Diagonalization

Given an $n \times n$ invertible symmetric matrix Q, the function $Q(v,w) = v^T Q w$ is an example of a nondegenerate symmetric bilinear form

More generally, a nondegenerate symmetric bilinear form on $\mathbb{Z}^n$ is a function $Q: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ satisfying:

- $Q(v,w) = Q(w,v) \quad \forall v,w \in \mathbb{Z}^n$ (symmetric)
- $\left.\begin{array}{l} Q(v_1+v_2,w) = Q(v_1,w) + Q(v_2,w) \\ Q(cv,w) = cQ(v,w) \end{array}\right\}$ linear in 1^{st} factor
- $Q(v,w) = 0 \ \forall w \in \mathbb{Z}^n \Rightarrow v = 0$ (nondegenerate)

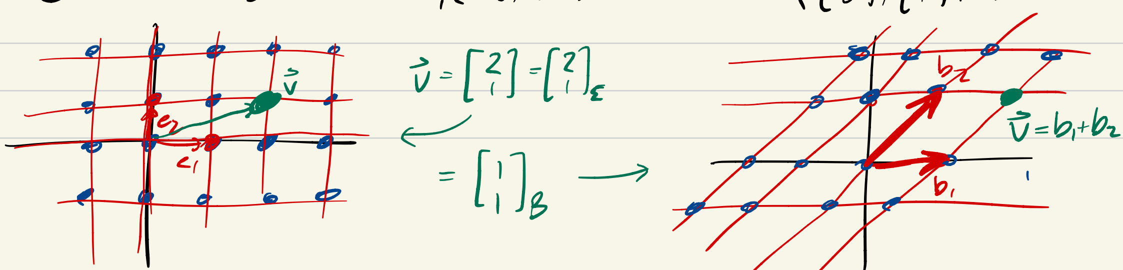
Any such function can be expressed as a matrix. But to write down the matrix, one needs a basis for $\mathbb{Z}^n$. If $B = \{b_1, \dots, b_n\}$, then the matrix for Q is written as Q_B and it is defined by:

(i,j)-th entry of $Q_B = Q(b_i, b_j)$

We have been using $\mathcal{E} = \{e_1, \dots, e_n\}$ (the standard basis)

If we choose a different basis, then the matrix will look different

Ex: Consider bases $\mathcal{E} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $B = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ for $\mathbb{Z}^2$



Consider the symmetric bilinear form Q, given by
$Q\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = ac + bd$

In $\mathcal{E}$,
$$\left.\begin{aligned} Q(e_1, e_1) &= 1 \\ Q(e_1, e_2) &= 0 \\ Q(e_2, e_2) &= 1 \end{aligned}\right\} \Rightarrow Q_{\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In $\mathcal{B}$,
$$\left.\begin{aligned} Q(b_1, b_1) &= Q\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1 \\ Q(b_1, b_2) &= Q\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 1 \\ Q(b_2, b_2) &= Q\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2 \end{aligned}\right\} \Rightarrow Q_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

<u>Sanity Check</u>: Let $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}}$

Then
$$Q(v,v) = \begin{bmatrix} 2 & 1 \end{bmatrix}_{\mathcal{E}} Q_{\mathcal{E}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}} = 5 \quad \checkmark$$
$$Q(v,v) = \begin{bmatrix} 1 & 1 \end{bmatrix}_{\mathcal{B}} Q_{\mathcal{B}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = 5$$

How are $Q_{\mathcal{E}}$ and $Q_{\mathcal{B}}$ related?

<u>Theorem</u>:

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for $\mathbb{Z}^n$. Let $P = [b_1 \cdots b_n]$
Then $Q_{\mathcal{B}} = P^T Q_{\mathcal{E}} P$

<u>Proof</u>: Note that $Pe_i = b_i \; \forall \, i$. Thus:

$$\begin{aligned} (i,j)\text{-th entry of } Q_{\mathcal{B}} &= Q(b_i, b_j) = b_i^T Q_{\mathcal{E}} b_j = (Pe_i)^T Q_{\mathcal{E}} (Pe_j) \\ &= e_i^T (P^T Q_{\mathcal{E}} P) e_j = (i,j)\text{-th entry of } P^T Q_{\mathcal{E}} P \end{aligned}$$

$$\Rightarrow Q_{\mathcal{B}} = P^T Q_{\mathcal{E}} P$$

$\blacksquare$

Def: A symmetric bilinear form Q is diagonalizable over $\mathbb{Z}$ if $\exists$ a basis $\mathcal{B}$ of $\mathbb{Z}^n$ such that $Q_{\mathcal{B}}$ is diagonal (i.e. $\exists$ matrix P such that $P^T Q_{\mathcal{E}} P$ is diagonal)

Ex: $Q = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ is diagonalizable over $\mathbb{Z}$:

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Then $P^T Q P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ ✓

We can view changing basis as a lattice isomorphism $\varphi : (\mathbb{Z}^n, Q_{\mathcal{E}}) \to (\mathbb{Z}^n, Q_{\mathcal{B}})$

$$e_i \longmapsto P e_i = b_i$$

[by construction, $Q_{\mathcal{B}}(\varphi(e_i), \varphi(e_j)) = Q_{\mathcal{B}}(b_i, b_j) = Q_{\mathcal{E}}(e_i, e_j)$]

$\rightsquigarrow$ Q is diagonalizable iff $\exists$ lattice isomorphism $\varphi : (\mathbb{Z}^n, Q_{\mathcal{E}}) \to (\mathbb{Z}^n, Q_{\mathcal{B}})$ s.t. $Q_{\mathcal{B}}$ is diagonal.

Note: If $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis for $\mathbb{Z}^n$ and $P = [b_1 \cdots b_n]$, then $\det P = \pm 1$.

Hence $\det Q_{\mathcal{B}} = \det(P^T Q_{\mathcal{E}} P) = \det P^T \det Q_{\mathcal{E}} \det P = \det Q_{\mathcal{E}}$

If $\det Q_{\mathcal{E}} = \pm 1$ and $Q_{\mathcal{E}}$ is diagonalizable, then

$\exists$ basis $\mathcal{B}$ s.t. $Q_{\mathcal{B}}$ is diagonal

& $|\det Q_{\mathcal{B}}| = 1$ $\Longrightarrow$ $Q_{\mathcal{B}} = \begin{bmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{bmatrix}$

<u>Def</u>: A symmetric bilinear form Q is
- <u>positive-definite</u> if $Q(v,v) > 0$ $\forall\, v \neq 0$
- <u>negative-definite</u> if $Q(v,v) < 0$ $\forall\, v \neq 0$

<u>Fact</u>: Let Q be diagonalizable. Then:
- Q is pos-def iff all evalues of $Q_{\mathcal{E}}$ are positive
  iff $\exists$ basis $\mathcal{B}$ s.t. $Q_{\mathcal{B}} = I$
- Q is neg-def iff all evalues of $Q_{\mathcal{E}}$ are negative.
  iff $\exists$ basis $\mathcal{B}$ s.t. $Q_{\mathcal{B}} = -I$.

<u>Def</u>: The <u>standard negative-definite lattice</u> is $(\mathbb{Z}^n, -I_n)$
The <u>standard positive-definite lattice</u> is $(\mathbb{Z}^n, I_n)$

Putting this all together:

- If a positive-definite lattice $(\mathbb{Z}^n, Q)$ with $\det Q = 1$ is diagonalizable then $\exists$ lattice isomorphism $\varphi : (\mathbb{Z}^n, Q) \longrightarrow (\mathbb{Z}^n, I)$

- If a negative-definite lattice $(\mathbb{Z}^n, Q)$ with $\det Q = \pm 1$ is diagonalizable then $\exists$ lattice isomorphism $\varphi : (\mathbb{Z}^n, Q) \longrightarrow (\mathbb{Z}^n, -I)$

Ex: $Q = \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}$.

- $\det Q = 1$
- evalues both negative

$\Rightarrow$ neg-def

Let's show Q is diagonalizable over $\mathbb{Z}$ by finding an isomorphism $\varphi : (\mathbb{Z}^2, Q) \longrightarrow (\mathbb{Z}^2, -I_2)$

Let $\varphi(f_1) = x_1 e_1 + x_2 e_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\quad \varphi(f_2) = y_1 e_1 + y_2 e_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\left.\begin{array}{l} \bullet\ -I(\varphi(f_1), \varphi(f_1)) = Q(f_1, f_1) = -2 \\ \qquad \| \\ -I(x_1 e_1 + x_2 e_2, x_1 e_1 + x_2 e_2) = -x_1^2 - x_2^2 \end{array}\right\} \Rightarrow x_1, x_2 = \pm 1$$

$$\begin{array}{l} \bullet\ -I(\varphi(f_2), \varphi(f_2)) = Q(f_2, f_2) = -5 \\ \qquad \| \\ \quad -y_1^2 - y_2^2 \end{array} \Longrightarrow (y_1, y_2) \in \{(\pm 1, \pm 2), (\pm 2, \pm 1)\}$$

$$\begin{array}{l} \bullet\ -I(\varphi(f_1), \varphi(f_2)) = Q(f_1, f_2) = 3 \\ \qquad \| \\ -I\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = -x_1 y_1 - x_2 y_2 \end{array}$$

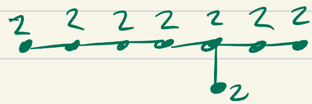
Set $x_1 = x_2 = 1$, $\quad y_1 = -2$, $y_2 = -1$ $\Rightarrow$
$$\varphi(f_1) = e_1 + e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\varphi(f_2) = -2e_1 - e_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$\Rightarrow$ Embedding exists

Note: $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right\} \Rightarrow P = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$$\Rightarrow P^T Q P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

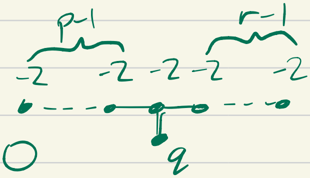
HW:

1. Let $E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \sim$ 

E_8 graph / plumbing

Show that E_8 is not diagonalizable

(Hint: Show $\nexists$ a lattice embedding into a suitable diagonal lattice)

2. Let Q be the incidence matrix for 

where $q \leq -2$, $p, r \geq 1$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 0$

For which values of p, q, r does $\exists$ embedding $(\mathbb{Z}^{p+r}, Q) \rightarrow (\mathbb{Z}^{p+r}, -I)$?

<u>Read</u> p. 11, paragraph 3 to p. 12 before Section 3.2 in Greene–Sabuka