exotic manifold?



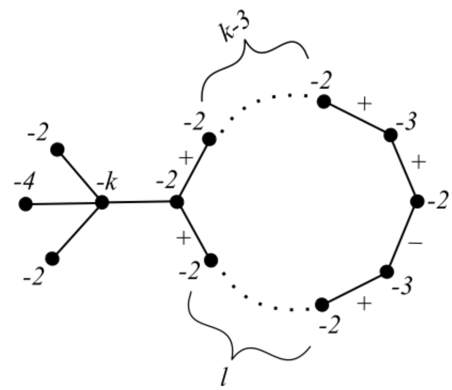
(a) Two plumbing trees (X_i, v_i)



(b) Joining the two plumbing trees to obtain $X = X_1 \bigvee_{v_1=v_2} X_2$. Consider the two distinguished vertices w_1 and w_2.



(c) Self-joining the plumbing tree X to obtain $X_{w_1=-w_2}$



(d) We can reverse orientations to move the negative intersection

Figure 5.4

To this end, we will prove a result that can help keep track of the Ozsváth-Szabó 4-manifold invariant when we perform such a cut-and-paste operation (c.f. the theorems of Section 2.4). Afterwards, we will narrow our view to non-simply connected plumbings whose boundaries have the simplest possible Heegaard Floer homology.

5.2.1 A gluing formula

Theorem 5.2.1. *Let Y be a $\mathbb{Q}S^1 \times S^2$. Suppose that $HF^-_{red}(Y, \mathfrak{s}) = 0$ in degree $-\frac{3}{2}$ for all torsion $\mathfrak{s} \in Spin^c(Y)$. Let P be negative definite manifold with $b_2 > 0$ and no 3-handles and let B be a $\mathbb{Q}S^1 \times D^3$ with no 3-handles such that $\partial P = \partial B = Y$. Suppose P is embedded in a 4-manifold X_1 with $b_2^+(X_1) \geq 2$. Let $Z = X_1 - P$ and $X_2 = Z \cup B$. Let $\mathfrak{t}_1 \in Spin^c(X_1)$ such that $c_1^2(\mathfrak{t}_1|_P) = -b_2(P)$. Then for any*

$\mathfrak{t}_2 \in Spin^c(X_2)$ *satisfying* $\mathfrak{t}_2|_Z = \mathfrak{t}_1|_Z$*, we have* $\Phi_{X_1,\mathfrak{t}_1} = \pm\Phi_{X_2,\mathfrak{t}_2}$.

Proof. Let W_1 denote the cobordism $P - B^4 : S^3 \to Y$, let W_2 denote the cobordism $B - B^4 : S^3 \to Y$, and let W denote the cobordism $Z - B^4 : Y \to S^3$. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be spin^c structures on X_1 and X_2, respectively, such that $\mathfrak{t}_2|_Z = \mathfrak{t}_1|_Z$ and $c_1^2(\mathfrak{t}_1|_P) = -b_2(P)$. It is easy to see that the degree shift of $F^\circ_{W_1,\mathfrak{t}_1|_{W_1}}$ is $\frac{1}{2}$. Similarly, since $c_1^2(\mathfrak{t}_2|_{W_2}) = 0$, the degree shift of $F^\circ_{W_2,\mathfrak{t}_2|_{W_2}}$ is $\frac{1}{2}$. Let $\mathfrak{s}_i = \mathfrak{t}_i|_{W_i}$ and $\mathfrak{s} = \mathfrak{t}_i|_W$.

We will now show that there is a unique spin^c structure on X_i that restricts to $\mathfrak{s}_i$ on W_i by considering the following long exact sequences and showing that (i^*, j^*) is injective.

$$\cdots \to H^1(W_i) \oplus H^1(W) \xrightarrow{(r^*,s^*)} H^1(Y) \to H^2(X_i) \xrightarrow{(i^*,j^*)} H^2(W_i) \oplus H^2(W) \to \cdots$$

$$\cdots \to H^1(W_i) \xrightarrow{r^*} H^1(Y) \xrightarrow{\delta^*} H^2(W_i, Y) \xrightarrow{Q} H^2(W_i) \to H^2(Y) \to \cdots$$

For $i = 1$, the map Q is injective (since it is the intersection form of P, which has no 3-handles). Thus r^* is surjective. This implies that the map (r^*, s^*) is surjective and so (i^*, j^*) is injective. For $i = 2$, $H^2(W_2, Y) = 0$, since B is a $\mathbb{Q}S^1 \times D^3$, and so we similarly obtain (i^*, j^*) is injective.

Furthermore, note that the restriction of any spin^c structure on W_i to Y is torsion. For $i = 2$, this is true since $H^2(W_2)$ is torsion and so the image of the map $H^2(W_2) \to H^2(Y)$ must be torsion. For $i = 1$, this is true by considering the long exact sequence

$$H^2(W_1;\mathbb{Q}) \xrightarrow{l^*} H^2(Y;\mathbb{Q}) \xrightarrow{m^*} H^3(W_1, Y;\mathbb{Q}) \to 0.$$

Since $H^2(Y;\mathbb{Q}) = H^3(W_1, Y;\mathbb{Q}) = \mathbb{Q}$, the map m^* must be an isomorphism. Thus l^* is the zero map and so the image of the map $H^2(W_1;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is torsion.

Let θ^- be the topmost (degree -2) generator of $HF^-(S^3)$. Then we can calculate the 4-manifold invariants of X_1 and X_2 by the formulas

$$\Phi_{X_i,\mathfrak{t}_i} = \pm F^{mix}_{W,\mathfrak{s}} \circ F^-_{W_i,\mathfrak{s}_i}(\theta^-).$$

By the degree shift calculated above, $F^-_{W_1,\mathfrak{s}_1}(\theta^-)$ and $F^-_{W_2,\mathfrak{s}_2}(\theta^-)$ both have degree $-\frac{3}{2}$.

Since W_1 is negative definite with $b_2^- > 0$ and $b_1 = 1$, by successively applying Proposition 9.4 in [59], it is easy to see that $F^\infty_{W_1,\mathfrak{s}_1}$ maps $HF^\infty(S^3)$ isomorphically onto one of the towers of $HF^\infty(Y, \mathfrak{s}|_Y)$. Since W_2 has $b_2^+ = b_2^- = 0$, we can repeatedly use Proposition 9.3 in [59] to obtain a similar result for $F^\infty_{W_2,\mathfrak{s}_2}$. Since these two cobordism maps have the same degree shift, they map onto the same towers. Furthermore, since $HF^-_{red}(Y, \mathfrak{s}_i|_Y) = 0$ in degree $-\frac{3}{2}$, we have the following commutative diagram.

$$\begin{array}{ccccc}
0 & \longrightarrow & HF^-_{-2}(S^3) & \xrightarrow{(j_1)_*} & HF^\infty_{-2}(S^3) \\
 & & \Big\downarrow F^-_{W_i,\mathfrak{s}_\mathfrak{i}} & & \Big\downarrow F^\infty_{W_i,\mathfrak{s}_\mathfrak{i}} \\
0 & \longrightarrow & HF^-_{-\frac{3}{2}}(Y,\mathfrak{s}_\mathfrak{i}|_Y) & \xrightarrow{(j_2)_*} & HF^\infty_{-\frac{3}{2}}(Y,\mathfrak{s}_\mathfrak{i}|_Y)
\end{array}$$

Since $(j_1)_*$ and $(j_2)_*$ are injective, the cobordism maps $F^-_{W_i,\mathfrak{s}_i}$ both map $HF^-_{-2}(S^3)$ isomorphically onto $HF^-_{-\frac{3}{2}}(Y,\mathfrak{s}|_Y)$. Thus $\Phi_{X_1,\mathfrak{t}_1} = \pm\Phi_{X_2,\mathfrak{t}_2}$. □

5.2.2 Families of L-spaces

We now explore a class of 3-manifolds that have the rational homology of $S^1 \times S^2$ and have the simplest possible Heegaard Floer homology. Abusing the usual terminology, we define the following.

Definition 5.2.2. *Let Y be a $\mathbb{Q}S^1 \times S^2$. Then Y is called an **L-space** if $HF_{red}(Y,\mathfrak{s}) = 0$ for all $\mathfrak{s} \in Spin^c(Y)$.*

Remark 5.2.3. To distinguish classical L-spaces (which are rational homology spheres) from these new L-spaces, we will refer to classical L-spaces as $\mathbb{Q}S^3$ L-spaces.

As a quick corollary to Theorem 5.2.1, if all hypotheses regarding X_1, X_2, P, B, and the spin^c structures hold, then we have the following.

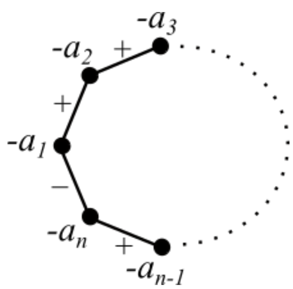
Corollary 5.2.4. *If Y is an L-space, then $\Phi_{X_1,\mathfrak{t}_1} = \pm\Phi_{X_2,\mathfrak{t}_2}$*

As with rational homology spheres, it is easy to see (using Theorem 10.1 of [61]) that the following are equivalent.

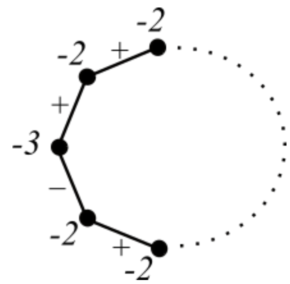
1. Y is a L-space
2. $\widehat{HF}(Y)$ is a free abelian group of rank $2|H_1(Y;\mathbb{Z})_T|$ (where A_T represents the torsion subgroup of the abelian group A)
3. The natural map $HF^+(Y,\mathfrak{s}) \to HF^-(Y,\mathfrak{s})$ is trivial for all $\mathfrak{s} \in \text{Spin}^c(Y)$.

We will use the second statement to construct families of non-simply connected plumbings whose boundaries are L-spaces. These constructions will rely on Lemma 5.2.13, whose proof is at the end of this section.

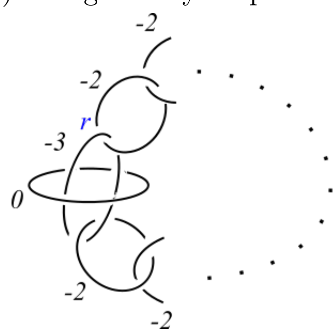
Proposition 5.2.5. *Let X denote a negative cyclic plumbing graph as in Figure 5.5a with weights $(-a_1, ..., -a_n)$, where $a_i \geq 2$ for all i and $a_1 \geq 3$. Then $Y = \partial X$ is an L-space.*



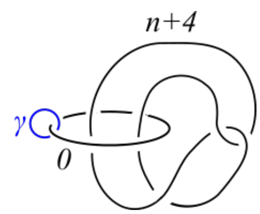
(a) A negative cyclic plumbing



(b) A negative cyclic plumbing



(c) Surgery diagram for Y



(d) Result after a blowup and $n-1$ blowdowns

Figure 5.5

Proof. First let X be a length n cyclic plumbing with $a_1 = 3$ and $a_i = 2$ for all $i \neq 1$, as depicted in Figure 5.5b. Let $Y = \partial X$.

Consider the surgery diagram of Y depicted in Figure 5.5c. By blowing up the intersection between labelled r with a $+1$-framed unknot and then sequentially blowing down the -1-framed unknots, we obtain the surgery diagram shown in Figure 5.5d (without the blue unknot labelled γ). From this surgery diagram, it is clear that $|H_1(Y;\mathbb{Z})_T| = n+4$. Now, let γ denote the simple closed curve that can be identified with a meridian of the 0-framed unknot. If we perform 0-surgery along γ, we obtain $(n+4)$-surgery on the right handed trefoil, which is known to be an L-space. If we perform -1-surgery along γ, we obtain the lens space $L(n+4,-1)$. Thus, the rank of $\widehat{HF}$ of each of these manifolds is $n+4$. Using a Heegaard Floer homology surgery exact triangle, this immediately implies that $\operatorname{rank}(\widehat{HF}(Y)) = 2(n+4)$ and thus Y is an L-space.

Now let X be a length n negative cyclic plumbing with weights $(-a_1, ..., -a_n)$ such that $a_i \geq 2$ for all i, and $a_1 \geq 3$. Let v_j be the vertex with weight $-a_j$. Let X_1^j denote the plumbing cycle obtained by decreasing the weight of the jth vertex to $-a_j - 1$ and let $X\backslash j$ denote the 3-manifold obtained by deleting the jth vertex. Let $Y = \partial X$, $Y\backslash j = \partial(X\backslash j)$, and $Y_1^j = \partial X_1^j$ and inductively assume that Y is an L-space. Let γ be a meridian of the surgery curve associated to v_j in the obvious surgery diagram of Y as shown in Figure 5.6. Then $Y\backslash j$ is obtained by 0-surgery along γ and Y_1^j is

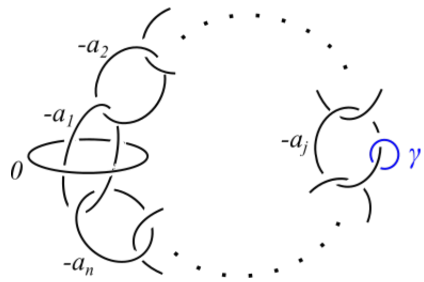


Figure 5.6: Surgery diagram of Y

obtained by 1-surgery along γ. Let $M_{Y\backslash j}$ be a linking matrix for $Y\backslash j$ such that the linking matrices $M_{Y_1^j}$ and M_Y of Y_1^j and Y are of following the forms.

$$M_{Y_1^j} = \left[\begin{array}{ccccc|c} & & & & & -1 \\ & & & & & 0 \\ & & M_{Y\backslash j} & & & \vdots \\ & & & & & 0 \\ & & & & & 1 \\ \hline -1 & 0 & \cdots & 0 & 1 & -a_j+1 \end{array}\right]$$

$$M_Y = \left[\begin{array}{ccccc|c} & & & & & -1 \\ & & & & & 0 \\ & & M_{Y\backslash j} & & & \vdots \\ & & & & & 0 \\ & & & & & 1 \\ \hline -1 & 0 & \cdots & 0 & 1 & -a_j \end{array}\right]$$

By Lemma 5.2.13 below, X and X_1^j are negative definite and by Lemma 2.5 in [23], $X\backslash j$ is negative definite. Thus M_Y, $M_{Y_1^j}$, and $M_{Y\backslash j}$ are negative definite and so the signs of $\det(M_Y)$ and $\det(M_{Y_1^j})$ are the same and differ from the sign of $\det(M_{Y\backslash j})$. It follows that $|\det(M_{Y_1^j})| = |\det(M_Y)| + |\det(M_{Y\backslash j})|$, which implies that $|H_1(Y_1^j)_T| = |H_1(Y)_T| + |H_1(Y\backslash j)|$ (c.f. Corollary 5.3.12 of [33]). By the Heegaard Floer homology surgery exact triangle, we have the following exact sequence.

$$\begin{array}{ccc} \widehat{HF}(Y) & \longrightarrow & \widehat{HF}(Y\backslash j) \\ & \nwarrow \qquad \swarrow & \\ & \widehat{HF}(Y_1^j) & \end{array}$$

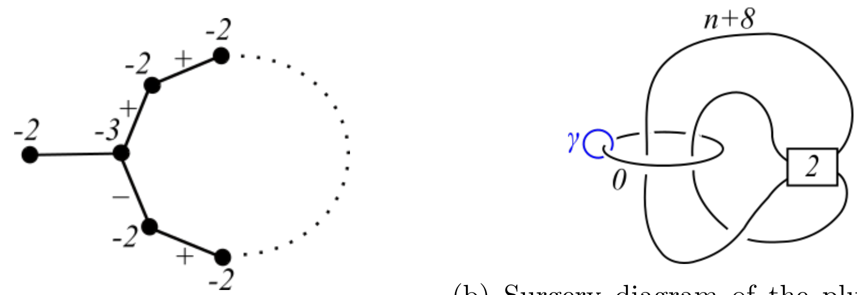
Since Y is an L-space, $\text{rank}(\widehat{HF}(Y)) = 2|H_1(Y)_T|$. Since $Y\backslash j$ is a connected sum of a lens space and $S^1 \times S^2$, it is also an L-space and thus satisfies $\text{rank}(\widehat{HF}(Y\backslash j)) = 2|H_1(Y\backslash j)|$. Thus, by applying the exact surgery triangle, we necessarily have that $\text{rank}(\widehat{HF}(Y_1^j)) = 2|H_1(Y_1^j)_T|$ and so Y_1^j is also an L-space. □

Remark 5.2.6. The boundaries of these negative cyclic plumbings are precisely the negative hyperbolic torus bundles over S^1 (c.f Example 2.2.7). Thus, all negative hyperbolic torus bundles are L-spaces.

Proposition 5.2.7. *Let X denote a plumbing graph with a negative cycle and trees emanating from the cycle. Moreover, assume X has no bad vertices, each weight is at most -2, and that there exists a vertex whose weight is strictly greater than negative its valence. Then ∂X is an L-space.*

Proof. If X is such a plumbing that has a vertex in the cycle whose weight is strictly greater than negative its valence, then the same inductive argument used in the proof of Proposition 5.2.5 proves the result. In this case, the cyclic plumbings of Proposition 5.2.5 serve as the base case of the induction.

Now consider the case in which there is a single arm of length one with weight -2 connected to the nth vertex, $a_n = -3$, and $a_i = -2$ for all $i \neq n$, as in Figure 5.7a. Consider the obvious surgery diagram gotten from the obvious handlebody diagram of the plumbing. We will perform the same moves as in the proof of Proposition 5.2.5. Namely: blow up the intersection between the -3-framed unknot and the -2-framed unknot that link positively with a $+1$-framed unknot; and sequentially blow down all of the -1-framed unknots. The result is the surgery diagram shown in Figure 5.7b (without the blue curve labelled γ). From this surgery diagram, it is clear that $|H_1(Y, \mathbb{Z})_T| = n+8$. Now, let γ denote a meridian of the 0-framed unknot as in Figure 5.7b. If we perform 0-surgery along γ, we obtain $(n+8)$-surgery on the $(5,2)$-torus knot; if we perform 1-surgery along γ, we obtain $(n+8)$-surgery on the right trefoil. These are known to be L-spaces by [37] for all $n \geq 2$. Thus, the rank of $\widehat{HF}$ of these two manifolds is $n+8$. Using a surgery triangle, this immediately implies that the rank of $\widehat{HF}$ of the cyclic plumbing is $2(n+8)$ and thus it is an L-space. Now using the same inductive argument as in the proof of Proposition 5.2.5, we obtain the result. □



(a) A cyclic plumbing with an arm

(b) Surgery diagram of the plumbed 3-manifold in (a)

Figure 5.7

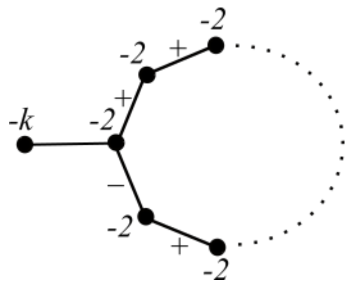
Proposition 5.2.8. *Let X be a plumbing graph with a single cycle such that each vertex has weight at most -2. Suppose the cycle is a negative cycle with weights $(-a_1, ..., -a_n)$, where $a_i \geq 2$, $a_1 = 2$, and $n \geq 6$. Further assume that there is a vertex with weight $-k$ and valence 1 sharing its edge with the vertex with framing $-a_1 = -2$. If $k \geq \lceil \frac{n-1}{2} \rceil$, then ∂X is an L-space.*

Proof. We will proceed by induction. Let X be as in the statement of the Proposition with $a_i = 2$ for all i, as in Figure 5.8a, and let $Y = \partial X$. Consider the obvious surgery diagram of Y gotten from the plumbing. First blow up the intersection between the bad vertex and the -2-framed unknot that links it positively, and then subsequently blow down all of the appearing -1-framed unknots. Then the unknot corresponding to the bad vertex has framing 0 as in Figure 5.8b. Next, perform a Rolfsen twist about the $-k$-framed unknot to obtain Figure 5.8c. Blow down 1-framed unknot and then continue to blow down the appearing 1-framed unknots, until we obtain the surgery diagram shown in Figure 5.8d. From this surgery diagram, it is clear that $|H_1(Y, \mathbb{Z})_T| = n - 4k$.

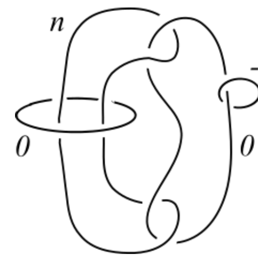
Let γ denote the simple closed curve that can be identified with a meridian of the 0-framed unknot as in Figure 5.8d. If we perform 0-surgery along γ, we obtain $(n-4k)$-surgery on the $(2k-1, -2)$-torus knot; if we perform 1-surgery along γ, we obtain $(n-4k)$-surgery on the $(2k+1, -2)$-torus knot. Since the $(2k+1, -2)$-torus knot has Seifert genus k and the $(2k-1, -2)$-torus knot has Seifert genus $k-1$, by [37], these 3-manifolds are both L-spaces if $n - 4k$ is at most $-2k+1$, or $n \leq 2k+1$. In this case, the rank of $\widehat{HF}$ of these two manifolds is $n - 4k$. Using a Heegaard Floer homology surgery exact triangle, this implies that $\text{rank}(\widehat{HF}(Y)) = 2(n-4k)$ and thus Y is an L-space.

By Lemma 5.2.13, any plumbing as in the statement of the Proposition is negative definite since $\frac{n-1}{2} \geq \frac{n+4}{4}$ when $n \geq 6$. Now, by applying induction with the Heegaard Floer homology surgery exact triangle as in the proof of Proposition 5.2.5, we obtain the result. $\square$

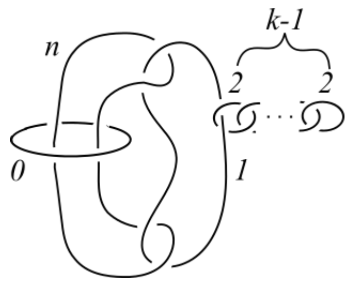
By once again using induction, the following follows from Proposition 5.2.8.



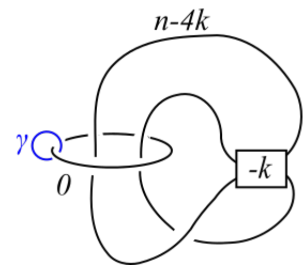
(a) A cyclic plumbing with an arm



(b) Surgery diagram obtained after blowing up and down



(c) Surgery diagram obtained Rolfsen twist



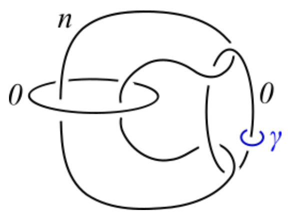
(d) Surgery diagram after blowdowns

Figure 5.8

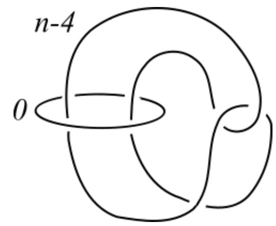
Corollary 5.2.9. *Let X be a plumbing graph with a negative cycle with weights $(-a_1, ..., -a_n)$ and trees emanating from the cycle. Furthermore, assume there is exactly one bad vertex, v, which is located in the cycle, has framing $-a_n$, and has valence $a_n + 1$. Finally assume that one of the vertices sharing an edge with v that is not in the cycle has weight at most $-\lceil \frac{n-1}{2} \rceil - (l-1)$, where l is the valence of this vertex and $n \geq 6$. Then ∂X is an L-space.*

Proposition 5.2.10. *Let X denote a negative cyclic plumbing of length 2 or 3 whose vertices all have weight -2. Then ∂X is an L-space*

Proof. Let $n = 2$ or 3. Consider the obvious surgery diagram of $Y = \partial X$. By blowing up the negative intersection with a $+1$-framed unknot and consecutively blowing down the resulting -1-framed unknots, we obtain the surgery description shown in Figure 5.9a. Consider γ as in the figure. By performing 0-surgery along γ, we obtain $S^1 \times S^2 \# L(-n, 1)$, which is an L-space whose $\widehat{HF}$ rank is $2n$. By performing -1-surgery along γ, we can blow down the -1-framed unknot, and then blow down the resulting $+1$-framed unknot to obtain the 3-manifold Y' with surgery diagram given in Figure 5.9b. Now, by choosing γ to be a meridian of the 0-framed unknot and using the surgery exact triangle, we can show, as in the previous proofs, that Y' is an L-space with $\text{rank}(\widehat{HF}(Y')) = 2(4-n)$. Thus by again applying the surgery triangle, we have that $\text{rank}(\widehat{HF}(Y)) = 2(4-n) + 2n = 2 = 2|H_1(Y; \mathbb{Z})_T|$. $\square$



(a) Surgery diagram corresponding to the boundary of the negative cyclic plumbing of length n whose weights are all -2



(b) Result of -1-surgery along γ in (a)

Figure 5.9

Remark 5.2.11. Notice that all of the plumbings used here are negative plumbings. It is also possible to use Kirby calculus to find positive plumbings whose boundaries are L-spaces; however, these plumbings are not negative definite. Thus the surgery triangle argument used in the above proofs cannot be easily used to construct negative definite, positive plumbings that have L-space boundaries. In fact, many such plumbings are simply not L-spaces. For a simple example of such a family of plumbings, see Section 5.3.2.

We end this section by proving that the plumbings constructed above are all indeed negative definite. This fact is needed in the inductive arguments used in the above proofs. We first prove a useful lemma. Recall that for a matrix $M = (m_{ij})_{1\le i,j\le n}$, the matrix M_l is defined by $M_l = (m_{ij})_{1\le i,j\le l}$.

Lemma 5.2.12. *Let P be a negative definite plumbing with intersection matrix $M = (m_{ij})_{1\le i,j\le n}$. Suppose v_n is a vertex of P whose weight is $-m_{nn}$ and $a|det(M)| \ge |det(M_{n-1})|$ for some integer a. Let P^1 be the plumbing obtained by plumbing a new vertex v^1 with weight $-k \le -(a+1)$ to v_n. Now inductively define P^j to be a plumbing built from P^{j-1} by plumbing a new vertex v^j to exactly one vertex in the set $\{v^1, \dots, v^{j-1}\}$. If v^i is not a bad vertex and has weight at most -2 for all $1 \le i \le j$, then P^j is negative definite.*

Proof. Let M^1 be the following matrix for the intersection form of P^1.

$$M^1 = \left[\begin{array}{ccc|c} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ & & & 1 \\ \hline 0 & \cdots\ 0 & 1 & -k \end{array}\right]$$

Since M is negative definite, $(-1)^l\det(M_l) > 0$ for all $1 \le l \le n$. Thus to prove M^1 is negative definite, we need only show that $(-1)^{n+1}\det(M^1) > 0$. Indeed, we have that

$$\begin{aligned}(-1)^{n+1}\det(M^1) &= (-1)^{n+1}((-k)\det(M) - \det(M_{n-1}))\\ &= (-1)^n k\det(M) - (-1)^{n-1}\det(M_{n-1})\\ &\geq (-1)^n(a+1)\det(M) - (-1)^{n-1}\det(M_{n-1})\\ &\geq (-1)^n\det(M) + (-1)^{n-1}\det(M_{n-1}) - (-1)^{n-1}\det(M_{n-1})\\ &= (-1)^n\det(M) > 0.\end{aligned}$$

Moveover notice that $|\det(M^1)| \geq |\det(M)|$. Let M^j denote an intersection matrix for the intersection form of P^j. Inductively assume that P^{j-1} is negative definite and $|\det(M^{j-1})| > |\det(M^{j-2})|$. For the base case, we use P^1 and $P^0 = P$. Since P^j is built from P^{j-1} by adding a vertex v^j with valence 1 and weight at most -2, by repeating the above calculation, we have that P^j is negative definite. □

Lemma 5.2.13. *Suppose P is a plumbing with a single cycle that is a negative cycle with weights $(-a_1, ..., -a_n)$. Let v_i be the vertex with weight $-a_i$.*

(a) If P has no bad vertices, then P is negative definite.

(b) Suppose all the vertices of P have weight at most -2 and that v_n is the only bad vertex of P. Further assume that the valence of v_n is a_n+1. If there is a vertex v not in the cycle that shares an edge with v_n and has weight $-k \leq -\lceil \frac{n+4}{4} \rceil$, then P is negative definite.

Proof. Let P a plumbing with a single cycle that is a negative cycle. Let the weights of the negative cycle be $(-a_1, \ldots, -a_n)$ and let the negative intersection in the cycle occur between the vertices with weights $-a_1$ and $-a_n$. Then

$$Q = (q_{ij}) = \left[\begin{array}{ccccccc|c} -a_1 & 1 & 0 & 0 & \cdots & 0 & -1 & \\ 1 & -a_2 & 1 & 0 & \cdots & 0 & 0 & \\ & & \ddots & & & & & \\ & & & \ddots & & & & * \\ & & & & \ddots & & & \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} & 1 & \\ -1 & 0 & \cdots & 0 & 0 & 1 & -a_n & \\ \hline & & & * & & & & * \end{array}\right]$$

is a matrix for the intersection form of P. Let $\mathbf{w} = (w_1, \ldots, w_n)^T$ be an arbitrary nonzero vector. Let $s_i = \sum_j q_{ij}$ denote the j^{th} row sum.

For (a), we will provide an argument similar to an argument used in the proof of Lemma 2.5 in [23]. Since there are no bad vertices, we necessarily have that $s_i \leq 0$ for all i, $s_1 \leq -2$, and $s_n \leq -2$. Thus

$$\begin{aligned}
\mathbf{w}^T Q \mathbf{w} &= \sum_{i,j} q_{ij} w_i w_j = \frac{1}{2} \sum_{i,j} q_{ij}(w_i^2 + w_j^2 - (w_i - w_j)^2) \\
&= \sum_{i,j} q_{ij} w_i^2 - \sum_{i<j} q_{ij}(w_i - w_j)^2 = \sum_i s_i w_i^2 - \sum_{i<j} q_{ij}(w_i - w_j)^2 \\
&= s_1 w_1^2 + s_n w_n^2 - q_{1n}(w_1 - w_n)^2 + \sum_{i \neq 1,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j) \neq (1,n)}} q_{ij}(w_i - w_j)^2 \\
&\leq -2w_1^2 - 2w_n^2 + (w_1 - w_n)^2 + \sum_{i \neq 1,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j) \neq (1,n)}} q_{ij}(w_i - w_j)^2 \\
&= -(w_1 + w_n)^2 + \sum_{i \neq 1,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j) \neq (1,n)}} q_{ij}(w_i - w_j)^2 \leq 0.
\end{aligned}$$

The last inequality is true since $s_i \leq 0$ for all i and $q_{ij} \geq 0$ for all $i \neq j$ and $(i,j) \neq (1,n)$. In particular, every term in this sum is nonpositive. If the last summation vanishes, then $w_i = w_j$ for all $i \neq j$ with $q_{ij} \neq 0$. Notice that for any $i \neq j$ we can find a sequence $i = i_1, i_2, \ldots, i_m = j$ such that $q_{i_k i_{k+1}} = 1$ for each k. This is because P is connected and $q_{1n} = -1$ can be avoided since it corresponds to the negative intersection in the cycle of P. Thus $w_1 = w_{i_1} = \cdots = w_{i_m} = w_j$. Since i and j were arbitrary, we have that $w_i = w_j$ for all i,j. In this case, the first term in the expression is strictly negative. Thus we have that $\mathbf{w}^T Q \mathbf{w} < 0$ and so P is negative definite.

Now assume X is an in the statement of (b). Then can express the intersection matrix as

$$Q = (q_{ij}) = \left[\begin{array}{ccccccc|c|c}
-a_1 & 1 & 0 & 0 & \cdots & 0 & -1 & 0 & \\
1 & -a_2 & 1 & 0 & \cdots & 0 & 0 & 0 & \\
 & & \ddots & & & & & 0 & \\
 & & & \ddots & & & & 0 & * \\
 & & & & \ddots & & & 0 & \\
0 & 0 & \cdots & 0 & 1 & -a_{n-1} & 1 & 0 & \\
-1 & 0 & \cdots & 0 & 0 & 1 & -a_n & 1 & \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 1 & -k & \\
\hline
 & & & * & & & & & * \\
\end{array}\right]$$

where $s_1 = -1$ and $k \geq \lceil \frac{n+4}{4} \rceil$. As above, we have that $s_i \leq 0$ for all i, and $s_n \leq -2$. If $\mathbf{w} = (w_1, \ldots, w_n, w_{n+1}, \ldots, w_m)^T$ (where m is the number of vertices of X) is any nonzero vector, then

$$\begin{aligned}
\mathbf{w}^T Q \mathbf{w} &= \sum_i s_i w_i^2 - \sum_{i<j} q_{ij}(w_i - w_j)^2 \\
&= s_1 w_1^2 + s_n w_n^2 - q_{1n}(w_1 - w_n)^2 + \sum_{i \neq 1,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j)\neq(1,n)}} q_{ij}(w_i - w_j)^2 \\
&\leq -2w_1^2 - w_n^2 + (w_1 - w_n)^2 + \sum_{i \neq 1,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j)\neq(1,n)}} q_{ij}(w_i - w_j)^2 \\
&\leq -2w_1^2 - w_n^2 + (w_1 - w_n)^2 + \sum_{i \neq 1,\ldots,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j)\neq(1,n)}} q_{ij}(w_i - w_j)^2
\end{aligned}$$

The last expression occurs when $a_i = 2$ for all $i \neq n$. Thus, if we can show that X is negative definite when $a_i = 2$ for all $i \neq n$, then we obtain the result. Assume from now on that $a_i = 2$ for all $i \neq n$. Let $Q_l = (q_{ij})_{1 \leq i,j \leq l}$. It is easy to check that $\det(Q_n) = 4(-1)^n$ and $\det(Q_{n-1}) = n(-1)^{n+1}$. Since the cyclic plumbing with intersection matrix Q_n is negative definite (by part (a)) and since $(k-1)|\det(Q_n)| = 4(k-1) \geq n = |\det(Q_{n-1})|$ whenever $k \geq \lceil \frac{n+4}{4} \rceil$, by Lemma 5.2.12, we have that Q must be negative definite. □

Remark 5.2.14. Similar proofs can show that other types of plumbings with bad vertices are negative definite. We only prove the case of Lemma 5.2.13(b) because we will work with such plumbings in Section 5.3.

5.3 Potentially exotic 4-manifolds

We will use the plumbings depicted in Figure 5.10 to construct potential examples of exotic 4-manifolds. In particular, we will construct manifolds that are homeomorphic to, but not obviously diffeomorphic to, $(2n-1)\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $n = 1, 2$ and $1 \leq k \leq 9n - 3$, and $(2n-1)\mathbb{C}P^2 \# (\lfloor \frac{11n+4}{4} \rfloor)\overline{\mathbb{C}P^2}$, where $2n - 1 - \lfloor \frac{11n+4}{4} \rfloor$ is not divisible by 16. In particular, for the first family of manifolds, if $n = 1$ and $k = 1$, we will have a potential exotic $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

The family of plumbings in Figure 5.10b were shown to have $\mathbb{Q}S^1 \times D^3$ replacements in Example 5.1.6. To see that the plumbings in Figure 5.10a have $\mathbb{Q}S^1 \times D^3$ replacements, start with the linear plumbing with weights $(-2, -1, -3, -2, ..., -2, -1)$ (whose boundary is simply $S^1 \times S^2$) and join the -1-weighted vertices. It is easy to check that the homological conditions of Proposition 5.1.4 are satisfied. Moreover, we will explicitly construct the $\mathbb{Q}S^1 \times D^3$ replacements and so we won't need to rely on Proposition 5.1.4.

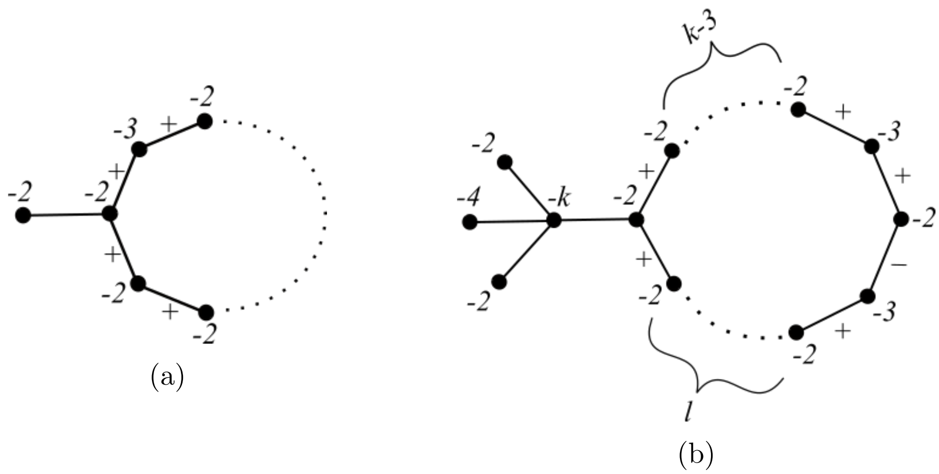


Figure 5.10: Plumbings that can be replaced by $\mathbb{Q}S^1 \times D^3$s.

Unfortunately, we will not be able to use Theorem 5.2.1 to compute the Ozsváth-Szabó 4-manifold invariants of the resulting 4-manifolds for reasons that will be explained at the end of each respective section. In both cases, we will start with an appropriate elliptic fibration containing a particular configuration of spheres.

Lemma 5.3.1. *For $n \geq 1$, there exists an elliptic fibration structure on $E(n)$ with an I_{9n}-fiber, $3n$ fishtail fibers, and a section. The fibration is depicted schematically in Figure 5.11.*

Proof. This can easily be seen via a monodromy factorization. The following is an argument similar to one found in [51]. According to [44], the monodromy of a fishtail fiber I_1 is $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and the monodromy of the singular fiber I_k is $a^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Now, let $b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Then, since $b = (ab)a(ab)^{-1}$, b also represents a fishtail fiber.

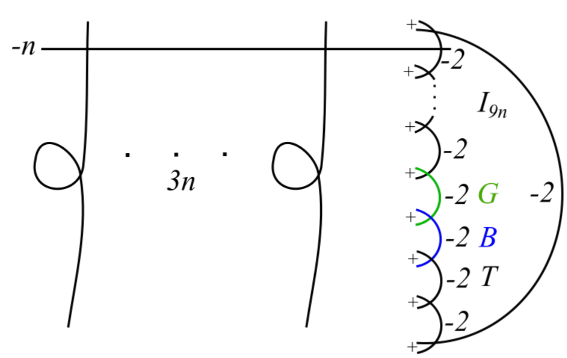


Figure 5.11: Elliptic fibration $E(n)$ with $3n$ fishtail fibers and an I_{9n} fiber

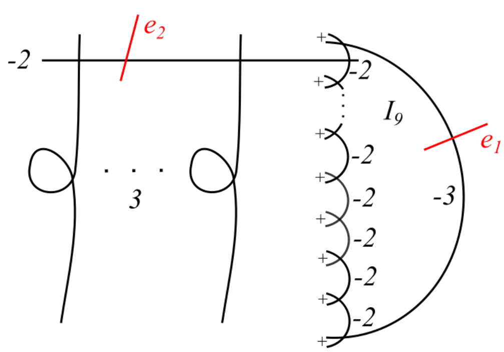
Now notice $a^{9n}(a^{-9n+3}ba^{9n-3})(a^{-9n+6}ba^{9n-6})\cdots(a^{-3}ba^{3})b = (a^3b)^{3n} = I$. □

5.3.1 A potential exotic $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$

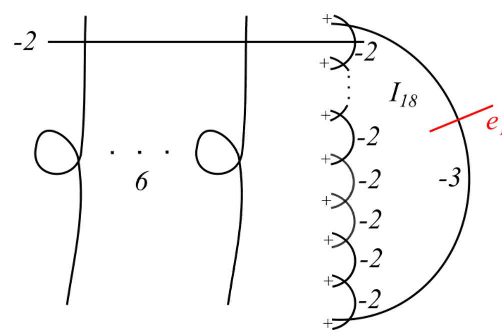
Let P_m denote the plumbing depicted in Figure 5.10a, where m is the number of spheres in the plumbing and let B_m denote its $\mathbb{Q}S^1 \times D^3$ replacement, which we will explicitly construct in Proposition 5.3.3 below.

Proposition 5.3.2. *P_m can be embedded in $E(n)\#(3-n)\overline{\mathbb{C}P^2}$, where $n = 1, 2$ and $3 \leq m \leq 9n+1$.*

Proof. Consider the configuration of spheres given by Figure 5.11 when $n = 1$. Blow up the rightmost -2-sphere in the I_9-fiber and the -1-section in generic points. Then we can easily see P_{10} embedded in $E(1)\#2\overline{\mathbb{C}P^2}$, as in Figure 5.12a. By smoothing out the intersection points of adjacent -2-spheres in the "blownup" I_9-fiber, we can obtain the plumbing P_i embedded in $E(1)\#2\overline{\mathbb{C}P^2}$ for $3 \leq i \leq 9$. Similarly, if $n = 2$, blow up the rightmost -2-sphere in the I_{18}-fiber to see P_{19} embedded in $E(2)\#\overline{\mathbb{C}P^2}$, as in Figure 5.12b. Again by smoothing out the intersection points of adjacent -2-spheres in the I_{18}-fiber, we can obtain the plumbing P_i embedded in $E(2)\#\overline{\mathbb{C}P^2}$ for $3 \leq i \leq 17$. □



(a) Spheres in $E(1)\#2\overline{\mathbb{C}P^2}$



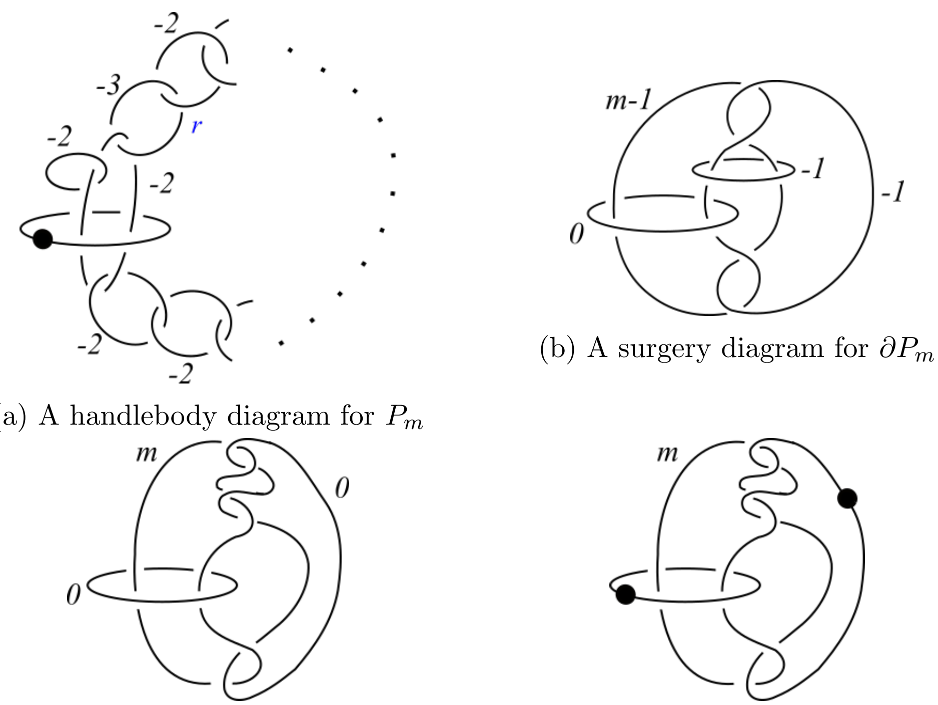
(b) Spheres in $E(2)\#\overline{\mathbb{C}P^2}$

Figure 5.12: Blowing up elliptic fibrations

Let ϕ_m be a diffeomorphism of ∂P_m and let $Z_{n,m} = E(n)\#(3-n)\overline{\mathbb{C}P^2} - \text{int}(P_m)$, where $n = 1, 2$ and $3 \leq m \leq 9n+1$. Moreover, let $X_{n,m} = Z_{n,m} \cup_{\phi_m} B_m$.

Proposition 5.3.3. *$X_{n,m}$ is homeomorphic to $(2n-1)\mathbb{C}P^2\#(9n+2-m)\overline{\mathbb{C}P^2}$, where $n = 1, 2$ and $3 \leq m \leq 9n+1$. In particular, if $n = 1$ and $m = 10$, then $X_{1,10}$ is homeomorphic to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.*

Proof. First consider the obvious handlebody diagram for P_m, as depicted in Figure 5.13a. Changing the dotted circle to a 0-framed unknot, we obtain a surgery diagram



(a) A handlebody diagram for P_m

(b) A surgery diagram for ∂P_m

(c) Result after a blowup and blowdowns

(d) Handlebody diagram for B_m

Figure 5.13: Kirby Calculus from P_m to B_m

for ∂P_m. Now, blow up the intersection labelled r with a 1-framed unknot and consecutively blow down the resulting -1-framed unknots, until we obtain the surgery description in Figure 5.13b. Next, blow down the horizontal -1-framed unknot in the middle of the diagram to obtain the surgery diagram in Figure 5.13c. Changing the 0-framed unknots to dotted circles, we obtain a handlebody diagram for a 4-manifold B_m with boundary $\partial B_m = \partial P_m$. By a simple homology calculation, it is clear the B_m is a $\mathbb{Q}S^1 \times D^3$.

Since $E(n)\#(3-n)\overline{\mathbb{C}P^2}$ is simply connected, the inclusion $\partial P_m \hookrightarrow Z_{n,m}$ induces a surjection $\pi_1(\partial P_m) \to \pi_1(Z_{n,m})$. Furthermore, since B_m is built out of 0-, 1-, and 2-handles, the inclusion $\partial B_m = \partial P_m \hookrightarrow B_m$ also induces a surjection $\pi_1(\partial P_m) \to \pi_1(B_m)$. By the Seifert-van Kampen theorem, we have $\pi_1(X_{n,m}) = \pi_1(Z_{n,m}) *_{\pi_1(\partial P)} \pi_1(B_m)$. Thus, in the amalgamation, the generators of $\pi_1(Z_{n,m})$ can be expressed in terms of the generators of $\pi_1(B_m)$. Therefore, if the generators of $\pi_1(B_m)$ bound disks in $Z_{n,m}$, then $\pi_1(X_{n,m})$ is trivial.

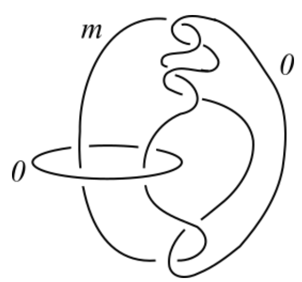
In Figure 5.13d, let a denote a meridian of the leftmost 1-handle and let b denote a meridian of the rightmost 1-handle. It is easy to see that $\pi_1(B_m) = \langle a, b \mid b^2ab^{-1}a^{-1} = 1\rangle$. Notice that by tracing through the Kirby calculus, b can be identified with a meridian of the -3-framed unknot of the handlebody diagram of P_m show in Figure 5.13a. Thus, it can be identified with the equator of the exceptional sphere labelled

e_1 in Figure 5.12 that is dangling off of the -3-sphere and so it bounds a disk in $Z_{n,m}$, namely a hemisphere of e_1.

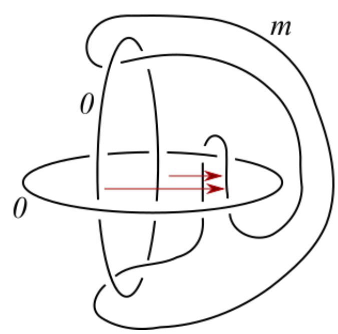
Similarly, a can be identified with the meridian of the 1-handle in the handlebody diagram of P_m. We wish to show that this meridian bounds a disk in $Z_{n,m}$. Consider the amalgamated product $1 = \pi_1(E(n)\#(3-n)\overline{\mathbb{C}P^2}) = \pi_1(Z_{n,m}) *_{\pi_1(\partial P_m)} \pi_1(P_m)$ given by the Seifert-van Kampen theorem. Since P is built out of 0-, 1-, and 2-handles, $\pi_1(\partial P_m)$ surjects onto $\pi_1(P_m)$. Let z be the free generator of $\pi_1(\partial P_m)$ that maps to a. Since a is a free generator of $\pi_1(P_m)$ and it maps to the identity in the amalgamation $\pi_1(Z_{n,m}) *_{\pi_1(\partial P_m)} \pi_1(P_m)$, the image of z in $\pi_1(Z_{n,m})$ must be trivial. Thus, a must bound a disk in $Z_{n,m}$ and so $X_{n,m}$ is simply connected.

Finally, we can easily calculate the Euler characteristics and signatures of these manifolds to be $\chi(X_{n,m}) = \chi(E(n)\#(3-n)\overline{\mathbb{C}P^2}) - \chi(P_m) + \chi(B_m) = 11n + 3 - m = \chi((2n-1)\mathbb{C}P^2\#(9n+2-m)\overline{\mathbb{C}P^2})$ and $\sigma(X_{n,m}) = \sigma(E(n)\#(3-n)\overline{\mathbb{C}P^2}) - \sigma(P_m) + \sigma(B_m) = -7n + 3 + m = \sigma((2n-1)\mathbb{C}P^2\#(9n+2-m)\overline{\mathbb{C}P^2})$. Since $-7n+3+m$ is not divisible by 16 for all n and m, $X_{n,m}$ is odd, and so by Freedman's theorem, $X_{n,m}$ is homeomorphic to $(2n-1)\mathbb{C}P^2\#(9n+2-m)\overline{\mathbb{C}P^2}$. $\square$

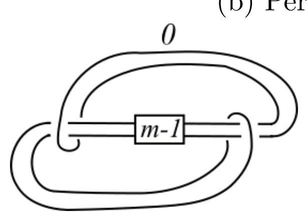
The reason why we cannot use Theorem 5.2.1 to compute the 4-manifold invariants of $X_{n,m}$ is because $HF^-_{red}(\partial P_m)$ is nonzero in degree $-\frac{3}{2}$. Consider the surgery diagram of ∂P_m in Figure 5.14a. Isotoping the link appropriately, we can obtain the diagram in Figure 5.14b. Performing the handleslides indicated by the red arrows, we obtain a 3-component link that includes a 0-framed unknot that is simply a meridian of one of the other knots. Cancelling these two knots, we are left with 0-surgery on a ribbon



(a) Surgery description of ∂P_m.



(b) Perform the indicated slides



(c) Result after handle cancellation

Figure 5.14: Realizing ∂P_m as 0-surgery on a ribbon knot

knot, as shown in Figure 5.14c.

Computing the Heegaard Floer homology using this surgery description is fairly routine. One obtains

$$HF^+(\partial P_m, \mathfrak{s}) = \mathcal{T}^+_{(-\frac{1}{2})} \oplus \mathcal{T}^+_{(\frac{1}{2})} \oplus \mathbb{Z}^2_{(-\frac{1}{2})}$$

where $\mathfrak{s}$ is the unique torsion spin^c structure on ∂P_m and $\mathcal{T}^+ = \mathbb{Z}[U, U^{-1}]/\mathbb{Z}[U]$. In particular, $HF^+_{red}(Y, \mathfrak{s}) = \mathbb{Z}^2$ lives in degree $-\frac{1}{2}$; equivalently, $HF^-_{red}(Y, \mathfrak{s}) = \mathbb{Z}^2$ lives in degree $-\frac{3}{2}$.

Moreover, it is unknown whether the replacement B_m admits a symplectic structure with strongly convex boundary. Thus we do not know if $X_{n,m}$ admits a symplectic structure, which would automatically prove that it is exotic when $n = 2$, since $3\mathbb{C}P^2\#(19-m)\overline{\mathbb{C}P^2}$ does not admit a symplectic structure.

5.3.2 A potential exotic $(2n-1)\mathbb{C}P^2\#(\lfloor \frac{11n+4}{4} \rfloor)\overline{\mathbb{C}P^2}$

Let P_n denote the plumbing depicted in Figure 5.10b, where $9n$ is the number of spheres in the cycle and $k = \lceil \frac{9n+5}{2} \rceil$. Let B_n denote the $\mathbb{Q}S^1 \times D^3$ replacement of P_n. We will explicitly construct this replacement in Proposition 5.3.5 below.

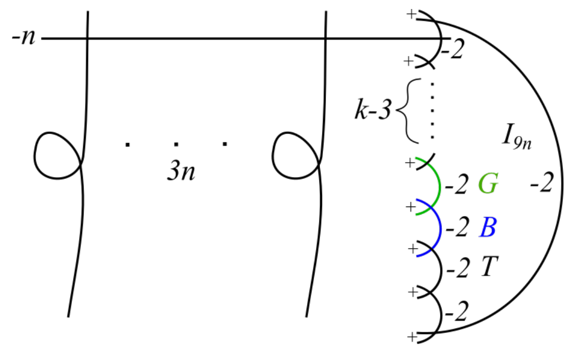
Proposition 5.3.4. *P_n can be embedded in a blowup of $E(n)$ for all $n \geq 1$.*

Proof. Consider the elliptic fibration of Lemma 5.3.1, which is depicted in Figure 5.15a. Let B denote the blue sphere, G denote the green sphere, and T denote the black sphere adjacent to B. Take a pushoff B' of B. Then B' intersects B geometrically -2-times and it intersects T geometrically $+1$-times. Now smooth the intersection point of B and T to obtain the sphere $B \cup T$. Then the algebraic intersection between B' and $B \cup T$ is -1. Moreover, using a local model, it is easy to see that we can isotope B' so that the geometric intersection between B' and $B \cup T$ is also -1. Next, blow up the intersection point of $B \cup T$ and G. Then $G.G = -3$, $B'.B' = -2$, and $(B \cup T).(B \cup T) = -3$ (see Figure 5.15b). Let

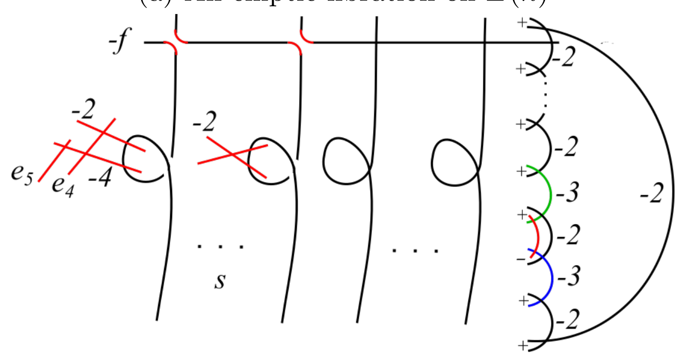
$$s = \begin{cases} \dfrac{7n}{4} & n = 0(\text{mod}4) \\ \dfrac{7n+1}{4} & n = 1(\text{mod}4) \\ \dfrac{7n+2}{4} & n = 2(\text{mod}4) \\ \dfrac{7n-1}{4} & n = 3(\text{mod}4) \end{cases} \qquad f = \begin{cases} \dfrac{9n+4}{2} & n = 0(\text{mod}4) \\ \dfrac{9n+5}{2} & n = 1(\text{mod}4) \\ \dfrac{9n+6}{2} & n = 2(\text{mod}4) \\ \dfrac{9n+3}{2} & n = 3(\text{mod}4) \end{cases}$$

Blow up the double points of s fishtail fibers and smooth the intersections of each of these fishtail fibers with the $-n$-section to obtain a sphere S with self-intersection $-n-2s$. Let e_1, e_2 denote two of the exceptional spheres resulting from the blowups. Then $S.e_i = 2$ for each i. Blow up one of the intersection points between e_i and S for

$i = 1, 2$. Label the new exceptional sphere intersecting e_2 and S by e_3. Blow up the intersection point of e_2 and e_3 and label the new exceptional sphere e_4. Finally, blow up e_2 in a generic point and label the new exceptional sphere e_5. Then $e_1.e_1 = -2$, $e_2.e_2 = -4$, $e_3.e_3 = -2$, and $S.S = -f$. We thus have the configuration depicted in Figure 5.15b. This configuration lives in $E(n)\#(s+5)\overline{\mathbb{C}P^2}$. If $n \equiv 1(\text{mod}4)$ or $n \equiv 2(\text{mod}4)$, then $f = \lceil \frac{9n+5}{2} \rceil = k$ and so P_n is embedded in $E(n)\#(s+5)\overline{\mathbb{C}P^2}$. Otherwise, blow up the $-f$-sphere one final time to obtain a sphere with self intersection $-(\lceil \frac{9n+3}{2} \rceil + 1) = -\lceil \frac{9n+5}{2} \rceil = -k$. In this case, P_n is embedded in $E(n)\#(s+6)\overline{\mathbb{C}P^2}$. $\square$



(a) An elliptic fibration on $E(n)$



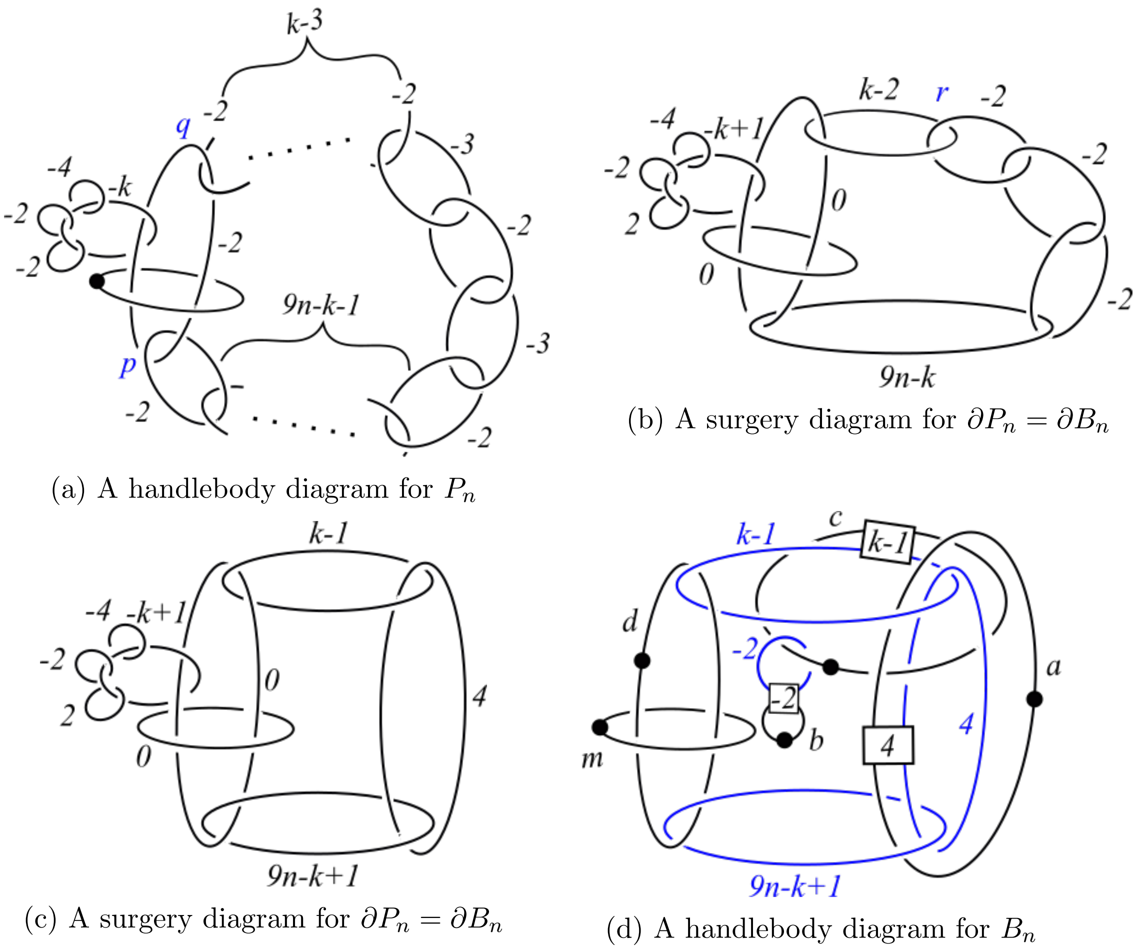
(b) Result after blowing up $E(n)$ $s+5$ times and smoothing $s+1$ points. f and s depend on n

Figure 5.15

Let $Z_n = E(n)\#(s+4)\overline{\mathbb{C}P^2} - \text{int}(P_n)$ if $n \equiv 1(\text{mod}4)$ or $n \equiv 2(\text{mod}4)$ and let $Z_n = E(n)\#(s+5)\overline{\mathbb{C}P^2} - \text{int}(P_n)$ if $n \equiv 0(\text{mod}4)$ or $n \equiv 3(\text{mod}4)$. Let ϕ_n be a diffeomorphism of ∂P_n and let $X_n = Z_n \cup_{\phi_n} B_n$.

Proposition 5.3.5. *For $n \geq 1$, if $2n - 1 - \lfloor \frac{11n+4}{4} \rfloor$ is not divisible by 16, then X_n is homeomorphic to $(2n-1)\mathbb{C}P^2\#(\lfloor \frac{11n+4}{4} \rfloor)\overline{\mathbb{C}P^2}$.*

Proof. First consider the obvious handlebody diagram for P_n shown in Figure 5.16a. We will now describe the Kirby moves needed to get to a handlebody diagram of



(a) A handlebody diagram for P_n

(b) A surgery diagram for $\partial P_n = \partial B_n$

(c) A surgery diagram for $\partial P_n = \partial B_n$

(d) A handlebody diagram for B_n

Figure 5.16

B_n. First, change the dotted circle to a 0-framed unknot to obtain a surgery diagram for ∂P_n. Next, perform a Rolfsen twist about one of the -2-framed unknots linking the $-k$-framed unknot so as to increase the latter's framing by 1. Now blow up the intersections labelled p and q in Figure 5.16a with $+1$-framed unknots. Consecutively blow down the -1-framed unknots that appear until we obtain the surgery diagram depicted in Figure 5.16b. Next, blow up the intersection labelled r with a $+1$-framed unknot and then blow down the -1-framed unknots that appear until we obtain Figure 5.16c. Slide the $(-k+1)$-framed unknot over the $(k-1)$-framed unknot, slide the 2-framed unknot over the -2-framed unknot, and finally slide the -4-framed unknot over the 4-framed unknot. Finally, change the resulting 0-framed unknots to dotted circles to obtain a handlebody diagram for B_n, as shown in Figure 5.16d. A quick homology calculation shows that B_n is a $\mathbb{Q}S^1 \times D^3$.

As in the proof of Proposition 5.3.3, to show that the X_n is simply connected, we need only show that the generators of $\pi_1(B_m)$ bound disks in Z_n. In Figure 5.16d, let m, a, b, c, and d denote the meridians of the 1-handles labelled m, a, b, c, and d in Figure 5.16d. It is easy to see that

$$\pi_1(B_n) = \langle m, a, b, c, d \mid amdm^{-1}, dc^{-(k-1)}a, cb^{-2}, a^4c^{-1}\rangle.$$

Moreover, it can be shown that this group is generated by the elements m, a, and ba^{-2}. By carefully tracing through the Kirby calculus, a can be identified with a meridian a' of the leftmost -4-framed unknot in the handlebody diagram of P_n in Figure 5.16a. Similarly, ab^{-2} can be identified with a loop l that links the same -4-sphere twice and the adjacent -2-sphere once. It is easy to see that a' can be identified with the equator of the exceptional sphere e_5 that is dangling off of the -4-sphere depicted in Figure 5.15b. Thus, a bounds a disk in Z_n, namely a hemisphere of e_5. Similary, l can be identified with the boundary of the disk given by the hemispheres of the exceptional spheres e_5 and e_4 depicted in Figure 5.15b joined together by a band. Thus, a and ba^{-2} both bound disks in Z_n. Furthermore, it is clear that m must also bound a disk in Z_n, by the reasoning given in the proof of Proposition 5.3.3. Thus, X_n is simply connected.

We can easily calculate the signature and Euler characteristic of X_n to be

$$\chi(X_n) = \begin{cases} \dfrac{19n+8}{4} & n = 0(\text{mod}4) \\ \dfrac{19n+5}{4} & n = 1(\text{mod}4) \\ \dfrac{19n+6}{4} & n = 2(\text{mod}4) \\ \dfrac{19n+7}{4} & n = 3(\text{mod}4) \end{cases} \qquad \sigma(X_n) = \begin{cases} -\dfrac{3n+8}{4} & n = 0(\text{mod}4) \\ -\dfrac{3n+5}{4} & n = 1(\text{mod}4) \\ -\dfrac{3n+6}{4} & n = 2(\text{mod}4) \\ -\dfrac{3n+7}{4} & n = 3(\text{mod}4) \end{cases}$$

These are the signatures and Euler characteristics of $(2n-1)\mathbb{C}P^2\#(\lfloor\frac{11n+4}{4}\rfloor)\overline{\mathbb{C}P^2}$. Moreover, if $\sigma(X_n) = (2n-1) - \lfloor\frac{11n+4}{4}\rfloor$ is not divisible by 16, then X_n is odd and so by Freedman's theorem, it is homeomorphic to $(2n-1)\mathbb{C}P^2\#(\lfloor\frac{11n+4}{4}\rfloor)\overline{\mathbb{C}P^2}$. □

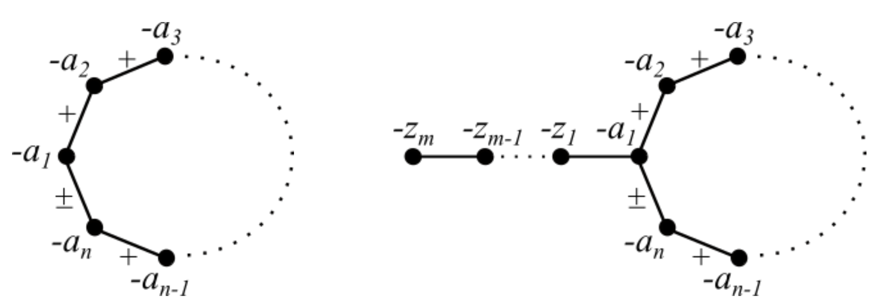
In this example, unlike the example of Section 5.3.1, ∂P_n is an L-space by Proposition 5.2.9 (since $k \geq \lceil\frac{9n-1}{2}\rceil + 3$). However, there does not exist a spin^c structure on the blowup of $E(n)$ whose first Chern class is characteristic and whose restriction to P_n has first Chern class squared equal to $-b_2(P_n)$. Thus, we cannot apply Theorem 5.2.1. Moreover, as in the previous example, it is unknown if the replacement B_m admits a symplectic structure with strongly convex boundary.

Chapter 6

Contact structures on plumbed 3-manifolds

In this chapter, we will explore Question 5.0.2. In particular, we will classify tight contact structures with no Giroux torsion on families of plumbed 3-manifolds whose associated graphs are not simply connected. Recall from Example 3.2.3 that such classifications can be helpful in constructing symplectic exotic 4-manifolds. Recall from Section 3.2.1, that a 3-manifold can admit infinitely many tight contact structures in the presence of incompressible tori. However, by a result of Gay [24], only tight contact structures with no Giroux torsion can be strongly symplectically fillable. Thus we will focus on classifying contact structures with no Giroux torsion.

As seen in Example 2.2.7, the boundaries of cyclic plumbings, as depicted in Figure 6.1a, are T^2-bundles over S^1. In [39], Honda classified tight contact structures on such manifolds and in particular those with no Giroux torsion. Before we recall this result, we introduce notation due to Golla and Lisca in [32] that will be used throughout this chapter.



(a) A cyclic plumbing with boundary $C_\pm$

(b) A cyclic plumbing with an arm with boundary $Y_\pm$

Figure 6.1: Plumbed 3-manifolds

Recall that a *blowup* of a sequence $b = (b_1, ..., b_k)$ of nonnegative integers is a sequence of the form $(b_1, ..., b_i + 1, 1, b_{i+1} + 1, ..., b_k)$, where $1 \leq i \leq n-1$. If b' is a sequence obtained from b by a finite number of blowups, we call b' a blowup of b. Given two sequences b and c of length k, we write $c \prec b$ if $c_i \leq b_i$ for all i. Let $b = (\underbrace{2, ..., 2}_{n_0}, m_1 + 3, \underbrace{2, ..., 2}_{n_1}, m_2 + 3, ..., m_k + 3, \underbrace{2, ..., 2}_{n_k})$. Then we define $\rho(b)$ to be the sequence $\rho(b) = (n_0 + 2, \underbrace{2, ..., 2}_{m_1}, n_1 + 3, \underbrace{2, ..., 2}_{m_2}, ..., n_{s-1} + 3, \underbrace{2, ..., 2}_{m_s}, n_s + 2)$.

Definition 6.0.1 (Golla-Lisca [32])**.** *A sequence of nonnegative integers* $a = (a_1, ..., a_n)$ *is* ***embeddable*** *if* $s \prec \rho(a)$ *for some blowup* s *of* $(0, 0)$*.*

Combining Honda's work in [39] and Golla-Lisca's work in [32], we have the following classification of Stein fillable contact structures on hyperbolic T^2-bundles over S^1.

Theorem 6.0.2 ([39], [32])**.** *Let* $C_\pm$ *be the boundary of the cyclic plumbing depicted in Figure 6.1a, where* $a_i \geq 2$ *for all* i *and* $a_1 \geq 3$*. Then, up to isotopy,*

- C_+ *admits exactly* $(a_1 - 1) \cdots (a_n - 1)$ *Stein fillable contact structures, two of which are universally tight, and*
- C_- *admits exactly* $(a_1 - 1) \cdots (a_n - 1)$ *Stein fillable virtually overtwisted contact structures and a unique universally tight contact structure with no Giroux torsion, which is Stein fillable if* $(a_1, ..., a_n)$ *is embeddable.*

Remark 6.0.3. $(a_1, ..., a_n)$ being embeddable is not a necessary condition for the universally tight contact structure on C_- to be Stein fillable. However, a necessary condition is described by Ding and Li in [9].

Now let $Y_\pm$ be the plumbed 3-manifold obtained as the boundary of the plumbing depicted in Figure 6.1b. The main result of this chapter is the following.

Theorem 6.0.4. *If* $a_i, z_j \geq 2$ *for all* i, j *and* $a_1 \geq 3$*, then, up to isotopy,*

- Y_+ *admits exactly* $(a_1 - 1) \cdots (a_n - 1)(z_1 - 1) \cdots (z_m - 1)$ *Stein fillable contact structures, and*
- Y_- *admits exactly* $(a_1 - 1) \cdots (a_n - 1)(z_1 - 1) \cdots (z_m - 1) + z_1(z_2 - 1) \cdots (z_m - 1)$ *tight contact structures with no Giroux torsion,* $(a_1 - 1) \cdots (a_n - 1)(z_1 - 1) \cdots (z_m - 1)$ *of which are Stein fillable. If* $(a_1, ..., a_n)$ *is embeddable, then all of these contact structures are Stein fillable.*

Remark 6.0.5. The proof of Theorem 6.0.4 can be modified to classify the tight contact structures with no Giroux torsion for $Y_\pm$ in more general settings. That is, one can remove the assumption $a_1 \geq 3$ in certain cases and prove analogous results.

The proof of Theorem 6.0.4 for Y_+ is fairly standard. It relies on convex surface theory to provide an upper bound for the number of tight contact structures with no Giroux torsion and then by producing explicit Stein fillings with distinct first Chern classes, we realize this upper bound by applying Lisca and Matic's result (Theorem 3.1.18). Y_-, on the other hand, admits additional contact structures that are not necessarily Stein (or even strongly symplectically) fillable (c.f. Remark 6.0.3). To show that these additional structures are distinct, we will need to use the following generalization of Theorem 3.1.18 to Stein cobordisms, which relies on Heegaard Floer homology with ω-twisted coefficients.

Theorem 6.0.6. *Suppose (Y, ξ) is a contact manifold and $[\omega] \in H^2(Y; \mathbb{R})$ is an element such that the contact invariant $c(\xi, [\omega])$ is nontrivial. Let (W, J_i) be a Stein cobordism from (Y, ξ) to (Y', ξ_i) for $i = 1, 2$. If the spin^c structures induced by J_1 and J_2 are not isomorphic, then ξ_1 and ξ_2 are nonisotopic tight contact structures.*

As mentioned in Section 3.1.4, one of the uses of the ω-twisted coefficient system is that it can detect tight contact structures that the untwisted contact invariant does not detect, namely weakly symplectically fillable contact structures that are not strongly symplectically fillable (e.g. see [26]). In particular, the following result is due to Ozsváth and Szabó.

Theorem 6.0.7 (Theorem 4.2 of [60])**.** *If (X, ω) is a weak symplectic filling of (Y, ξ), then the contact invariant $c(\xi; [\omega]|_Y)$ is non-trivial.*

Coupling this result with Theorem 6.0.6, we have the following corollary, which will be used in the proof of Theorem 6.0.4.

Corollary 6.0.8. *If (Y, ξ) is weakly symplectically fillable and (W, J_i) is a Stein cobordism from (Y, ξ) to (Y', ξ_i) for $i = 1, 2$ such that the spin^c structures induced by J_1 and J_2 are not isomorphic, then ξ_1 and ξ_2 are nonisotopic tight contact structures.*

This chapter is organized as follows. Section 6.1 contains the proof of Theorem 6.0.6. Section 6.2 discusses a method (analogous to Proposition 3.1.17) for distinguishing Stein structures on a cobordism using rotation numbers. Section 6.3 contains a quick review of convex surface theory and relevant theorems of Giroux and Honda. Section 6.4 contains the proof of Theorem 6.0.4. In Section 6.5, we provide explicit descriptions of the Stein fillings of some of the additional fillable contact structures on Y_-. Finally, in Section 6.6, we collect some results regarding continued fractions that are needed throughout the chapter.

6.1 The Ozsváth-Szabó contact invariant with ω-twisted coefficients

In this section, we will recall the definition of the contact invariant with ω-twisted coefficients, as defined in [60], and use it to prove Theorem 6.0.6 below, which will in turn be used in the proof of Theorem 6.0.4. We will assume the reader is familiar with Heegaard Floer homology with twisted coefficients and the contact invariant (see [61], [63]).

Let Y be a three-manifold and fix a cohomology class $[\omega] \in H^2(Y;\mathbb{R})$. We can then view $\mathbb{Z}[\mathbb{R}]$ as a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$-module via the ring homomorphism $[\gamma] \to T^{\langle \gamma \cup \omega, [Y] \rangle}$, where T^r denotes the group ring element associated to $r \in \mathbb{R}$. Using this coefficient system, we denote the *ω-twisted Floer homology* by $\underline{\widehat{HF}}(Y;[\omega])$. Let $W : Y \to Y'$ be a cobordism and let $[\Omega] \in H^2(W;\mathbb{R})$. Then for each $\mathfrak{s} \in$Spin$^c(W)$, we obtain an induced map $\underline{F}_{W,\mathfrak{s};[\Omega]} : \underline{\widehat{HF}}(Y, \mathfrak{s}|_Y; [\Omega]|_Y) \to \underline{\widehat{HF}}(Y', \mathfrak{s}|_{Y'}; [\Omega]|_{Y'})$, which is well-defined up to multiplication by $\pm T^c$ for some $c \in \mathbb{R}$. See [60] for more details.

Given a contact structure ξ on Y, we can define the *ω-twisted contact invariant* $c(Y;[\omega]) \in \underline{\widehat{HF}}(-Y, \mathfrak{t}_\xi; [\omega])$, where $\mathfrak{t}_\xi$ denotes the canonical spin^c structure on Y determined by ξ. This element is well-defined up to sign and multiplication by invertible elements in $\mathbb{Z}[H^1(Y;\mathbb{Z})]$. We denote its equivalence class by $[c(\xi;[\omega])]$.

The following theorem follows by the proof of Theorem 3.6 in [29]. We will use it to prove the main result of this section, Theorem 6.0.6 below, which can be thought of as a generalization Lisca-Matic's Theorem 3.1.18 in Section 3.1.3.

Theorem 6.1.1. *Let (Y,ξ) and (Y',ξ') be contact manifolds and let (W,J) be a Stein cobordism from (Y,ξ) to (Y',ξ') which is obtained by Legendrian surgery on some Legendrian link in Y. If $\mathfrak{t}$ is the canonical spin^c structure on W for the complex structure J, then for any $[\Omega] \in H^2(W;\mathbb{R})$,*

$$[F_{\overline{W},\mathfrak{s};[\Omega]}(c(\xi';[\Omega]|_{Y'}))] = \begin{cases} [c(\xi;[\Omega]|_Y)] \text{ if } \mathfrak{s} = \mathfrak{t} \\ 0 \text{ if } \mathfrak{s} \neq \mathfrak{t} \end{cases}$$

where $\overline{W}$ denotes W with the opposite orientation, thought of as a cobordism from $-Y'$ to $-Y$.

Theorem 6.0.6. *Suppose (Y,ξ) is a contact manifold and $[\omega] \in H^2(Y;\mathbb{R})$ is an element such that $c(\xi,[\omega])$ is nontrivial. Let (W,J_i) be a Stein cobordism from (Y,ξ) to (Y',ξ_i) for $i = 1,2$. If the spin^c structures induced by J_1 and J_2 are not isomorphic, then ξ_1 and ξ_2 are nonisotopic tight contact structures.*

Proof. Since W is Stein, it has no 3-handles and so $H^3(W,Y) = 0$. Thus by considering the long exact sequence of the pair, there exists an element $[\Omega] \in H^2(W;\mathbb{R})$ satisfying $[\Omega]|_Y = [\omega]$. Let $\mathfrak{s}_1, \mathfrak{s}_2 \in \text{Spin}^c(W)$ such that $\mathfrak{s}_i|_Y = \mathfrak{t}_\xi$ and $\mathfrak{s}_i|_{Y'} = \mathfrak{t}_{\xi_i}$ for $i = 1,2$. Consider the cobordism maps $F_{\overline{W},\mathfrak{s}_i;[\Omega]} : \underline{\widehat{HF}}(-Y', \mathfrak{t}_{\xi_i}; [\Omega]|_{Y'}) \to \underline{\widehat{HF}}(-Y, \mathfrak{t}_\xi; [\Omega]|_Y)$.

By Theorem 6.1.1, $[F_{\overline{W},\mathfrak{s}_i;[\Omega]}(c(\xi_i;[\Omega]|_{Y'}))] = [c(\xi;[\Omega]|_Y)]$ if $\mathfrak{s}_\mathfrak{i} = \mathfrak{t}_\mathfrak{i}$, where $\mathfrak{t}_\mathfrak{i}$ is the canonical spin^c structure associated to J_i. Thus $[c(\xi_1;[\Omega]|_{Y'})]$ and $[c(\xi_2;[\Omega]|_{Y'})]$ are both nontrivial. Moreover, $[F_{\overline{W},\mathfrak{s}_i;[\Omega]}(c(\xi_i;[\Omega]|_{Y'}))] = 0$ whenever $\mathfrak{s}_i \neq \mathfrak{t}_i$. In particular, $[F_{\overline{W},\mathfrak{t}_i;[\Omega]}(c(\xi_j;[\Omega]|_{Y'}))] = 0$ when $i \neq j$. Thus, since $\mathfrak{t}_1 \neq \mathfrak{t}_2$, we have that $[c(\xi_1;[\Omega]|_{Y'})] \neq [c(\xi_2;[\Omega]|_{Y'})]$. □

Remark 6.1.2. The proof of Theorem 6.0.6 actually shows something stronger than $[c(\xi_1;[\Omega]|_{Y'})] \neq [c(\xi_2;[\Omega]|_{Y'})]$. Namely, the contact elements $c(\xi_1;[\Omega]|_{Y'})$ and $c(\xi_2;[\Omega]|_{Y'})$ live in different summands of $\underline{\widehat{HF}}(-Y;[\Omega]|_Y)$ and are thus linearly independent.

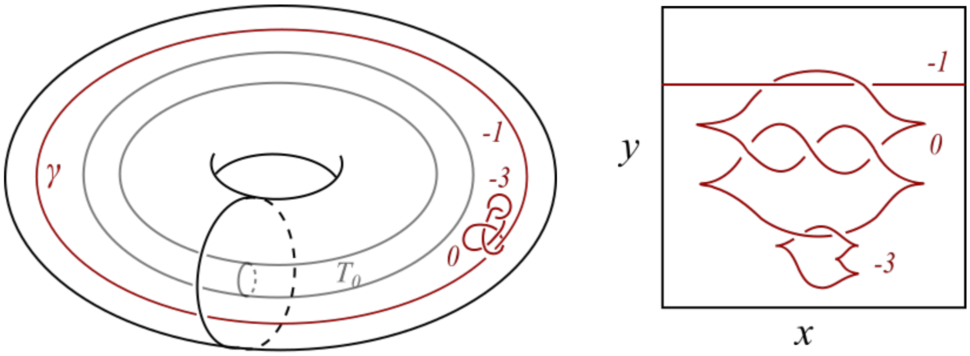
6.2 Legendrian surgery in $T^2 \times I$

In this section, we will describe a method to distinguish contact structures obtained by Legendrian surgery on 3-manifolds containing a particular contact $T^2 \times [0,1]$. This method will be used in the proof of Theorem 6.0.4 found in Section 6.4. Give $T^2 \times [0,1]$ the coordinates $((x,y),t)$ and define the contact structure ξ on $T^2 \times [0,1]$ to be the kernel of the 1-form $\alpha = \sin(\phi(t))dx + \cos(\phi(t))dy$, where $\phi'(t) > 0$, $\phi(0) = -\frac{\pi}{2}$, and $\phi(1) = \frac{\pi}{2}$. Then there exists a torus $T_{t_0} = T^2 \times \{t_0\}$, such that $\phi(t_0) = 0$ so that the contact form restricted to T_{t_0} is $\alpha = dy$.

Consider the standard diagram of $T^2 \times [0,1]$ as embedded in $\mathbb{R}^3$ depicted in Figure 6.2a (without the red surgery curves), where $T_1 = T^2 \times \{1\}$ is the outer torus and $T_0 = T^2 \times \{0\}$ is the inner torus. Moreover, let $S^1 \times \{pt\} \times \{pt\}$ be the longitudinal direction and $\{pt\} \times S^1 \times \{pt\}$ be the meridional direction in this diagram. Then we can draw T_{t_0} as a square, with its edges identified, such that the horizontal edges of the square are the x-direction and the vertical edges of the square are the y-direction. Let $\gamma = S^1 \times \{pt\} \times \{t_0\}$ and define the 0-framing associated to γ to be surface framing of γ in the surface T_{t_0}. Denote this framing by $\mathcal{F}$. Then any knot smoothly isotopic to γ in $T^2 \times (0,1)$ has a well-defined 0-framing, namely the image of $\mathcal{F}$ under the isotopy. For any nullhomologous knot in $T^2 \times [0,1]$, the 0-framing is given by the Seifert surface framing.

As in the case of Legendrian knots in $(\mathbb{R}^3, \xi_{st})$, we can project any Legendrian curve $L \subset (T^2 \times (0,1), \xi)$ to T_{t_0}. We call this the *front projection* of L. If $L \subset T^2 \times (0,1)$, then the projection will have no vertical tangencies, since $\frac{dy}{dx} = -\tan(\phi(t)) \neq \infty$ for all $t \in (0,1)$. It will, however, contain semi-cubical cusps and away from these cusp points L can be recovered by $\frac{dy}{dx} = -\tan(\phi(t))$. In particular, at a crossing the strand with smaller slope is in front. For example, Figure 6.2b shows a front projection of the link depicted in Figure 6.2a. We will only concern ourselves with nullhomologous knots that can be contained in a 3-ball and knots that are smoothly isotopic to γ.

Give $\mathbb{R}^3$ the coordinates (u,v,w) so that $\xi_{st} = \ker(dw + udv)$ and let $\tilde{\xi}_{st}$ be the image of ξ_{st} under the projection $\mathbb{R}^3 \to \mathbb{R} \times (\mathbb{R}^2/\mathbb{Z}^2) \cong (0,1) \times T^2$. It is easy to



(a) A surgery link L in $T^2 \times [0,1]$

(b) The front projection of L. The framings are smooth framings.

Figure 6.2

see that $(T^2 \times (0,1), \xi)$ is isotopic to $(T^2 \times (0,1), \tilde{\xi}_{st})$. In particular, the contact planes of $\tilde{\xi}_{st}$ and ξ twist in similar fashions. Thus, for a front projection K of a nullhomologous Legendrian knot that can be contained in a 3-ball, the Thurston-Bennequin number $tb(K)$ and the rotation number $r(K)$ can be defined and computed in the same way for Legendrian knots in $(\mathbb{R}^3, \xi_{st})$. That is, $tb(K) = w(K) - \frac{1}{2}c(K)$ and $r(K) = \frac{1}{2}(c_d(K) - c_u(K))$, where $w(K)$ is the writhe of K, $c(K)$ is the total number of cusps of K, $c_d(K)$ is the number of down cusps of K, and $c_u(K)$ is the number of up cusps of K. Now let $\tilde{\gamma}$ be a Legendrian knot that is smoothly isotopic to γ (and is thus not nullhomologous). As above, the twisting number along $\tilde{\gamma}$ with respect to $\mathcal{F}$, which we denote by $tb(\tilde{\gamma}, \mathcal{F})$, can also be computed using the formula $tb(\tilde{\gamma}, \mathcal{F}) = w(\tilde{\gamma}) - \frac{1}{2}c(\tilde{\gamma})$. For simplicity, we will drop the decoration $\mathcal{F}$ and simply write $tb(\tilde{\gamma})$. Next, since $\frac{\partial}{\partial t} \in \xi$ is a nonvanishing vector field, we can define the rotation number of $\tilde{\gamma}$ with respect to $\frac{\partial}{\partial t}$, denoted by $r_{\partial/\partial t}(\tilde{\gamma})$, to be the signed number of times that the tangent vector field to $\tilde{\gamma}$ rotates in ξ relative to $\frac{\partial}{\partial t}$ as we traverse $\tilde{\gamma}$. For simplicity, we will write $r(\tilde{\gamma}) = r_{\partial/\partial t}(\tilde{\gamma})$. It is once again easy to see that we can compute $r(\tilde{\gamma})$ using the formula $r(\tilde{\gamma}) = \frac{1}{2}(c_d(\tilde{\gamma}) - c_u(\tilde{\gamma}))$. In particular, for the Legendrian knot γ, we have that $tb(\gamma) = 0 = r(\gamma)$.

Now suppose $T^2 \times [0,1]$ is embedded in a closed tight contact 3-manifold (Y, ξ) such that $c_1(\xi) = 0$ and $\xi|_{T^2 \times [0,1]}$ is isotopic to the contact structure above. Further suppose (W, J) is a Stein cobordism from (Y, ξ) to (Y', ξ') obtained by attaching 2-handles $\{h_i\}_{i=1}^n$ along Legendrian knots $\{K_i\}_{i=1}^n$, where each K_i is either contained in a 3-ball or is smoothly isotopic to γ in $T^2 \times (0,1)$. Moverover, suppose these knots have respective smooth framings $\{tb(K_i) - 1\}_{i=1}^n$. Assume we can extend $\frac{\partial}{\partial t}$ to a nonvanishing vector field $v \in \xi$ (which trivializes ξ as a 2-plane bundle). Let $w \in TW|_Y$ be an outward normal vector field to Y. Then the frame (v, Jv, Jw) gives a trivialization τ of TY. Following the arguments of Proposition 2.3 in [35], we prove the following.

Lemma 6.2.1. *$c_1(W, J, \tau)$ can be represented by a cocycle whose value on h_i is equal to $r(K_i)$.*

Proof. By [12], we can thicken Y to a Stein cobordism $Y \times [0, 1]$ from (Y, ξ) to itself. We can extend τ of TY to a complex trivialization of $T(Y \times [0, 1])$ using the inward pointing normal vector field $-\frac{\partial}{\partial s}$ (which agrees with w on Y), where s is the coordinate on $[0, 1]$. To form W, we attach the 2-handles h_i to $Y \times \{1\}$. By definition, $c_1(W, J, \tau)$ measures the failure to extend the trivialization of $T(Y \times [0, 1])$ over h_i for all i. For each i, viewing $h_i \cong D^2 \times D^2 \subset i\mathbb{R}^2 \times \mathbb{R}^2$, we can build a complex trivialization of Th_i. First trivialize $T(D^2 \times 0)|_{\partial D^2}$ by using the tangent vector field a to ∂D^2 and the outward normal vector field b. We can then extend this trivialization to a complex trivialization (a^*, b^*) of Th_i (see [35] for details). Now, when we attach h_i to Y, a is identified with a tangent vector field to K_i and b is identified with $-\frac{\partial}{\partial s}|_{K_i}$. Thus a^* and v both span ξ when restricted to TY and thus together they span a complex line bundle L_1 on $(Y \times I) \cup W$. Moreover, b^* and $-\frac{\partial}{\partial s}$ fit together to span a complementary trivial line bundle L_2. Since $T((Y \times I) \cup W) = L_1 \oplus L_2$, the cochain associated to $c_1(W, J, \tau)$ evaluated on h_i is clearly given by the rotation number of a in ξ relative to $\frac{\partial}{\partial t}$. □

We will use Lemma 6.2.1 in the following context. Suppose 2-handles are attached along a Legendrian link $L = K_1 \sqcup \cdots \sqcup K_n \subset T^2 \times (0, 1) \subset Y$ with respective framings $-a_i$ to obtain Y', where each K_i is either contained in a 3-ball or is smoothly isotopic to γ. Further supposed that there exists a front projection L' of L such that $tb(K_i') \geq -a_i + 1$ for all i. Then for each i, we can stabilize K_i' $(tb(K_i') + a_i - 1)$-times to obtain a Legendrian knot satisfying $tb(K_i') = -a_i + 1$. There are two kinds of stabilizations (i.e with an up cusp or a down cusp), which affect the rotation numbers differently. Thus, for each i, there are $tb(K_i') + a_i - 1$ different stabilizations possible for K_i'. As a quick example, notice that the link in Figure 6.2b has two stabilization possibilities. Now by Lemma 6.2.1, these different kinds of stabilizations yield distinct Stein cobordisms. Moreover, if the hypothesis of Theorem 6.0.6 is satisfied, then the induced contact structures on Y' are nonisotopic.

6.3 Results from convex surface theory

We will assume that the reader is familiar with convex surface theory due to Giroux [30] and we will list some key results about bypass attachments due to Honda [38] which will be used throughout the rest of the paper. For a nice exposition on the basics of convex surface theory, see [28]. First recall that, by Giroux [30], any embedded orientable surface Σ (that is either closed or has Legendrian boundary with nonpositive twisting number) in a contact 3-manifold can be perturbed to be *convex*. This is equivalent to the existence of a collection of curves $\Gamma_\Sigma \subset \Sigma$ called the *dividing set* that satisfies certain properties (see [28]). If T^2 is a convex torus, then by Giroux's criterion (Theorem 3.1 in [38]), Γ_{T^2} consists of (an even number of) parallel dividing curves. Identifying T^2 with $\mathbb{R}^2/\mathbb{Z}^2$, the slope s of the dividing curves is called the *boundary slope*. By Giroux's Flexibility Theorem in [30], T^2 can be further perturbed (relative to Γ_{T^2}) so that the characteristic foliation consists of a 1-parameter family of closed curves called *Legendrian rulings*. Each of these curves has the same slope r, called the *ruling slope*. In this case, each component of $T^2 \backslash \Gamma_{T^2}$ contains a line of singular points of slope s called a *Legendrian divide*. A convex torus that is in this form is said to be in *standard form*.

Theorem 6.3.1 (Flexibility of Legendrian rulings [38])**.** *Assume T^2 is a convex torus in standard form, and, using $\mathbb{R}^2/\mathbb{Z}^2$ coordinates, has boundary slope s and ruling slope r. Then by a C^0-small perturbation near the Legendrian divides, we can modify the ruling slope from $r \neq s$ to any other $r' \neq s$ (including ∞).*

Proposition 6.3.2 ([38])**.** *Assume $T^2 \times I$ has convex boundary in standard form and the boundary slope on $T^2 \times \{i\}$ is s_i for $i = 0, 1$. Then we can find convex tori parallel to $T^2 \times \{0\}$ with any boundary slope s in $[s_1, s_0]$ (including ∞ if $s_0 < s_1$).*

Theorem 6.3.3 (The Farey Tessellation [38])**.** *Assume T is a convex torus in standard form with $\#\Gamma_T = 2$ and boundary slope s. If a bypass is attached along a Legendrian ruling curve of slope $r \neq s$ to the "front" of T, then the resulting convex torus T' will have $\#\Gamma_{T'} = 2$ and its boundary slope s' is obtained from the Farey tessellation as follows. Let $[r, s]$ be the arc on $\partial\mathbb{D}$ (where $\mathbb{D}$ is the disc model of the hyperbolic plane) running from r to s counterclockwise. Then s' is the point in $[r, s]$ closest to r with an edge to s. If the bypass is attached to the "back" of T, then we use the same algorithm except we use the interval $[s, r]$.*

Theorem 6.3.4 (The Imbalance Principle [38])**.** *Suppose Σ and Σ' are two disjoint convex surfaces and let A be a convex annulus whose interior is disjoint from both Σ and Σ' and whose boundary is Legendrian with one component on each surface. If $|\Gamma_\Sigma \cdot \partial A| > |\Gamma_{\Sigma'} \cdot \partial A|$, then by the Giroux Flexibility Theorem [30], there exists a bypass for Σ on A.*

Lemma 6.3.5 (The Edge Rounding Lemma [38])**.** *Let Σ_1 and Σ_2 be convex surfaces with collared Legendrian boundaries which intersect transversely inside an ambient contact manifold along a common boundary Legendrian curve. Assume the neighborhood of the common boundary Legendrian is locally isomorphic to the neighborhood $N_\epsilon = \{x^2 + y^2 \leq \epsilon\}$ of $M = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ with coordinates (x, y, z) and contact 1-form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, for some $n \in \mathbb{Z}^+$, and that $\Sigma_1 \cap N_\epsilon = \{x = 0, 0 \leq y \leq \epsilon\}$ and $\Sigma_2 \cap N_\epsilon = \{y = 0, 0 \leq x \leq \epsilon\}$. If we join Σ_1 and Σ_2 along $x = y = 0$ and round the common edge so that the orientations of Σ_1 and Σ_2 are compatible and induce the same orientation after rounding, the resulting surface is convex, and the dividing curve $z = \frac{k}{2n}$ on Σ_1 will connect to the dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on Σ_2, where $k = 0, 1, \cdots, 2n - 1$.*

We will use these tools in the following context. Let Σ be a pair of pants and consider a contact 3-manifold $S^1 \times \Sigma$. Identify each boundary component of $-\partial(S^1 \times \Sigma)$ with $\mathbb{R}^2/\mathbb{Z}^2$ by setting $(0,1)^T$ as the direction of the S^1-fiber and $(1,0)^T$ as the direction given by $-\partial(\{pt\} \times \Sigma)$. Let T_0 and T_1 be convex tori isotopic to two different boundary components of $S^1 \times \Sigma$ and suppose these tori have boundary slopes $\frac{b}{a}$ and $\frac{t}{s}$, respectively, where $a, s > 0$. Moreover, assume both dividing sets have $2k$ curves. By Theorem 6.3.1, we can arrange that the Legendrian rulings on both tori have infinite slope. Suppose there exists a convex "vertical" annulus A whose boundary components lie on Legendrian rulings of each torus. If $a \neq s$, then by the Imbalance Principle, there exists a bypass along either T_0 or T_1. If $a = s$ then there either exists a bypass along both T_0 and T_1 or there are no bypasses. If there do exist bypasses and $k > 1$, then attaching the bypasses decreases k by 1, but leaves the boundary slope unchanged. If $k = 1$, then attaching the bypasses decreases the boundary slopes as described in Theorem 6.3.3. If there do not exist bypasses, then we may use the Edge Rounding Lemma four times to produce a new torus T made up of T_0, T_1, and two parallel copies of A. Notice that T now contains exactly 2 dividing curves, each of which wraps around T $(kb + kt + 1)$-times in the S^1-direction and ka-times in the $-\partial(\{pt\} \times \Sigma)$-direction. Thus the boundary slope of T is $\frac{kb+kt+1}{ka} = \frac{b}{a} + \frac{t}{a} + \frac{1}{ka}$.

6.4 Proof of Theorem 6.0.4

6.4.1 Decomposing $Y_\pm$

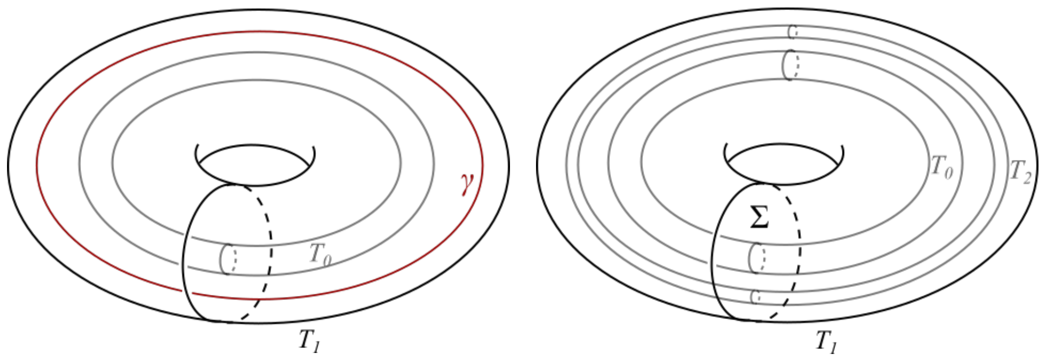
Let $C_\pm$ denote a plumbed 3-manifold obtained as the boundary of a length $n > 1$ cyclic plumbing, $Z_\pm$, as depicted in Figure 6.1a, where $a_i \geq 2$ for all i. Recall from Example 2.2.7, $C_\pm$ is a T^2-bundle over S^1. In particular, if we endow $T^2 \times [0,1] = \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]$ with the coordinates $(\mathbf{x}, t) = (x, y, t)$, then by [54], $C_\pm$ is of the form $T^2 \times [0,1]/(\mathbf{x}, 1) \sim (\pm B\mathbf{x}, 0)$, where

$$B = B(a_1, ..., a_n) = \begin{bmatrix} p & q \\ -p' & -q' \end{bmatrix}, \frac{p}{q} = [a_1, ..., a_n], \text{ and } \frac{p'}{q'} = [a_1, ..., a_{n-1}].$$

Note that since $\det B = 1$, we have $p'q - q'p = 1$.

Let $Y_\pm$ denote the plumbed 3-manifold obtained as the boundary of the plumbing, $X_\pm$, depicted in Figure 6.1b, which has a cycle of length $n > 1$ and where $a_i, z_j \geq 2$ for all i, j and $a_1 \geq 3$. Let $T \subset Y$ be a torus associated with the plumbing operation that plumbs together the $-a_1$- and $-a_n$-framed vertices. Cutting along this torus, we obtain a manifold, $Y'_\pm$ with two torus boundary components, T_0 and T_1. It is easy to see that $Y'_\pm$ is a Seifert fibered space over the annulus with a single singular fiber F, given by the arm with framings $(-z_1, ..., -z_n)$. This structure can be built explicitly using the methods of [56].

Let $T^2 \times [0,1] = \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]$ have the coordinates (x, y, t), let $T_0 = T^2 \times \{0\}$ and $T_1 = -T^2 \times \{1\}$, and identify T_i with $\mathbb{R}^2/\mathbb{Z}^2$ by $\partial_y = (1,0)^T = \mu$ and $\partial_x =$



(a) The surgery curve γ in $T^2 \times [0,1]$ (b) $Y'_{\pm} - V \cong S^1 \times \Sigma$

Figure 6.3

$(0,1)^T = \lambda$. Note that the orientation we are using on T_1 is opposite the orientation used above (this convention will be useful later when we apply Honda's classifications of tight contact structures). Then $Y'_{\pm}$ can be obtained by starting with $T^2 \times [0,1]$ and performing $-\frac{r}{s} = [-z_1, ..., -z_m]$-surgery along a curve $S^1 \times \{pt\} \times \{pt\} \subset T^2 \times (0,1)$ (See Figure 6.3a). The framing is defined with respect to the S^1-fiber, λ. The core of the solid torus obtained after surgery along this unknot is the singular fiber F. Let V be a tubular neighborhood of F. Then $Y'_{\pm} - V \cong S^1 \times \Sigma$, where Σ is a pair of pants (See Figure 6.3b). Identify ∂V with $\mathbb{R}^2/\mathbb{Z}^2$ by choosing $(1,0)^T$ as the meridional direction and $(0,1)^T$ as the longitudinal direction and let T_2 denote the boundary component of $-\partial(S^1 \times \Sigma)$ that is glued to ∂V. Identify T_i with $\mathbb{R}^2/\mathbb{Z}^2$ by setting $(1,0)^T = \mu$ as the direction given by $-\partial(\{pt\} \times \Sigma)$ and $(0,1)^T = \lambda$ as the direction given by the S^1-fiber. Note that this identification agrees with the identification we made for T_0 and T_1 above. With this identification, the gluing map $A : T_1 \to T_0$ is now given by

$$A = \begin{bmatrix} p & -q \\ -p' & q' \end{bmatrix}$$

where $\frac{p}{q} = [a_1, ..., a_n]$ and $\frac{p'}{q'} = [a_1, ..., a_{n-1}]$. Moreover, the gluing map $g : \partial V \to -\partial(S^1 \times \Sigma)$ is given by

$$g = \begin{bmatrix} r & r' \\ -s & -s' \end{bmatrix}$$

where $\frac{r'}{s'} = [z_1, ..., z_{m-1}]$. In particular, $\det(g) = r's - s'r = 1$. With these conventions set up, we have $-\partial(Y'_{\pm} - V) = T_0 \cup T_1 \cup T_2$.

6.4.2 The upper bound

Let $Y = Y_{\pm}$. We will distinguish between these two cases when necessary. Let ξ be a tight contact structure on Y with no Giroux torsion. Using the notation from Section

6.4.1, let T be an incompressible convex torus that we can cut along to obtain Y' and let Γ_T denote the dividing set. After cutting along T, let Γ_{T_i} denote the image of the dividing set on T_i for $i = 0, 1$. With the coordinates described in Section 6.4.1, let $\Gamma_{T_0} = a\mu + b\lambda$, where $(a, b) = 1$. Since T_0 and T_1 are identified by the map $\pm A$, the dividing set on T_1 is of the form $\Gamma_{T_1} = -(bq + aq')\mu - (bp + ap')\lambda$ for Y_+ and $\Gamma_{T_1} = (bq + aq')\mu + (bp + ap')\lambda$ for Y_-. Now isotope the singular fiber F so that it is Legendrian and has very negative twisting number $-m << 0$, relative to a fixed framing. Then we may take V to be a standard tubular neighborhood of F with convex boundary so that the slope of the dividing set is $-\frac{1}{m}$ and $\#\Gamma_{\partial V} = 2$ (See section 2.3.2 of [28]). Thus the dividing set on $T_2 \subset -\partial(S^1 \times \Sigma)$ is of the form $\Gamma_{T_2} = (-mr + r')\mu - (-ms + s')\lambda$ and $\#\Gamma_{T_2} = 2$.

The slopes of these three dividing curves are as follows:

$$s(\Gamma_{T_0}) = \frac{b}{a} \qquad s(\Gamma_{T_1}) = \frac{bp + ap'}{bq + aq'} \qquad s(\Gamma_{T_2}) = -\frac{ms - s'}{mr - r'}$$

Notice, for all relatively prime a and b, $\frac{bp+ap'}{bq+aq'}$ is a reduced fraction, since $(\alpha q - \beta q')(bp+ap')+(\beta p' - \alpha p)(bq+aq') = 1$, where α, β are integers such that $\alpha a+\beta b = 1$. Furthermore, $-1 \leq s(\Gamma_{T_2}) < 0$. We can view A^{-1} as a real-valued function that maps the slopes on T_0 to the slopes on T_1 given by $f(x) = \frac{xp+p'}{xq+q'}$. Since f is a decreasing function of each interval of its domain, we have the relationship between slopes on T_0 and T_1 shown in Figure 6.4.

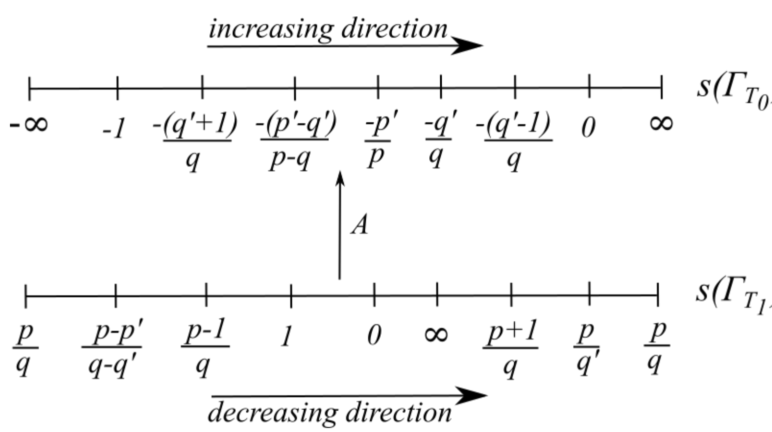


Figure 6.4: Relationship between slopes on T_0 and T_1 via the gluing map

By the flexibility of Legendrian rulings (Theorem 6.3.1), we may arrange so that the Legendrian rulings on each torus has slope ∞ as long as the dividing sets do not have infinite slope. A convex annulus connecting two tori along such Legendrian rulings is called a *vertical annulus*. Whenever possible, we will assume that the Legendrian rulings have infinite slope. Throughout this section, we assume that all tori and annuli are convex.

We have the following three cases:

- $|a| < |bq + aq'|$ if and only if $-\infty < s(\Gamma_{T_0}) < -\frac{q'+1}{q}$ or $-\frac{q'-1}{q} < s(\Gamma_{T_0}) < \infty$;
- $|a| > |bq + aq'|$ if and only if $-\frac{q'+1}{q} < s(\Gamma_{T_0}) < -\frac{q'-1}{q}$; and
- $|a| = |bq + aq'|$ if and only if $s(\Gamma_{T_0}) = -\frac{q'\pm 1}{q}$.

Let $2k$ be the number of dividing curves on T_0 and T_1. If $a \neq 0$ and $bq + aq' \neq 0$, then take a vertical annulus A between T_0 and T_1. Then $|\Gamma_{T_0} \cdot \partial A| = |2ka|$ and $|\Gamma_{T_1} \cdot \partial A| = |2k(bq + aq')|$. By the Imbalance Principle (Theorem 6.3.4), if we are in the first case, then there exists a bypass along T_1. If we are in the second case, then there exists a bypass along T_2. If we are in the third case, then there are either bypasses along both tori or there are no bypasses. If $a = 0$ (or $bq + aq' = 0$), then we can take an annulus between a Legendrian divide of T_0 (or T_1) and a Legendrian ruling of T_1 (or T_0) and use the Imbalance Principle to see that there is a bypass along T_0 (or T_1). We will explore these cases in the following two propositions.

Proposition 6.4.1. *If $a_i, z_j \geq 2$ for all i, j and $a_1 \geq 3$, then we can choose T so that $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$.*

Proof. Suppose $a \neq 0$, $bq + aq' \neq 0$, and $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2k$ for some $k > 0$. Take a vertical annulus A between T_0 and T_1. If $|a| \neq |bq + aq'|$, then by the Imbalance Principle (Theorem 6.3.4) there exists a bypass along a Legendrian divide of either T_0 or T_1 on A. Without loss of generality, assume $|bq + aq'| > |a|$. Then we may attach a bypass to T_0, giving us a new torus T_0' isotopic to T_0 such that $s(\Gamma_{T_0'}) = s(\Gamma_{T_0})$ and $|\Gamma_{T_0'} \cdot \partial A| = |2(k-1)a|$. Thus, there exists an incompressible torus T' isotopic to T in Y such that if we cut along T' to obtain Y', the new boundary tori T_0' and T_1' have the same boundary slopes as T_0 and T_1, but with two fewer dividing curves. Continuing this recutting process, we are able to arrange that $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$.

If $|a| = |bq + aq'|$, then $s(\Gamma_{T_0}) = \frac{b}{a} = -\frac{q'\pm 1}{q}$ and so $s(\Gamma_{T_1}) = \frac{p\pm 1}{q}$ (Note that these fractions may not be reduced, but their reduced fractions will still have the same denominators by Lemma 6.6.2 in Section 6.6). Assume the fractions are reduced. Take a vertical annulus between T_0 and T_1. If there exist bypasses along T_0 and T_1, then we can attach the bypasses to lower k and recut Y along one of these new tori. If we can continue this until $k = 1$, then we are done. Suppose there exists a $k > 1$ such that there are no more bypasses. Then we may use the Edge Rounding Lemma (Lemma 6.3.5) to obtain a torus parallel to $-T_2$ with two dividing curves of slope $\frac{p-q'-2}{q} + \frac{1}{kq} > \frac{p-q'-2}{q}$ (if $\frac{b}{a} = -\frac{q'+1}{q}$) or $\frac{p-q'+2}{q} + \frac{1}{kq} > \frac{p-q'+2}{q}$ (if $\frac{b}{a} = -\frac{q'-1}{q}$). By Lemma 6.6.1 in Section 6.6, both of these slopes are greater than 1 since $a_1 \geq 3$. Thus, there is a torus, T_2', parallel to T_2 with slope less than -1. Since $s(\Gamma_{T_2}) > -1$, by Proposition 6.3.2, we can find a torus "between" T_2' and T_2 with boundary slope -1 and two dividing curves. With abuse of notation, call this new torus T_2. Now take a vertical annulus A between T_0 and T_2. Then $|\Gamma_{T_2} \cdot \partial A| = 2$ and $|\Gamma_{T_0} \cdot \partial A| = |2kq|$. Thus by the Imbalance Principle, we may add bypasses to T_0 and lower k until it is equal to 1. Recut Y along this new torus to obtain the result. If $-\frac{q'\pm1}{q}$ is not reduced, then the same argument holds, since after edge rounding, we will obtain a torus of slope even greater than $\frac{p-q'-2}{q}+\frac{1}{kq}$ or $\frac{p-q'+2}{q}+\frac{1}{kq}$.

If $a=0$, then $s(\Gamma_{T_0})=\infty$ and $s(\Gamma_{T_1})=\frac{p}{q}$. Take a vertical annulus A from a Legendrian divide of T_0 to a Legendrian ruling of T_1. Then $|\Gamma_{T_0}\cdot\partial A|=0$ and $|\Gamma_{T_1}\cdot\partial A|=|2kq|$. We can thus add bypasses along T_1 until $k=1$. Recutting along this new torus, we obtain the result. We can similarly obtain the result if $bq+aq'=0$. □

Proposition 6.4.2. *If $a_i, z_j \geq 2$ for all i, j and $a_1 \geq 3$, then we can choose T and V so that $s(\Gamma_{T_0})=-1$, $s(\Gamma_{T_1})=\frac{p-p'}{q-q'}$, and $s(\Gamma_{T_2})=-1$.*

Proof. First note that if we are able to arrange that either $s(\Gamma_{T_0})=-1$ or $s(\Gamma_{T_1})=\frac{p-p'}{q-q'}$, then we can easily obtain the result. Indeed, if we find a torus T_0' parallel to T_0 with $s(\Gamma_{T_0'})=-1$, then we can recut Y to obtain $s(\Gamma_{T_1})=\frac{p-p'}{q-q'}$ (or vice versa). We can then take a vertical annulus between T_0 and T_2 and, by the Imbalance Principle, add bypasses and use the Farey tessellation (Theorem 6.3.3) to decrease $s(\Gamma_{T_2})$ to -1.

First suppose $a=0$, so that $s(\Gamma_{T_0})=\infty$ and $s(\Gamma_{T_1})=\frac{p}{q}$. Take an annulus from a Legendrian divide of T_0 to a Legendrian ruling of T_1. Then we can add bypasses to T_1 to get a torus T_1' with $s(\Gamma_{T_1'})=1$. Thus, by Proposition 6.3.2, there exists a torus between T_1 and T_1' with slope $\frac{p-p'}{q-q'}$. We obtain a similar result if $bq+aq'=0$. We now assume $a\neq0$ and $bq+aq'\neq0$.

Suppose $-1<s(\Gamma_{T_0})\leq-\frac{q'+1}{q}$ and $\frac{p-1}{q}\leq s(\Gamma_{T_1})<\frac{p-p'}{q-q'}$. Take a vertical annulus A between T_0 and T_2. Suppose there exists a bypass on A for either T_0 or T_2, or both. Then attach the bypasses, lowering the boundary slopes, and repeat the process. If we eventually reach $s(\Gamma_{T_0})=-1$, then we are done. Suppose we reach a step in which there are no more bypasses. Then since $-1<s(\Gamma_{T_0}),s(\Gamma_{T_2})<0$, we can use the Edge Rounding Lemma to find a torus $-T_1'$ parallel to $-T_1$ with boundary slope greater than -2. Thus $s(\Gamma_{T_1'})<2$. By Lemma 6.6.1 in Section 6.6, $s(\Gamma_{T_1})\geq\frac{p-1}{q}\geq2$. Thus, by Proposition 6.3.2, there exists another torus parallel to T_1 with slope 2. With abuse of notation, call this new torus T_1. Now take a vertical annulus between T_1 and T_0 and use the Imbalance Principle to add bypasses to T_0 until $s(\Gamma_{T_0})=-1$.

Next suppose $-\frac{q'+1}{q}<s(\Gamma_{T_0})<-\frac{q'-1}{q}$ (and $s(\Gamma_{T_1})>\frac{p-1}{q}$ or $s(\Gamma_{T_1})<\frac{p+1}{q}$). Then $|a|>|bq+aq'|$ and so if we take a vertical annulus between T_0 and T_1, we can add a bypass to T_0, increasing its boundary slope using the Farey tessellation (Theorem 6.3.3), recut, and repeat. Since 0 and -1 share an edge in the Farey tessellation, we will eventually obtain $-1\leq s(\Gamma_{T_0})\leq-\frac{q'+1}{q}$, which is handled above.

Now suppose $-\infty<s(\Gamma_{T_0})<-1$ or $-\frac{q'-1}{q}<s(\Gamma_{T_0})<\infty$. Then $|a|<|bq+aq'|$. Taking a vertical annulus between T_0 and T_1, by the Imbalance Principle, we can add a bypass to T_1, recut, and repeat. Now, since $\frac{p-p'}{q-q'}<s(\Gamma_{T_1})<\frac{p+1}{q}$, by adding bypasses, recutting, and repeating, we eventually obtain $1 \leq s(\Gamma_{T_1}) \leq \frac{p-p'}{q-q'}$ (and $-1 < s(\Gamma_{T_0}) \leq -\frac{p'-q'}{p-q}$), which is handled above.

Finally suppose $s(\Gamma_{T_0}) = -\frac{q'-1}{q}$ and $s(\Gamma_{T_1}) = \frac{p+1}{q}$. Take a vertical annulus between T_0 and T_1. Then there either exists bypasses along both tori or along neither, since by Lemma 6.6.2 in Section 6.6, these slopes have the same denominator. If there do exist bypasses, we may add a bypass to T_0 to decrease its slope. Recut along this new torus to obtain the case $-\frac{q'+1}{q} < s(\Gamma_{T_0}) < -\frac{q'-1}{q}$, which is handled above. If there do not exist bypasses, then as in the proof of Proposition 6.4.1, we can use the Edge Rounding Lemma and Proposition 6.3.2 to obtain $s(\Gamma_{T_2}) = -1$. Now, take a vertical annulus between T_0 and T_2 and use the Imbalance Principle to add bypasses to T_0 until $s(\Gamma_{T_0}) = -1$. □

Remark 6.4.3. In this proof we started with $s(\Gamma_{T_2}) = -\frac{ms-s'}{mr-r'}$, for $m >> 0$, and ended up with $s(\Gamma_{T_2}) = -1$ after attaching bypasses. Thus, by Proposition 6.3.2, there is a convex torus T_2' isotopic to T_2 with boundary slope $-\frac{s-s'}{r-r'}$. Equivalently, viewed from V, $s(\Gamma_{T_2'}) = -1$ and $s(\Gamma_{T_2}) = -\frac{r-s}{r'-s'}$. Thus, V contains a toric annulus $T_2 \times [1,2]$ such that $s(\Gamma_{T_2\times\{1\}}) = -\frac{r-s}{r'-s'}$ and $s(\Gamma_{T_2\times\{2\}}) = -1$. This fact will be used in the proof of Proposition 6.4.7.

The following propositions consider *basic slices* contained in $T^2 \times I$ and related notions. See Section 4.3 in [38] or Section 2.3 in [28] for relevant definitions and results involving basic slices.

Proposition 6.4.4. *There are no vertical Legendrian curves with twisting number 0 in $Y_+' - V$.*

Proof. By Proposition 6.4.2, we can assume that $s(\Gamma_{T_0}) = -1$, $s(\Gamma_{T_1}) = \frac{p-p'}{q-q'}$, and $s(\Gamma_{T_2}) = -1$. Suppose there is a vertical Legendrian curve γ with twisting number 0. Take vertical annuli from γ to T_i for all i. Then γ does not intersect Γ_A and so we may use the Imbalance Principle to add bypasses to each torus until $s(T_i) = \infty$ for all i. We will apply Lemma 4.13 in [28] to produce overtwisted disks, contradicting tightness.

There are three copies of $T^2 \times I$ embedded in $Y_+' - V = S^1 \times \Sigma$, namely $T_i \times I$, where $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ has slope ∞ for all i. The toric annulus $T_1 \times I$ has $k \geq 2$ basic slices, $T_1 \times [0, \frac{1}{k}], ..., T_1 \times [\frac{k-1}{k}, 1]$, while $T_0 \times I$ and $T_2 \times I$ both have a single basic slice. Furthermore, note that since $s(\Gamma_{T_1}) = \frac{p-p'}{q-q'} > \frac{p-1}{q} \geq 2$ (by Lemma 6.6.1), $T_1 \times \{\frac{k-3}{k}\}$ has boundary slope 2, $T_1 \times \{\frac{k-2}{k}\}$ has boundary slope 1, and $T_0 \times \{\frac{k-1}{k}\}$ has boundary slope 0.

First, we show that $T_1 \times [\frac{k-1}{k}, 1]$ and $T_0 \times I$ must have the same sign (after choosing the sign convention to be given by selecting $(0,1)^T$ as the vector associated to $T_i \times \{1\}$ for all i). Suppose that $T_1 \times [\frac{k-1}{k}, 1]$ has sign ϵ_1 and $T_0 \times I$ has sign ϵ_0. Assume $T_2 \times [0,1]$ also has sign ϵ_1. The case in which $T_2 \times [0,1]$ has sign ϵ_0 is analogous. By Honda's

Gluing Theorem (Theorem 4.25 in [38]), $T_1 \times [\frac{k-2}{k}, \frac{k-1}{k}]$ must also have sign ϵ_1. By Honda's Shuffling Lemma (Lemma 4.14 in [38]), we may assume that $T_1 \times [\frac{k-3}{k}, \frac{k-2}{k}]$ also has sign ϵ_1. By Lemma 4.13 in [28], there exists a vertical annulus between $T_1 \times \{\frac{k-2}{k}\}$ and T_2 with no boundary parallel dividing curves. Thus, we can cut along this vertical annulus and use the Edge Rounding Lemma to find a torus T_0' parallel to $T_0 \times \{1\}$ with boundary slope -1. We now have a toric annulus $T_0 \times [0, 2]$ with boundary slopes $s(\Gamma_{T\times\{i\}}) = -1$ (for $i = 0, 2$) and I-twisting equal to π.

Recut Y along $T_1 \times \{\frac{k-2}{k}\}$ and thicken $T_0 \times [0, 2]$ to a toric annulus $T_0 \times [-1, 2]$, where $s(\Gamma_{T_0\times\{-1\}}) = -\frac{p'-q'}{p-q}$ and $T_0 \times [-1, -\frac{1}{2}]$ is a basic slice with sign $-\epsilon_1 = \epsilon_0$ (i.e. $T_0 \times [-1, -\frac{1}{2}]$ is the image of $T_1 \times [\frac{k-3}{k}, \frac{k-2}{k}]$ under the recutting). Moreover, $T_0 \times [-1, 2]$ has I-twisting greater than π and so $T_0 \times [-1, 2]$ admits exactly two tight contact structures (see Section 5.2 in [38]). Based on the definitions of these two contact structures, the signs of the basic slices $T_0 \times [-1, -\frac{1}{2}]$ and $T_0 \times [1, 2]$ must be different. Thus $\epsilon_0 = \epsilon_1$.

Since the basic slices of $T_1 \times [\frac{k-2}{k}, 1]$ and $T_0 \times I$ all have the same sign, by Lemma 4.13 in [28], there exists a vertical annulus between $T_1 \times \{\frac{k-2}{k}\}$ and $T_0 \times \{0\}$ that has no boundary-parallel dividing curves. Thus, by the Edge Rounding Lemma, we can obtain a torus, T_2' parallel to $T_2 \times \{1\}$ of slope -1. Thus, there must exist a torus T_2'' between $T_2 \times \{1\}$ and T_2' with boundary slope $-\frac{s'}{r'}$. On ∂V, the slope of this dividing set is 0. Any contact structure on $V \cong S^1 \times D^2$ with boundary slope 0 contains an overtwisted disk. □

Remark 6.4.5. The proof above shows that it is possible for $Y_-' - V$ to have vertical Legendrian curves with twisting number 0. In this case, the gluing theorem is valid for the case $\epsilon_0 \neq \epsilon_1$.

Proposition 6.4.6. *If $a_i, z_j \geq 2$ for all i, j and $a_1 \geq 3$, then Y_+ admits at most $(a_1 - 1) \cdots (a_n - 1)(z_1 - 1) \cdots (z_m - 1)$ tight contact structures with no Giroux torsion.*

Proof. For convenience, let $Y = Y_+$. By Proposition 6.4.2, we may assume $s(\Gamma_{T_0}) = -1$, $s(\Gamma_{T_1}) = \frac{p-p'}{q-q'}$, and $s(\Gamma_{T_2}) = -1$. Take a vertical annulus A between T_0 and T_2. If there exists boundary parallel dividing curves on A, then we can add bypasses and obtain a torus parallel to T_2 with infinite slope, which contradicts Proposition 6.4.4. Thus, there do not exist boundary parallel dividing curves. Cutting along A and edge rounding, we obtain a torus T_1' parallel to T_1 with boundary slope 1. Moreover, the toric annulus between T_1 and T_1' must have minimal I-twisting, since otherwise there would exist a torus with infinite boundary slope, contradicting Proposition 6.4.4.

Let $S^1 \times \Sigma' \subset S^1 \times \Sigma = Y' - V$ have boundary $-T_0 \cup -T_1' \cup -T_2$. Then $S^1 \times \Sigma = (S^1 \times \Sigma') \cup (T^2 \times [0, 1])$, where $T^2 \times \{0\} = T_1$ and $T^2 \times \{1\} = T_1'$. To find an upper bound on the number of tight contact structures with no Giroux torsion on Y' we need only find the number of such structures on the pieces $S^1 \times \Sigma, T^2 \times I$, and V.

Since $s(\Gamma_{T_2}) = -1$, we have that $s(\Gamma_{\partial V}) = -\frac{r-s}{r'-s'} = -[z_m, ..., z_1 - 1]$. The proof of the latter equality can be found in Section 6.6 (Lemma 6.6.4). Thus by Theorem

2.3 in [38], V admits $(z_1-1)\cdots(z_m-1)$ tight contact structures. Changing the coordinates on $T^2\times[0,1]$ by reversing the orientation on T^2, we obtain $s(\Gamma_{T_1}) = -\frac{p-p'}{q-q'} = -[a_1,...,a_n-1]$ and $s(\Gamma_{T_1'}) = -1$. The proof of the former equality can also be found in Section 6.6 (Lemma 6.6.3). By Theorem 2.2 in [38], there are $(a_1-1)\cdots(a_n-1)$ tight contact structures with no Giroux torsion on $T^2\times[0,1]$. By Proposition 6.4.4, $S^1\times\Sigma'$ has no vertical Legendrian curves of twisting number 0 and so by Lemma 5.1-4c in [39], $S^1\times\Sigma$ admits $2-(1-1+1)=1$ tight contact structure.

Therefore, Y' admits at most $1\cdot(a_1-1)\cdots(a_n-1)(z_1-1)\cdots(z_m-1)$ tight contact structures with no Giroux torsion. Gluing the ends of Y' together via A to obtain Y, we have that Y also admits at most $(a_1-1)\cdots(a_n-1)(z_1-1)\cdots(z_m-1)$ tight contact structures with no Giroux torsion. □

Proposition 6.4.7. *If $a_i, z_j \geq 2$ for all i,j and $a_1 \geq 3$, then Y_- admits at most $(a_1-1)\cdots(a_n-1)(z_1-1)\cdots(z_m-1)+z_1(z_2-1)\cdots(z_m-1)$ tight contact structures with no Giroux torsion.*

Proof. Let $Y=Y_-$. Once again, assume $s(\Gamma_{T_0})=-1$, $s(\Gamma_{T_1})=\frac{p-p'}{q-q'}$, and $s(\Gamma_{T_2})=-1$. Using the notation and arguments of Proposition 6.4.6, if there does not exist a vertical Legendrian curve with twisting number 0 in $S^1\times\Sigma'$, then Y admits $(a_1-1)\cdots(a_n-1)(z_1-1)\cdots(z_m-1)$ tight contact structures with no Giroux torsion.

From now on, assume $S^1\times\Sigma'$ contains a vertical Legendrian curve with twisting number 0. Then as in the proof of Proposition 6.4.4, after recutting there exists an embedded toric annulus $T_0\times[-1,2]\subset S^1\times\Sigma$ such that $s(\Gamma_{T_0\times\{-1\}})=-\frac{p'-q'}{p-q}$, $s(\Gamma_{T_0\times\{0\}})=-1$, $s(\Gamma_{T^2\times\{1\}})=\infty$, $s(\Gamma_{T^2\times\{2\}})=-1$, and $T^2\times[-1,2]$ has I-twisting greater than π (and less then 2π). See Figure 6.5a. By Section 5.2 in [38], there are two tight contact structures on $T_0\times[-1,2]$. Decompose $S^1\times\Sigma$ into $S^1\times\Sigma=(S^1\times\Sigma'')\cup(T_0\times[-1,2])$. Notice that $S^1\times\Sigma''$ does not contain a vertical Legendrian with twisting number 0, since otherwise, by repeating the above process, we would obtain a toric annulus with I-twisting equal to 2π. Thus by Lemma 5.1-4c in [39], $S^1\times\Sigma''$ admits one tight contact structure. Therefore, Y admits at most $2(z_1-1)\cdots(z_m-1)$ tight contact structures with no Giroux torsion. If $z_1=2$, then this number is the same as $z_1(z_2-1)\cdots(z_m-1)$ and we are done. Assume $z_1>2$.

Since $s(\Gamma_{T_2})=-1$, we have $s(\Gamma_{\partial V})=-\frac{r-s}{r'-s'}=-[z_m,...,z_2,z_1-1]$. Let $T_2'\subset S^1\times\Sigma$ be a torus isotopic to T_2 such that $s(\Gamma_{T_2'})=\infty$ and let V' be the corresponding thickening of V. Then $s(\Gamma_{\partial V'})=-\frac{r}{r'}=-\frac{r}{\overline{s}}=-[z_m,...,z_1]$, where $\overline{s}$ is the unique integer such that $1<\overline{s}<r$ and $s\overline{s}\equiv 1 \mathrm{mod} r$. By [38], V and V' admit $(z_1-1)\cdots(z_m-1)$ and $z_1(z_2-1)\cdots(z_m-1)$ tight contact structures, respectively.

Let $B=T_2\times[0,1]\subset S^1\times\Sigma$ be the basic slice satisfying $T_2\times\{0\}=T_2$ and $T_2\times\{1\}=T_2'$ (i.e. $B=V'-V$). Then B admits 2 tight contact structures, which depend on the sign of the relative Euler class. We claim that exactly $(z_1-2)(z_2-1)\cdots(z_m-1)$ of the $z_1(z_2-1)\cdots(z_m-1)$ tight contact structures of V' have the property that the sign of the basic slice B can be either positive or negative after shuffling (Lemma 4.14 of [38]). Thicken B to the toric annulus $T_2 \times [0,2] = B \cup T_2 \times [1,2] \subset V'$, which has boundary slopes $s(\Gamma_{T_2\times\{0\}}) = -\frac{r}{r'}$, $s(\Gamma_{T_2\times\{1\}}) = -\frac{r-s}{r'-s'}$, and $s(\Gamma_{T_2\times\{2\}}) = -1$ (this thickening exists by Remark 6.4.3). Consider the first *continued fraction block* (see Section 4.4.5 in [38]) of $T_2 \times [0,2]$, which is a toric annulus $T_2 \times [0, \frac{3}{2}]$ satisfying $s(\Gamma_{T_2\times\{\frac{3}{2}\}}) = -\frac{r-(z_1-1)s}{r'-(z_1-1)s'}$. This block admits z_1 tight contact structures, of which only two do not have the desired property, namely the two contact structures whose basic slices all have the same sign. Thus $T_2 \times [0, \frac{3}{2}]$ admits $z_1 - 2$ contact structures that satisfy the desired property. Moreover, $T_2 \times [\frac{3}{2}, 2]$ admits $(z_2-1)\cdots(z_m-1)$ nonisotopic tight contact structures which remain nonisotopic in $T_2 \times [0,2]$. Thus, there are $(z_1-2)(z_2-1)\cdots(z_m-1)$ tight contact structures on $T_2 \times [0,2]$ (and thus on V') such that the sign of the basic slice B can be either positive or negative after shuffling.

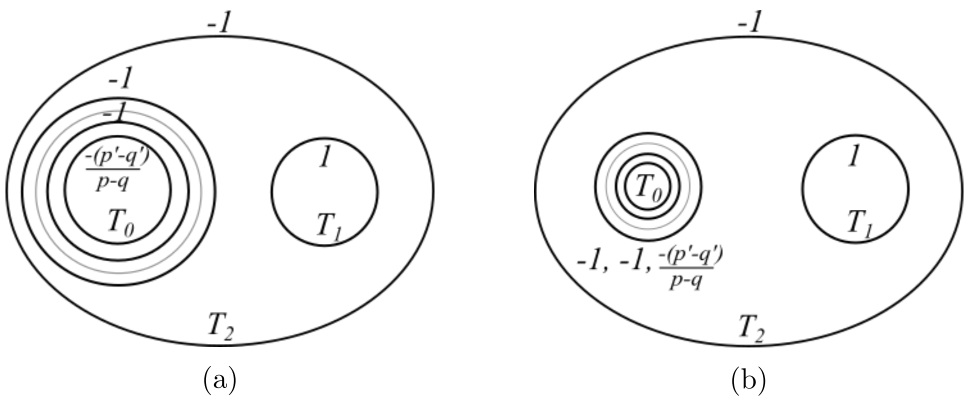


Figure 6.5: Consider a section of $S^1 \times \Sigma$, which is a pair of pants. The numbers represent the slopes of the dividing curves of the corresponding tori. The gray circles represent tori with boundary slope ∞. In (b), the slopes below T_0 correspond to the black circles from outer- to inner-most.

Finally we claim that the corresponding $2(z_1-2)(z_2-1)\cdots(z_m-1)$ tight contact structures on Y pair off isotopically, yielding a total of $2(z_1-1)\cdots(z_m-1)-(z_1-2)(z_2-1)\cdots(z_m-1) = z_1(z_2-1)\cdots(z_m-1)$ tight contact structures on Y. Fix one such contact structure ξ on Y. Reset the decomposition of Y so that $s(\Gamma_{T_0}) = -\frac{p'-q'}{p-q}$, $s(\Gamma_{T_1}) = 1$, $s(\Gamma_{T_2}) = -1$, and there exist toric annuli $T_i \times [0,1]$ such that $T_i \times \{0\} = T_i$ and $s(\Gamma_{T_i\times\{1\}}) = \infty$ for all i, as in the proof of Proposition 6.4.4. By Remark 6.4.5, using the notation of Proposition 6.4.4, the signs of $T_0 \times [0,1]$ and $T_1 \times [\frac{k-2}{k}, 1]$ are different. Choose the sign of B to be the same as the sign of $T_1 \times [\frac{k-2}{k}, 1]$. Then following to proof of Proposition 6.4.4, there is a toric annulus $T_0 \times [-1,2]$ with I-twisting greater than π such that $s(\Gamma_{T_0\times\{-1\}}) = -\frac{p'-q'}{p-q}$, $s(\Gamma_{T_0\times\{0\}}) = -1$, $s(\Gamma_{T_0\times\{1\}}) = \infty$, and $s(\Gamma_{T_0\times\{2\}}) = -1$. See Figure 6.5a. Thus $\xi|_{T_0\times[-1,2]}$ is one of two possible tight contact structures.

By edge rounding and recutting, we will be able to isotope ξ so that $\xi|_{T_0\times[-1,2]}$ becomes the other contact structure, which we denote by $\xi'|_{T_0\times[-1,2]}$. To do this, ignore the basic slice $T_0 \times [1,2]$, take vertical annuli from $T_0 \times \{1\}$ to T_1 and from $T_0 \times \{1\}$ to T_2, and add bypasses to T_1 and T_2 to obtain toric annuli $T_i \times [0,1]$ such that $T_i \times \{0\} = T_i$ and $s(\Gamma_{T_i\times\{1\}}) = \infty$ for $i = 1, 2$. Notice that $T_2 \times [0,1] = B$. Now, by shuffling, choose the opposite sign for B than we chose previously. Then by applying Lemma 4.13 in [28], we can edge round (similar to the proof of Proposition 6.4.4) and thicken the toric annulus $T_1 \times [0,1]$ to $T_1 \times [0,2]$, where $s(\Gamma_{T_1\times\{2\}}) = 1$ and $T_1 \times [0,2]$ has I-twisting equal to π. As previously, recut Y along $T_1 \times \{2\}$, relabel the toric annulus as $T_0 \times [-3,-1]$, and glue it to $T_0 \times [-1,0]$. See Figure 6.5b. Based on the definition of the two contact structures in question (see Section 5.2 in [38]), it is easy to see that $\xi|_{T_0\times[-3,-1]} = \xi'|_{T_0\times[-1,2]}$. □

6.4.3 The lower bound

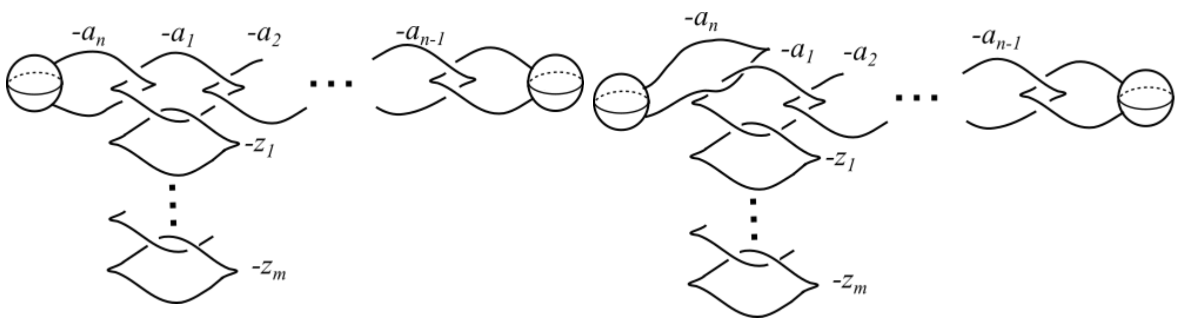
Proposition 6.4.8. *If $a_i, z_j \geq 2$ for all i, j and $a_1 \geq 3$, then Y_+ admits exactly $(a_1 - 1)\cdots(a_n - 1)(z_1 - 1)\cdots(z_m - 1)$ Stein fillable contact structures.*

Proof. We can easily construct $(a_1 - 1)\cdots(a_n - 1)(z_1 - 1)\cdots(z_m - 1)$ Stein fillable contact structures for Y_+, by drawing suitable handlebody diagrams. Start with the obvious handlebody diagram of the plumbing X_+ and make every unknot Legendrian with $tb = -1$ and $r = 0$, as in Figure 6.6a. Then to ensure we obtain a Stein structure, we must stabilize each $-a_i$-framed unknot $a_i - 2$ times and each $-z_i$-framed unknot $z_i - 2$ times. There are $a_i - 1$ (resp. $z_i - 1$) ways to stabilize each unknot. Thus, there are a total of $(a_1 - 1)\cdots(a_n - 1)(z_1 - 1)\cdots(z_m - 1)$ ways to stabilize the entire diagram. Since different kinds of stabilizations yield different rotation numbers, the resulting Stein structures have different first Chern classes and so the induced contact structures on the boundary are pairwise nonisotopic (by Theorem 1.2 in [48]). Thus there are at least $(a_1-1)\cdots(a_n-1)(z_1-1)\cdots(z_m-1)$ Stein fillable contact structures on Y_+. Coupling this with Proposition 6.4.6, we obtain the result. □

Proposition 6.4.9. *If $a_i, z_j \geq 2$ for all i, j and $a_1 \geq 3$, then Y_- admits exactly $(a_1-1)\cdots(a_n-1)(z_1-1)\cdots(z_m-1)+z_1(z_2-1)\cdots(z_m-1)$ tight contact structures with no Giroux torsion, $(a_1-1)\cdots(a_n-1)(z_1-1)\cdots(z_m-1)$ of which are Stein fillable. If $(a_1, ..., a_n)$ is embeddable, then all of these contact structures are Stein fillable.*

Proof. Let $Y = Y_-$. As seen in the proof of Proposition 6.4.7, any tight contact structure ξ on Y satisfies one of two disjoint properties. $\xi|_{S^1\times\Sigma}$ either contains or does not contain vertical Legendrian curves with twisting number 0. Notice that $\xi|_{S^1\times\Sigma}$ contains a vertical Legendrian curve with twisting number 0 if and only if there is an embedded toric annulus with I-twisting equal to π.

As in the proof of Proposition 6.4.8, we can easily exhibit $(a_1 - 1)\cdots(a_n - 1)(z_1 - 1)\cdots(z_m - 1)$ pairwise nonisotopic contact structures as the boundaries of



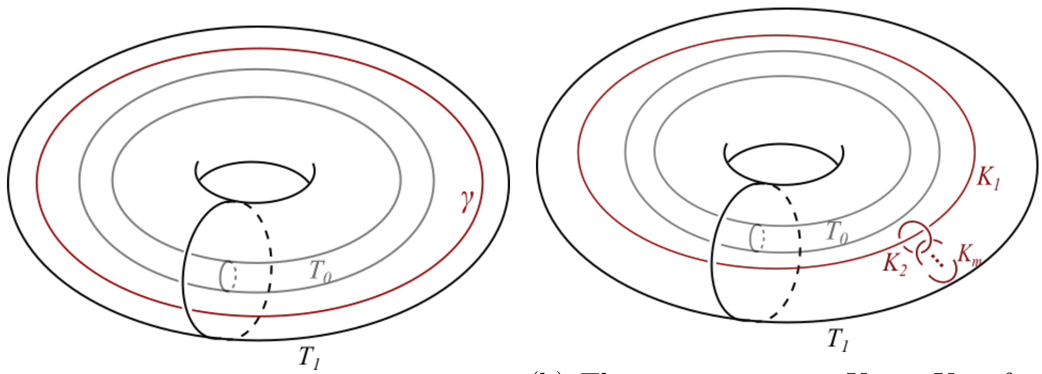
(a) handlebody diagram of X_+ (b) handlebody diagram of X_-

Figure 6.6: handlebody diagrams of $X_\pm$ with smooth framings

the Stein domains obtained by stabilizing the obvious handlebody diagram depicted in Figure 6.6b. We now argue that these contact structures (restricted to $S^1 \times \Sigma$) do not contain vertical Legendrian curves with twisting number 0. Let $C = C_-$ denote the T^2-bundle over S^1 obtained as the boundary of the cyclic plumbing depicted in Figure 6.1a. Equip C with a contact structure ξ that is induced by a Stein structure on the plumbing. Such a Stein domain has a handle description consisting of the 1-handle and the horizontal chain of unknots (with additional stabilizations) depicted in Figure 6.6b. By performing Legendrian surgery along the vertical chain of unknots (with additional stabilizations) in Figure 6.6b, we obtain Y along with one of the contact structures in question. Moreover, all of the contact structures in question can be obtained this way. By Theorem 3.1 in [32], the induced contact structure on C is virtually overtwisted. By [39], such contact torus bundles are minimally twisting (i.e. there do not exist embedded toric annuli with I-twisting equal to π). Thus the contact structures in question must also be minimally twisting and so cannot contain vertical Legendrian curves with twisting number 0.

The remaining $z_1(z_2-1)\cdots(z_m-1)$ contact structures are not minimally twisting. They are obtained by performing Legendrian surgery on C with respect to the unique universally tight contact structure with no Giroux torsion, ξ_C (c.f. [39]). Recall, using the notation of Section 6.4.1, that $C \cong T^2 \times [0,1]/(x,1) \sim (-Bx, 0)$, and Y is obtained from C by performing surgery along $\gamma = S^1 \times \{\text{pt}\} \times \{\text{pt}\}$, as depicted in Figure 6.7a (using the conventions set up in Section 6.2). To perform this surgery, we remove a solid torus neighborhood of γ and glue in solid torus V via the map $g = \begin{bmatrix} r & r' \\ -s & -s' \end{bmatrix}$, where $\frac{r}{s} = [z_1, ..., z_m]$, $\frac{r'}{s'} = [z_1, ..., z_{m-1}]$, and $-\partial(Y - V)$ is identified with $\mathbb{R}^2/\mathbb{Z}^2$ as in Section 6.4.1.

It is known (see [38]) that ξ_C is the kernel of the 1-form $\alpha = \sin(\phi(t))dx + \cos(\phi(t))dy$, where $C = T^2 \times [0,1]/\sim$ has coordinates (x, y, t), $\phi'(t) > 0$, and $\pi \leq \sup_{t \in \mathbb{R}/\mathbb{Z}} \phi(t+1) - \phi(t) < 2\pi$. Furthermore, since C is a hyperbolic torus bundle, the first inequality is strict. Thus there exists a toric annulus $T^2 \times [0,1]$ embedded in C



(a) The surgery curve γ in C

(b) The surgery curves $K_1, ..., K_m$ after slam dunk

Figure 6.7

with contact form $\alpha = \sin(\phi(t))dx + \cos(\phi(t))dy$, where $\phi(0) = -\frac{\pi}{2}$ and $\phi(1) = \frac{\pi}{2}$.

Following the methods and notation of Section 6.2, let $\gamma = S^2 \times \{pt\} \times \{t_0\}$ be a longitude on the torus T_{t_0}. Then γ Legendrian and has twisting number 0 (with respect to the 0-framing). This is the curve we will perform surgery on to obtain Y. Perform consecutive (reverse) slam dunk moves starting with γ to obtain a link $K_1 \sqcup \cdots \sqcup K_m$ in C (see Figure 6.7b), where $K_1 = \gamma$ and K_i has framing $-z_i$ for all i. Then Y is obtained by $-z_i$-surgery on K_i for all i. Take the front projection of $K_1 \sqcup \cdots \sqcup K_m$ satisfying $tb(K_1) = 0$, $tb(K_i) = -1$ for $i \geq 2$, and $r(K_i) = 0$ for all i (see Figure 6.8). Stabilize K_1 z_1-times and stabilize K_i $(z_i - 1)$-times for $i \geq 2$. By Lemma 6.2.1, since $c_1(\xi_C) = 0$, by attaching Stein 2-handles along each K_i, we obtain $z_1(z_2 - 1)\cdots(z_m - 1)$ distinct Stein cobordisms (W, J_i) from (C, ξ_C) to contact manifolds (Y, ν_i), where $1 \leq i \leq z_1(z_2-1)\cdots(z_m-1)$. In [8], it is shown that (C, ξ_C) is weakly symplectically fillable. Thus by Corollary 6.0.8, the contact structures ν_i are all tight and pairwise nonisotopic. Finally, if $(a_1, ..., a_n)$ is embeddable, then by [32], (C, ξ_C) is Stein fillable. If (X, J) is a Stein filling, then we can glue each cobordism to (X, J) to obtain Stein fillings of (Y, ν_i) for all i. □

Remark 6.4.10. Although $(a_1, ..., a_n)$ being embeddable is not a necessary condition for (C, ξ_C) to be Stein fillable, a necessary condition is described in [9]. Thus, Theorem 6.0.6 is a necessary part of the proof of Proposition 6.4.9.

6.5 Some explicit Stein fillings of Y_-

In this section, we will give a general description of the Stein fillings of (Y_-, ν_i), where the ν_i are the $z_1(z_2 - 1)\cdots(z_m - 1)$ contact structures described in the proof of Proposition 6.4.9. This description is similar to the description of the symplectic fillings of the canonical contact structure on lens spaces described by Lisca in [47]

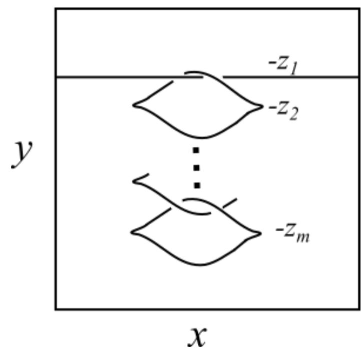


Figure 6.8: The front projection of $K_1 \sqcup \cdots \sqcup K_m$ with smooth framings

(c.f. Section 4.5). We first give smooth descriptions of the Stein fillings of hyperbolic T^2-bundles over S^1 found in [32]. Let $C = C_-$ be the boundary of the negative cyclic plumbing with framings $(-a_1, ..., -a_n)$ shown in Figure 6.1a such that $(a_1, ..., a_n)$ is embeddable. Consider the obvious surgery description of C. Then by blowing up with 1-framed unknots and continually blowing down any resulting -1-framed unknots, we can obtain a surgery description of the so-called *dual graph* (c.f. [54]) with positive framings $(d_1, ..., d_k)$. Denote the unknot with framing d_i by K_i. Since $(a_1, ..., a_n)$ is embeddable, there exists a blowup $(c_1, ..., c_k)$ of $(0, 0)$ such that $c_i \leq d_i$ for all i. If we blow down K_i $(d_i - c_i)$-times, we obtain the surgery description of C in Figure 6.9.

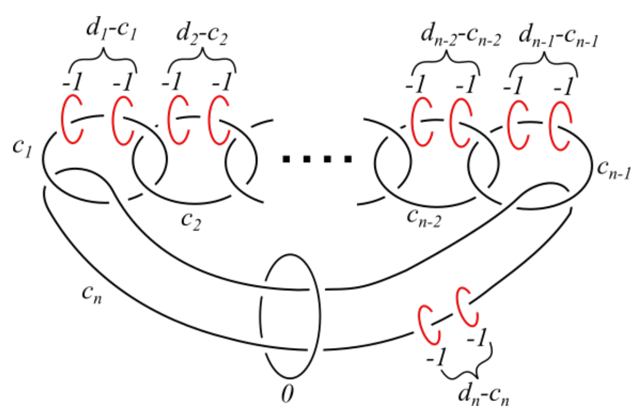
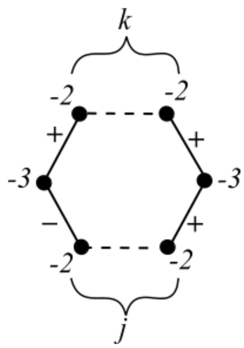


Figure 6.9: A surgery diagram of C

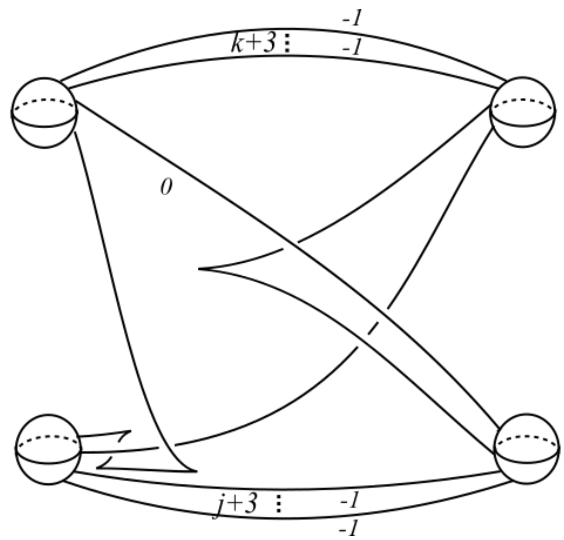
To obtain a handlebody diagram of the smooth filling of C, blow down the sequence $(c_1, ..., c_k)$ appropriately until the chain becomes two 0-framed unknots. Notice, we will be left with the 0-framed Borromean rings along with the image of the -1-framed red curves, which are now complicated knots with various negative framings. Finally, change the two 0-framed unknots resulting from blowing down the sequence $(c_1, ..., c_k)$ to dotted circles. Then the resulting 4-manifold, D, is bounded by C. In [32], D is given a Stein structure that induces the universally tight contact structure ξ_C on C.

Similarly, let $Y = Y_-$ be the boundary of the negative cyclic plumbing shown in Figure 6.1b. Consider its obvious surgery diagram and follow the above steps for the cycle portion. We will then obtain Figure 6.9 along with a chain of unknots with framings $(-z_2, ..., -z_m)$ dangling from the image of the $-z_1$-framed unknot, which links $K_1 \sqcup \cdots \sqcup K_n$ in a complicated way. Once again, to obtain the smooth filling of Y, on which we can place $z_1(z_2 - 1) \cdots (z_m - 1)$ Stein structures, blow down the sequence $(c_1, ..., c_k)$ appropriately until the chain becomes two 0-framed unknots and then change those unknots to dotted circles. To see the various Stein structures, one would need to arrange the diagram appropriately. Since the induced contact structures are obtained via Legendrian surgery on (C, ξ_C), by the remarks in the proof of Proposition 6.4.9, these contact structures are indeed the ν_i.

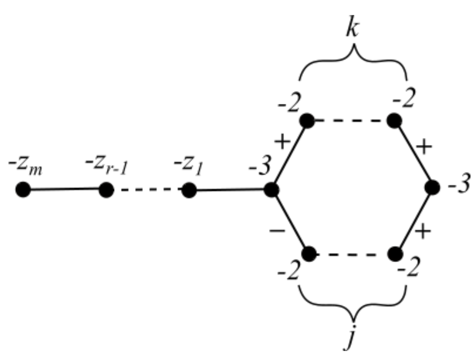
Drawing these diagrams is intractable in general, but in simple situations, it is manageable. For example, consider the cyclic plumbing in Figure 6.10a. Denote its boundary by C. Consider the obvious surgery diagram of the boundary. By blowing up with two $+1$-unknots located on either side of the leftmost -3-unknot and then blowing down consecutive -1-framed unknots, we obtain the Borromean rings with framings $0, k + 3$, and $j + 3$. Next, blow up the unknots with framings $k + 3$ and $j + 3$, $(k + 3)$-times and $(j + 3)$-times, respectively. Finally turn the resulting two 0-framed unknots into dotted circle notation. By isotoping appropriately, we obtain the handlebody diagram depicted in Figure 6.10b. By the arguments in [32], C admits a Stein structure that induces the universally tight contact structure ξ_C. By computing the d_3-invariants of the three possible contact structures on C, it is easy to see that the Stein diagram we have drawn in Figure 6.10b induces ξ_C. Similarly, if we begin with the plumbing Y_- in Figure 6.10c and apply the same moves, we obtain the handlebody diagram depicted in Figure 6.10d. After stabilizing appropriately, we obtain $z_1(z_2 - 1) \cdots (z_m - 1)$ nonisotopic tight contact structures on Y_-. Since these contact structures are obtained by Legendrian surgery on (C, ξ_C), thus these contact structures are the ν_i.



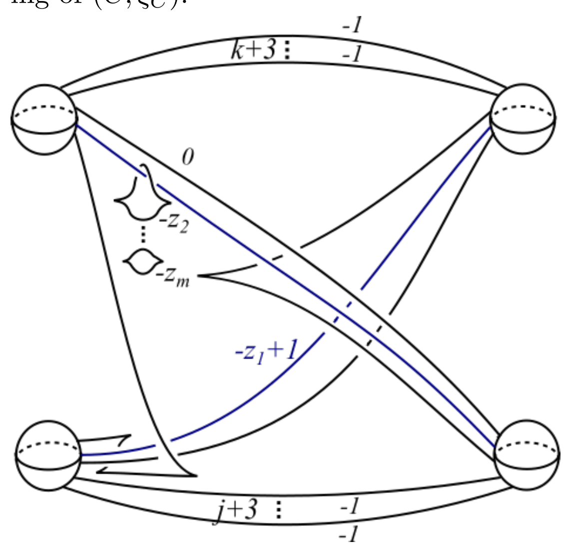
(a) A cyclic plumbing. Denote its boundary by C.



(b) Stein handlebody diagram of the filling of (C, ξ_C).



(c) A cyclic plumbing with an arm. Denote its boundary by Y



(d) Stein handlebody diagrams (without all stabilizations) of the fillings of (Y, ν_i) for all i.

Figure 6.10: Stein fillings with smooth framings

6.6 Some continued fraction computations

Here we prove the some minor facts about continued fractions that are used throughout Section 6.4. Let $\frac{p}{q} = [a_1, ..., a_n]$ and $\frac{p'}{q'} = [a_1, ..., a_{n-1}]$, where $a_i \geq 2$ for all $1 \leq i \leq n$.

Lemma 6.6.1. *If $a_1 \geq 3$, then $p \geq 2q + 1 > q + q' + 1$.*

Proof. Let t be the unique integer satisfying $\frac{q}{t} = [a_2, ..., a_n]$. Since $t, q' < q$, we have that $p = a_1 q - t = (a_1 - 2)q + q + (q - t) \geq 2q + 1 > q + q' + 1$. □

Lemma 6.6.2. *$\frac{q' \pm 1}{q}$ is a reduced fraction if and only if $\frac{p \mp 1}{q}$ is a reduced fraction. Moreover, we have that $(q' \pm 1, q) = (p \mp 1, q)$.*

Proof. Since $p'q - q'p = 1$, we have that $p'q - (q'+1)p = -(p-1)$ and $p'q - (q'-1)p = p + 1$. Thus, if d divides any two elements of $\{q, q' + 1, p - 1\}$, it must divide the third. Similarly, if d divides any two elements of $\{q, q' - 1, p + 1\}$, it must divide the third. The result follows. □

Lemma 6.6.3. $\frac{p-p'}{q-q'} = [a_1, ..., a_n - 1]$.

Proof. We will prove this by induction on q. First, let $q = 2$ and let $p > 2$ be odd. Then $\frac{p}{2} = [\frac{p+1}{2}, 2]$ and $\frac{p'}{q'} = \frac{p+1}{2}$ (and, in particular, $q' = 1$). Then $\frac{p-p'}{q-q'} = \frac{p-1}{2} = [\frac{p+1}{2}, 1]$. Now assume the result is true for all fractions satisfying $q \leq k - 1$. Let $\frac{p}{k} = [a_1, ..., a_n]$ so that $\frac{p'}{k'} = [a_1, ..., a_{n-1}]$. Furthermore, let t and t' be integers such that $\frac{k}{t} = [a_2, ..., a_n]$ and $\frac{k'}{t'} = [a_2, ..., a_{n-1}]$. Then by the inductive hypothesis, $\frac{k-k'}{t-t'} = [a_2, ..., a_n - 1]$. Now, $p = a_1 k - t$ and $p' = a_1 k' - t'$. Thus, $\frac{p-p'}{k-k'} = \frac{a_1(k-k')-(t-t')}{k-k'} = a_1 - \frac{t-t'}{k-k'} = [a_1, ..., a_n - 1]$. □

Lemma 6.6.4. $\frac{p-q}{p'-q'} = [a_n, ..., a_1 - 1]$.

Proof. By Lemma 6.6.4, we have that $\frac{p-p'}{q-q'} = [a_1, ..., a_n - 1]$. Thus, $[a_1 - 1, ..., a_n - 1] = \frac{p-p'}{q-q'} - 1 = \frac{p-p'-q+q'}{q-q'}$. Let $\frac{p-p'-q+q'}{k} = [a_n - 1, ..., a_1 - 1]$, where k is the unique integer satisfying $1 < k < p - p' - q + q'$ and $k(q - q') \equiv 1 \mod (p - p' - q + q')$. We claim $k = p' - q'$. Indeed, $(p' - q')(q - q') = (p - p' - q + q')q' + 1$ (since $pq' + 1 = p'q$). Thus $[a_n, ..., a_1 - 1] = 1 + \frac{p-p'-q+q'}{p'-q'} = \frac{p-q}{p'-q'}$. □

Bibliography

[1] Aceto, P. Rational homology cobordisms of plumbed 3-manifolds. *preprint* (2015).

[2] Akhmedov, A., and Park, B. D. Exotic smooth structures on small 4-manifolds. *Invent. Math. 173*, 1 (2008), 209–223.

[3] Akhmedov, A., and Park, B. D. Exotic smooth structures on small 4-manifolds with odd signatures. *Invent. Math. 181*, 3 (2010), 577–603.

[4] Cannas da Silva, A. *Lectures on symplectic geometry*, vol. 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[5] Casson, A., and Harer, J. Some homology lens spaces which bound rational homology balls. *Pacific Journal of Mathematics 96(1)* (1981), 23–36.

[6] Cieliebak, K., and Eliashberg, Y. *From Stein to Weinstein and back*, vol. 59 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012. Symplectic geometry of affine complex manifolds.

[7] Colin, V., Giroux, E., and Honda, K. Finitude homotopique et isotopique des structures de contact tendues. *Publ. Math. Inst. Hautes Études Sci.*, 109 (2009), 245–293.

[8] Ding, F., and Geiges, H. Symplectic fillability of tight contact structures on torus bundles. *Algebr. Geom. Topol. 1* (2001), 153–172.

[9] Ding, F., and Li, Y. Strong symplectic fillability of contact torus bundles. *arXiv:1608.00649* (2016).

[10] Eliashberg, Y. Filling by holomorphic discs and its applications. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, vol. 151 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1990, pp. 45–67.

[11] Eliashberg, Y. Topological characterization of Stein manifolds of dimension > 2. *Internat. J. Math. 1*, 1 (1990), 29–46.

[12] Eliashberg, Y. M. Complexification of contact structures on 3-dimensional manifolds. *Uspekhi Mat. Nauk 40*, 6(246) (1985), 161–162.

[13] Endo, H., Mark, T. E., and Horn-Morris, J. V. Monodromy substitutions and rational blowdowns. *Journal of Topology 4 360* (2011), 227–253.

[14] Etnyre, J. Planar open book decompositions and contact structures. *International Mathematics Research Notices* (2004), 4255–4267.

[15] Etnyre, J. B. Symplectic convexity in low-dimensional topology. *Topology Appl. 88*, 1-2 (1998), 3–25. Symplectic, contact and low-dimensional topology (Athens, GA, 1996).

[16] Etnyre, J. B. Lectures on open book decompositions and contact structures. In *Floer homology, gauge theory, and low-dimensional topology*, vol. 5 of *Clay Math. Proc.* Amer. Math. Soc., Providence, RI, 2006, pp. 103–141.

[17] Fintushel, R., Park, B. D., and Stern, R. J. Reverse engineering small 4-manifolds. *Algebr. Geom. Topol. 7* (2007), 2103–2116.

[18] Fintushel, R., and Stern, R. J. Rational blowdowns of smooth 4-manifolds. *Journal of Differential Geometry 46(2)* (1997), 181–235.

[19] Fintushel, R., and Stern, R. J. Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$. *J. Amer. Math. Soc. 19*, 1 (2006), 171–180.

[20] Fintushel, R., and Stern, R. J. Six lectures on four 4-manifolds. In *Low dimensional topology*, vol. 15 of *IAS/Park City Math. Ser.* Amer. Math. Soc., Providence, RI, 2009, pp. 265–315.

[21] Fintushel, R., and Stern, R. J. Pinwheels and nullhomologous surgery on 4-manifolds with $b^+ = 1$. *Algebr. Geom. Topol. 11*, 3 (2011), 1649–1699.

[22] Freedman, M. H. The topology of four-dimensional manifolds. *J. Differential Geom. 17*, 3 (1982), 357–453.

[23] Gay, D., and Mark, T. E. Convex plumbings and Lefschetz fibrations. *Journal of Symplectic Geometry 11(3)* (2013), 363–375.

[24] Gay, D. T. Four-dimensional symplectic cobordisms containing three-handles. *Geom. Topol. 10* (2006), 1749–1759 (electronic).

[25] Geiges, H. *An introduction to contact topology*, vol. 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[26] Ghiggini, P. Ozsváth-Szabó invariants and fillability of contact structures. *Math. Z. 253*, 1 (2006), 159–175.

[27] Ghiggini, P., Lisca, P., and Stipsicz, A. I. Classification of tight contact structures on small Seifert 3-manifolds with $e_0 \geq 0$. *Proc. Amer. Math. Soc. 134*, 3 (2006), 909–916 (electronic).

[28] Ghiggini, P., and Schönenberger, S. On the classification of tight contact structures. In *Topology and geometry of manifolds (Athens, GA, 2001)*, vol. 71 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 2003, pp. 121–151.

[29] Ghiggini, P., and Van Horn-Morris, J. Tight contact structures on the Brieskorn spheres $-\Sigma(2,3,6n-1)$ and contact invariants. *J. Reine Angew. Math. 718* (2016), 1–24.

[30] Giroux, E. Convexité en topologie de contact. *Comment. Math. Helv. 66*, 4 (1991), 637–677.

[31] Giroux, E. Géométrie de contact: de la dimension trois vers les dimensions supérieures. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)* (2002), Higher Ed. Press, Beijing, pp. 405–414.

[32] Golla, M., and Lisca, P. On Stein fillings of contact torus bundles. *Bull. Lond. Math. Soc. 48*, 1 (2016), 19–37.

[33] Gompf, R., and Stipsicz, A. *4-manifolds and Kirby Calculus*, vol. 20 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 1999.

[34] Gompf, R. E. A new construction of symplectic manifolds. *Ann. of Math. (2) 142*, 3 (1995), 527–595.

[35] Gompf, R. E. Handlebody construction of Stein surfaces. *Ann. of Math. (2) 148*, 2 (1998), 619–693.

[36] Gromov, M. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math. 82*, 2 (1985), 307–347.

[37] Hedden, M. On knot Floer homology and cabling. II. *Int. Math. Res. Not. IMRN*, 12 (2009), 2248–2274.

[38] Honda, K. On the classification of tight contact structures. I. *Geom. Topol. 4* (2000), 309–368.

[39] Honda, K. On the classification of tight contact structures. II. *J. Differential Geom. 55*, 1 (2000), 83–143.

[40] Honda, K., Kazez, W. H., and Matić, G. Convex decomposition theory. *Int. Math. Res. Not. 2002*, 2 (2002), 55–88.

[41] Johnson, D. L. Homeomorphisms of a surface which act trivially on homology. *Proc. Amer. Math. Soc. 75*, 1 (1979), 119–125.

[42] Karakurt, c. g., and Starkston, L. Surgery along star-shaped plumbings and exotic smooth structures on 4-manifolds. *Algebr. Geom. Topol. 16*, 3 (2016), 1585–1635.

[43] Kirby, R. C., and Siebenmann, L. C. *Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[44] Kodaira, K. On compact analytic surfaces. II, III. *Ann. of Math. (2) 77 (1963), 563–626; ibid. 78* (1963), 1–40.

[45] Kronheimer, P., Mrowka, T., Oszváth, P., and Szabó, Z. Monopoles and lens space surgeries. *Annals of Mathematics (2) 165(2)* (2007), 457–546.

[46] Lisca, P. Lens spaces, rational balls and the ribbon conjecture. *Geom. Topol. 11* (2007), 429–472.

[47] Lisca, P. On symplectic fillings of lens spaces. *Transactions of the American Mathematical Society 360* (2008), 765–799.

[48] Lisca, P., and Matić, G. Tight contact structures and Seiberg-Witten invariants. *Invent. Math. 129*, 3 (1997), 509–525.

[49] Lisca, P., and Matić, G. Stein 4-manifolds with boundary and contact structures. *Topology Appl. 88*, 1-2 (1998), 55–66. Symplectic, contact and low-dimensional topology (Athens, GA, 1996).

[50] Loi, A., and Piergallini, R. Compact Stein surfaces with boundary as branched covers of B^4. *Invent. Math. 143*, 2 (2001), 325–348.

[51] Michalogiorgaki, M. Rational blow-down along Wahl type plumbing trees of spheres. *Algebraic and Geometric Topology 7* (2007), 1327–1343.

[52] Milnor, J. On manifolds homeomorphic to the 7-sphere. *Ann. of Math. (2) 64* (1956), 399–405.

[53] Moise, E. E. *Geometric topology in dimensions* 2 *and* 3. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, Vol. 47.

[54] Neumann, W. D. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. Amer. Math. Soc. 268*, 2 (1981), 299–344.

[55] Ohta, H., and Ono, K. Simple singularities and topology of symplectically filling 4-manifold. *Comment. Math. Helv. 74* (1999), 575–590.

[56] Orlik, P. *Seifert manifolds.* Lecture Notes in Mathematics, Vol. 291. Springer-Verlag, Berlin-New York, 1972.

[57] Orlik, P., and Wagreich, P. Algebraic surfaces with k^*-action. *Acta Mathematics 138(1)* (1977), 43–81.

[58] Ozbagci, B., and Stipsicz, A. *Surgery on Contact 3-manifolds and Stein surfaces*, vol. 13 of *Bolyai Society Mathematical Studies.* Springer Berlin Heidelberg, 2004.

[59] Ozsváth, P., and Szabó, Z. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math. 173*, 2 (2003), 179–261.

[60] Ozsváth, P., and Szabó, Z. Holomorphic disks and genus bounds. *Geom. Topol. 8* (2004), 311–334.

[61] Ozsváth, P., and Szabó, Z. Holomorphic disks and three-manifold invariants: properties and applications. *Ann. of Math. (2) 159*, 3 (2004), 1159–1245.

[62] Ozsváth, P., and Szabó, Z. Heegaard Floer homology and contact structures. *Duke Math. J. 129*, 1 (2005), 39–61.

[63] Ozsváth, P., and Szabó, Z. Holomorphic triangles and invariants for smooth four-manifolds. *Adv. Math. 202*, 2 (2006), 326–400.

[64] Ozsváth, P., and Szabó, Z. Lectures on Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, vol. 5. Amer. Math. Soc., Providence, RI, 2006, pp. 29–70.

[65] Park, J. Seiberg-Witten invariants of generalised rational blow-downs. *Bull. Austral. Math. Soc. 56* (1997), 363–384.

[66] Park, J. Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$. *Inventiones Mathematicae 159(3)* (2005), 657–667.

[67] Park, J. Exotic smooth structures on $3\mathbb{C}P^2 \sharp 8\overline{\mathbb{C}P}^2$. *Bull. Lond. Math. Soc. 39*, 1 (2007), 95–102.

[68] Plamenevskaya, O. Contact structures with distinct Heegaard Floer invariants. *Mathematical Research Letters 11(4)* (2004), 547–561.

[69] Roberts, L. P. Rational blow-downs in Heegaard-Floer homology. *Commun. Contemp. Math. 10*, 4 (2008), 491–522.

[70] Stipsicz, A., and Szabó, Z. An exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$. *Geometry and Topology 9* (2005), 813–832.

[71] Stipsicz, A., Szabó, Z., and Wahl, J. Rational blow-downs and smoothing of surface singularities. *Journal of Topology 1(2)* (2008), 477–517.

[72] Stipsicz, A. I. Symplectic 4-manifolds, Stein domains, Seiberg-Witten theory and mapping class groups. In *Interactions between low-dimensional topology and mapping class groups*, vol. 19 of *Geom. Topol. Monogr.* Geom. Topol. Publ., Coventry, 2015, pp. 173–200.

[73] Symington, M. Symplectic rational blowdowns. *Journal of Differential Geometry 50(3)* (1998), 505–518.

[74] Symington, M. Generalized symplectic rational blowdowns. *Algebr. Geom. Topol. 1* (2001), 503–518.

[75] Weinstein, A. Contact surgery and symplectic handlebodies. *Hokkaido Math. J. 20*, 2 (1991), 241–251.

[76] Wu, H. Legendrian vertical circles in small Seifert spaces. *Commun. Contemp. Math. 8*, 2 (2006), 219–246.

[77] Yasui, K. Small exotic rational surfaces without 1- and 3-handles. *Trans. Amer. Math. Soc. 362*, 11 (2010), 5893–5907.

Index