

# Discrete-time variance-optimal hedging in affine stochastic volatility models

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## Abstract

We consider variance-optimal hedging when trading is restricted to a finite time set. Using Laplace transform methods, we derive semi-explicit formulas for the variance-optimal initial capital and hedging strategy in affine stochastic volatility models. For the corresponding minimal expected squared hedging error, we propose a closed-form approximation as well as a simulation approach. The results are illustrated by computing the relevant quantities in a time-changed Lévy model.

Key words: variance-optimal hedging, discrete time, stochastic volatility, affine processes, Laplace transform approach

## 1 Introduction

A classical question in Mathematical Finance is how the issuer of an option can hedge her risk by trading in the underlying. To tackle this problem in incomplete markets, we consider *variance-optimal hedging*, cf. e.g. [18, 22, 5] and the references therein for a survey of the extensive literature. Variance-optimal hedging of a contingent claim  $H$  means that one minimizes the *expected squared hedging error*

$$E((v_0 + \varphi \cdot S_T - H)^2)$$

over all initial endowments  $v_0$  and trading strategies  $\varphi$ , where  $\varphi \cdot S_T$  represents the cumulated gains resp. losses from trading  $\varphi$  up to the expiry date  $T$  of the claim. In this article

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we consider the above problem in affine stochastic volatility models. These generalize Lévy processes by allowing for volatility clustering and are capable of recapturing most of the stylized facts observed in stock price time series.

For Lévy processes variance-optimal hedging has been dealt with using PDE methods by [6] and by employing Laplace transform techniques in [10, 4]. The approach of [10] has subsequently been extended to affine models by [17, 13, 12] if the discounted asset price is a martingale and by [14] in the general case. However, whereas [10, 4] incorporate both *continuous* and *discrete* rebalancing, the results for affine processes have focused on continuous trading so far.

The present study complements these results by showing how to deal with discrete-time variance-optimal hedging in affine models. Since only finitely many trades are feasible in reality, this analysis is important in order to answer the following questions:

1. How should discrete rebalancing affect the *investment decisions* of the investor, i.e. to what extent should she adjust her hedging strategy?
2. How can one quantify the additional risk resulting from discrete trading, i.e. by how much does the hedging error increase?

The general structure of variance-optimal hedging in discrete time has been thoroughly investigated by [21]. However, examples of (semi-) explicit solutions seem to be limited to the results of [10, 4] for Lévy processes and [2] for some specific diffusion models with stochastic volatility. Here we show how to extend the Laplace transform approach of [10] to general affine stochastic volatility models. Similarly as in [13, 12] we focus on the case where the discounted asset price process is a martingale. Numerical experiments using the results of [10] and [14] indicate that the effect of a moderate drift rate on hedging problems is rather small.

This article is organized as follows. In Section 2 we summarize for the convenience of the reader the general structural results of [21] on variance-optimal hedging in discrete time, reduced to the case where the underlying asset is a martingale. Subsequently we explain how the Laplace transform approach can be used in general discrete-time models in order to obtain integral representations of the objects of interest. Section 4 turns to the computation of the integrands from Section 3 in affine stochastic volatility models. We show how to compute all integrands in closed form for the optimal initial capital and hedging strategy. This parallels results for continuous-time hedging in [10, 4] and [13, 12] and for discrete-time hedging in [10, 4]. Somewhat surprisingly, the expressions for the corresponding hedging error turn out to be considerably more involved than in the continuous-time case and cannot be computed in closed form. We propose two approaches to circumvent this problem: First, we determine a closed form approximation, whose error becomes negligible as the number of trades tends to infinity. As an alternative, we put forward a simple Monte-Carlo scheme to approximate the hedging error via simulation. Section 5 contains some numerical examples for the time-changed Lévy models introduced by [3].

## 2 Discrete-time variance-optimal hedging

Let  $T > 0$  be a fixed time horizon,  $N \in \mathbb{N}$  and  $\mathcal{T}_0 := \{t_0, t_1, \dots, t_N\}$ ,  $\mathcal{T} := \mathcal{T}_0 \setminus \{0\}$ , where  $t_n = nT/N$  for  $n = 0, \dots, N$ . Denote by  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}_0}, P)$  a filtered probability space with discrete time set  $\mathcal{T}_0$ . For simplicity, we assume that the initial  $\sigma$ -field  $\mathcal{F}_0$  is trivial. As for an introduction to financial mathematics in this discrete setup, the reader is referred to the textbook of Lamberton and Lapeyre [15]. The logarithm  $X$  of the discounted stock price process

$$S = S_0 \exp(X_t), \quad S_0 \in \mathbb{R}_+, \quad t \in \mathcal{T}_0,$$

is supposed to be the second component of an adapted process  $(y, X)$ , where  $X_0$  is normalized to zero. The first component  $y$  models stochastic volatility or, more accurately, stochastic activity in the model. Throughout, we suppose that

$$E(S_T^2) < \infty,$$

as well as

$$E(\Delta S_{t_n}^2 | \mathcal{F}_{t_{n-1}}) > 0, \quad t \in \mathcal{T}, \quad (2.1)$$

to rule out degenerate cases. Our goal is to compute the variance-optimal hedge for a given contingent claim  $H$  in the following sense.

**Definition 2.1** We say that  $(v_0, \varphi)$  is an *admissible endowment/strategy pair*, if  $v_0 \in \mathbb{R}$  and  $\varphi = (\varphi_t)_{t \in \mathcal{T}}$  is a predictable process (i.e.  $\varphi_t$  is  $\mathcal{F}_{t-1}$ -measurable) such that

$$\varphi \cdot S_T := \sum_{t \in \mathcal{T}} \varphi_t \Delta S_t \in L^2(P).$$

An admissible endowment/strategy pair  $(v_0^*, \varphi^*)$  is called *variance-optimal* for a contingent claim with discounted payoff  $H \in L^2(P)$  at time  $T$ , if it minimizes the *expected squared hedging error*

$$(v_0, \varphi) \mapsto E((v_0 + \varphi \cdot S_T - H)^2)$$

over all admissible endowment/strategy pairs  $(v_0, \varphi)$ . In this case, we refer to  $v_0^*$  as the *variance-optimal initial capital* and call  $\varphi^*$  *variance-optimal hedging strategy*.

As noted in the introduction, we restrict ourselves to the case where the stock price is a martingale.

**Assumption 2.2** The stock price process  $S$  is a square-integrable martingale.

In this case, the variance-optimal capital and strategy can be represented as follows.

**Proposition 2.3** Let  $H \in L^2(P)$ . Then the variance-optimal endowment/strategy pair for  $H$  is given by

$$v_0^* = V_0, \quad \varphi_{t_n}^* = \frac{E(V_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V_{t_{n-1}} S_{t_{n-1}}}{E(S_{t_n}^2 | \mathcal{F}_{t_{n-1}}) - S_{t_{n-1}}^2}, \quad t_n \in \mathcal{T},$$

where

$$V_{t_n} := E(H | \mathcal{F}_{t_n}), \quad t_n \in \mathcal{T}_0$$

denotes the option price process of  $H$ . The corresponding minimal expected squared hedging error is given by

$$J_0 := E(V_T^2 - V_0^2) - \sum_{t_n \in \mathcal{T}} E(\varphi_{t_n}^* (E(V_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V_{t_{n-1}} S_{t_{n-1}})).$$

PROOF. This follows from [21, Section 4.1] by making use of the martingale properties of  $S$  and  $V$ .  $\square$

Notice that if the initial capital is fixed at  $v_0 \in \mathbb{R}$  rather than being part of the optimization problem, the same strategy  $\varphi^*$  is still optimal if  $S$  is a martingale (cf. [21, Section 4.1]). However, the corresponding hedging error increases by  $(v_0 - V_0)^2$  in this case.

### 3 The Laplace transform approach

In order to derive formulas that can be computed in concrete models we use the *Laplace transform approach*, which has been introduced to variance-optimal hedging by [10]. The key assumption on the contingent claim is the existence of an integral representation in the following sense.

**Assumption 3.1** Suppose that the payoff function of the claim is of the form  $H = f(S_T)$  for some function  $f : (0, \infty) \rightarrow \mathbb{R}$ , such that

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^z l(z) dz,$$

for  $l : \mathbb{C} \rightarrow \mathbb{C}$  and  $R \in \mathbb{R}$  such that  $x \mapsto l(R + ix)$  is integrable and  $E(\exp(2RX_T)) < \infty$ .

**Example 3.2** Most European options admit an integral representation of this kind. E.g. for the European call with payoff function  $f(s) = (s - K)^+$ , we have

$$f(s) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz,$$

for any  $R > 1$  by [10, Lemma 4.1]. More generally, the *Bromwich inversion formula* as in [10, Theorem A.1] ascertains that  $l$  is typically given by the *bilateral Laplace transform* of  $x \mapsto f(\exp(x))$ , cf. [10] for more details and examples.

Henceforth, we only consider contingent claims satisfying Assumption 3.1. In this case, Proposition 2.3 can also be written in integral form.

**Theorem 3.3** We have  $H \in L^2(P)$  and the corresponding option price process is given by

$$V_{t_n} = \int_{R-i\infty}^{R+i\infty} V(z)_{t_n} l(z) dz, \quad t_n \in \mathcal{T}_0,$$

for the square-integrable martingales

$$V(z)_{t_n} := E(S_T^z | \mathcal{F}_{t_n}), \quad t_n \in \mathcal{T}_0.$$

Moreover, the variance-optimal hedging strategy for  $H$  can be represented as

$$\varphi_{t_n}^* = \int_{R-i\infty}^{R+i\infty} \frac{E(V(z)_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V(z)_{t_{n-1}} S_{t_{n-1}}}{E(S_{t_n}^2 | \mathcal{F}_{t_{n-1}}) - S_{t_{n-1}}^2} l(z) dz, \quad t_n \in \mathcal{T}.$$

PROOF. The first assertion follows from Assumption 3.1 and Fubini's theorem along the lines of [13, Lemma 3.3 and Proposition 3.4]. The second can be derived analogously using the Cauchy-Schwarz inequality and  $E(S_{t_n}^2) < \infty$ ,  $t_n \in \mathcal{T}$ .  $\square$

For the hedging error, Proposition 2.3 and Theorem 3.3 yield the following similar integral representation.

**Corollary 3.4** For  $t_n \in \mathcal{T}$  and  $z_1, z_2 \in R + i\mathbb{R}$ , let

$$\begin{aligned} J_1(z_1, z_2) &:= V(z_1)_T V(z_2)_T - V(z_1)_0 V(z_2)_0, \\ J_2(t_n, z_1, z_2) &:= \frac{(E(V(z_1)_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V(z_1)_{t_{n-1}} S_{t_{n-1}})}{E(S_{t_n}^2 | \mathcal{F}_{t_{n-1}}) - S_{t_{n-1}}^2} \times \\ &\quad \times (E(V(z_2)_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V(z_2)_{t_{n-1}} S_{t_{n-1}}). \end{aligned}$$

If, for  $t_n \in \mathcal{T}$ ,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(|J_2(t_n, R + ix_1, R + ix_2)|) |l(R + ix_1)| |l(R + ix_2)| dx_1 dx_2 < \infty, \quad (3.1)$$

the minimal expected squared hedging error is given by

$$\begin{aligned} J_0 &= \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} E(J_1(z_1, z_2)) l(z_1) l(z_2) dz_1 dz_2 \\ &\quad - \sum_{t_n \in \mathcal{T}} \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} E(J_2(t_n, z_1, z_2)) l(z_1) l(z_2) dz_1 dz_2. \end{aligned}$$

## 4 Application to affine stochastic volatility models

Theorem 3.3 and Corollary 3.4 show that in order to compute semi-explicit formulas of the discrete variance-optimal capital, hedging strategy and hedging error, one must be able to compute conditional exponential moments of the process  $X$ . This suggests to consider models whose moment generating function  $E(\exp(uX_{t_n}))$  is known in closed form. Here we use *affine processes* in the sense of [7].

**Assumption 4.1** Suppose that  $(y_t, X_t)_{t \in \mathcal{T}_0}$  is the restriction to discrete time of a semimartingale which is *regularly affine* w.r.t  $y$  in the sense of [7, Definitions 2.1 and 2.5]. This means that the characteristic function of  $(y, X)$  has exponentially affine dependence on  $y$ , i.e. there exist  $\Psi_j : \mathcal{T} \times i\mathbb{R}^2 \rightarrow \mathbb{C}$ ,  $j = 0, 1$  such that, for  $t \geq s$  and  $(u_1, u_2) \in i\mathbb{R}^2$ ,

$$E(e^{u_1 y_t + u_2 X_t} | \mathcal{F}_s) = \exp(\Psi_0(t-s, u_1, u_2) + \Psi_1(t-s, u_1, u_2) y_s + u_2 X_s). \quad (4.1)$$

**Example 4.2** By [7, Theorems 2.7 and 2.12], a continuous-time semimartingale  $(y, X)$  is affine if and only if its local dynamics expressed in terms of the infinitesimal generator resp. the differential characteristics depend on  $y$  in an affine way. Moreover, the functions  $\Psi_0, \Psi_1$  can be determined by solving some generalized Riccati equations.

[11] shows that a large number of stochastic volatility models from the empirical literature fit into this framework. Examples include the models of Heston [9] and Barndorff-Nielsen and Shephard [1] as well as their extensions to time-changed Lévy models by [3]. A particular specification of this general class of models is given by the following *OU-time-change model*:

$$X_t = L \int_0^t y_s ds, \quad dy_t = -\lambda y_t dt + dZ_t, \quad y_0 > 0,$$

for a mean reversion speed  $\lambda > 0$ , a Lévy process  $L$  with Lévy exponent  $\psi^L$  and an increasing Lévy process  $Z$  with Lévy exponent  $\psi^Z$ , i.e.

$$E(e^{uL_t}) = \exp(t\psi^L(u)), \quad E(e^{uZ_t}) = \exp(t\psi^Z(u)), \quad \forall u \in i\mathbb{R}.$$

In this case,  $(y, X)$  is affine by [11, Section 4.4] and in view of [11, Corollary 3.5], we have

$$\begin{aligned} \Psi_1(t, u_1, u_2) &= e^{-\lambda t} u_1 + \frac{1 - e^{-\lambda t}}{\lambda} \psi^L(u_2), \\ \Psi_0(t, u_1, u_2) &= \int_0^t \psi^Z(\Psi_1(s, u_1, u_2)) ds. \end{aligned}$$

If  $y$  is chosen to be a *Gamma-OU process* with stationary Gamma( $a, b$ ) distribution (see e.g. [20] for more details), we have  $\psi^Z(u) = \lambda a u / (b - u)$  and  $\Psi_0$  can be determined in closed form as well. By e.g. [12, Proposition 3.6], we have

$$\Psi_0(t, u_1, u_2) = \begin{cases} \frac{a\lambda}{b\lambda - \psi^L(u_2)} \left( b \log \left( \frac{b - \Psi_1(t, u_1, u_2)}{b - u_1} \right) + t\psi^L(u_2) \right), & \text{if } b\lambda \neq \psi^L(u_2), \\ -a\lambda \left( \frac{b}{\lambda u_1 - \psi^L(u_2)} (e^{\lambda t} - 1) + t \right), & \text{if } b\lambda = \psi^L(u_2), \end{cases}$$

where  $\log$  denotes the *distinguished logarithm* in the sense of [19, Lemma 7.6], i.e. the branch is chosen such that the resulting function is continuous in  $t$ .

To compute exponential moments of  $X$  such as  $V(z)_t = E(S_T^z | \mathcal{F}_t)$ ,  $z \in R + i\mathbb{R}$ , we need Equation (4.1) to remain valid on a suitable extension of  $i\mathbb{R}^2$ . The following sufficient condition is taken from [13].

**Assumption 4.3** Suppose that for all  $t_n \in \mathcal{T}_0$ , the mappings  $(u_1, u_2) \mapsto \Psi_j(t_n, u_1, u_2)$ ,  $j = 0, 1$  admit analytic continuations to the strip

$$\mathcal{S} := \{z \in \mathbb{C}^2 : \operatorname{Re}(z) \in (-\infty, (M \vee 0) + \varepsilon) \times ((2R \wedge 0) - \varepsilon, (2R \vee 2) + \varepsilon)\},$$

for some  $\varepsilon > 0$  and  $M := \sup\{2\Psi_1(T - t_n, 0, r) : r \in [R \wedge 0, R \vee 0], t_n \in \mathcal{T}_0\}$ .

The existence of the analytic extensions in Assumption 4.3 is difficult to verify in general. For affine diffusion processes, [8, Theorem 3.3] shows that it suffices to establish that solutions to the corresponding Riccati equations exist on  $[0, T]$ . In the presence of jumps, the situation is more involved and one has to work on a case-by-case basis. For time-changed Lévy processes, this has been carried out in detail by [12].

**Example 4.4** By the proof of [12, Theorems 3.3, 3.4], Assumption 4.3 holds in the OU-time-change models from Example 4.2, if the Lévy exponents  $\psi^L$  and  $\psi^Z$  admit analytic extensions to  $\{z \in \mathbb{C} : \operatorname{Re}(z) \in ((2R \wedge 0) - \varepsilon, (2R \vee 2) + \varepsilon)\}$  resp.  $\{z \in \mathbb{C} : \operatorname{Re}(z) \in (-\infty, M + \varepsilon)\}$  for some  $\varepsilon > 0$ . E.g. if  $L$  is chosen to be a NIG process with Lévy exponent

$$\psi^L(u) = u\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2}),$$

for  $\mu \in \mathbb{R}$ ,  $\delta, \alpha > 0$ ,  $\beta \in (-\alpha, \alpha)$  in the Gamma-OU-time-change model from Example 4.2, one easily shows that the Lévy exponents  $\psi^Z$  and  $\psi^L$  admit analytic extensions to  $\{z \in \mathbb{C} : \operatorname{Re}(z) \in (-\infty, b)\}$  resp.  $\{z \in \mathbb{C} : \operatorname{Re}(z) \in (-\alpha - \beta, \alpha - \beta)\}$ . Consequently, checking the validity of Assumption 4.3 amounts to verifying

$$M < b, \quad 2R > -\alpha - \beta, \quad 2R \vee 2 < \alpha - \beta,$$

for  $M = ((1 - e^{-\lambda T})/\lambda)2 \max\{\psi^L(R \wedge 0), \psi^L(R \vee 0)\} \geq 0$  in this case.

By [7, Theorem 2.16(ii)], Assumption 4.3 implies that the exponential moment formula (4.1) holds for all  $z \in \mathcal{S}$ . In particular,  $S$  is square-integrable. We proceed by providing sufficient and essentially necessary conditions that ensure the validity of the martingale assumption 2.2 and the non-degeneracy condition (2.1).

**Assumption 4.5** Assume that the martingale conditions

$$\Psi_0\left(\frac{T}{N}, 0, 1\right) = \Psi_1\left(\frac{T}{N}, 0, 1\right) = 0 \tag{4.2}$$

are satisfied and suppose that for

$$\delta_0 := \Psi_0\left(\frac{T}{N}, 0, 2\right), \quad \delta_1 := \Psi_1\left(\frac{T}{N}, 0, 2\right),$$

we have  $\delta_0, \delta_1 \geq 0$  and

$$\delta_0 > 0 \quad \text{or} \quad \delta_1 y_t > 0 \text{ a.s. for all } t \in \mathcal{T}. \tag{4.3}$$

**Example 4.6** For OU-time-change models, the martingale conditions (4.2) read as  $\psi^L(1) = 0$ , i.e.  $\exp(L)$  has to be a martingale. E.g. in the NIG-OU models from Example 4.4, this means

$$\mu = \delta \left( \sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2} \right).$$

As for the non-degeneracy condition (4.3), the term  $\delta_0 + \delta_1 y$  is actually bounded away from zero in most applications.

1. In OU-time-change models satisfying the conditions of Example 4.4,  $\Psi_1(s, 0, 2) = \psi^L(2)(1 - \exp(-\lambda s))/\lambda > 0$ , unless  $L$  is deterministic. Moreover,  $\Psi_0(T/N, 0, 2) = \int_0^{T/N} \psi^Z(\Psi_1(s, 0, 2)) ds$ , which is also positive by e.g. [19, Theorem 21.5]. Since  $(y_t)_{t \in \mathcal{T}_0}$  is bounded from below by  $\exp(-\lambda T)y_0 > 0$ , the term  $\delta_0 + \delta_1 y$  is bounded away from zero in this case. In particular, (4.3) is satisfied.
2. Now suppose that the Ornstein-Uhlenbeck process  $y$  is replaced by a square-root process

$$dy_t = \kappa(\eta - y_t)dt + \sigma\sqrt{y_t}dW_t, \quad y_0 > 0,$$

where  $\kappa, \eta, \sigma > 0$  and  $W$  denotes a standard Brownian motion. Subject to certain regularity conditions (cf. [12, Assumption 4.2]), the proof of [12, Theorems 4.3, 4.4] and a comparison argument show that  $\Psi_1(s, 0, 2) > 0$  for  $s > 0$ . This in turn yields  $\Psi_0(t, 0, 2) = \kappa\eta \int_0^t \Psi_1(s, 0, 2) ds > 0$  for  $t > 0$  and hence  $\delta_0, \delta_1 > 0$ . Since  $y$  is positive, this shows that  $\delta_0 + \delta_1 y$  is bounded away from zero for these *CIR-time-change models* as well and (4.3) holds.

From now on, Assumptions 4.3 and 4.5 are supposed to be in force. Combined with Theorem 3.3, Assumption 4.3 allows us to compute the variance-optimal initial capital  $v_0^*$  and the variance-optimal hedging strategy  $\varphi_t^*$  at time  $t$  by performing single numerical integrations.

**Theorem 4.7** For  $t_n \in \mathcal{T}_0$  and  $z \in R + i\mathbb{R}$ , we have

$$V(z)_{t_n} = S_{t_n}^z \exp(\Psi_0(T - t_n, 0, z) + \Psi_1(T - t_n, 0, z)y_{t_n}).$$

Moreover, for  $t_n \in \mathcal{T}$ ,

$$\varphi_{t_n}^* = \int_{R-i\infty}^{R+i\infty} \frac{V(z)_{t_{n-1}}}{S_{t_{n-1}}} \left( \frac{\exp(\kappa_0(t_n, z) + \kappa_1(t_n, z)y_{t_{n-1}}) - 1}{\exp(\delta_0 + \delta_1 y_{t_{n-1}}) - 1} \right) l(z) dz,$$

where, for  $j = 0, 1$ ,

$$\begin{aligned} \delta_j &= \Psi_j \left( \frac{T}{N}, 0, 2 \right), \quad j = 0, 1, \\ \kappa_j(t, z) &:= \Psi_j \left( \frac{T}{N}, \Psi_1(T - t, 0, z), z + 1 \right) - \Psi_j \left( \frac{T}{N}, \Psi_1(T - t, 0, z), z \right). \end{aligned}$$



PROOF. The formula for  $V(z)$  follows immediately from Assumption 4.3 and [7, Theorem 2.16(ii)]. Analogously, we obtain

$$E(S_{t_n}^2 | \mathcal{F}_{t_{n-1}}) - S_{t_{n-1}}^2 = S_{t_{n-1}}^2 (e^{\delta_0 + \delta_1 y_{t_{n-1}}} - 1) \quad (4.4)$$

and

$$E(V(z)_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V(z)_{t_{n-1}} S_{t_{n-1}} = V(z)_{t_{n-1}} S_{t_{n-1}} (e^{\kappa_0(t_n, z) + \kappa_1(t_n, z) y_{t_{n-1}}} - 1), \quad (4.5)$$

with

$$\begin{aligned} \kappa_0(t, z) &= \Psi_0\left(\frac{T}{N}, \Psi_1(T-t, 0, z), z+1\right) + \Psi_0(T-t, 0, z) - \Psi_0\left(T-t + \frac{T}{N}, 0, z\right), \\ \kappa_1(t, z) &= \Psi_1\left(\frac{T}{N}, \Psi_1(T-t, 0, z), z+1\right) - \Psi_1\left(T-t + \frac{T}{N}, 0, z\right). \end{aligned}$$

By the martingale property of  $V(z)_{t_n}$ , we have

$$V(z)_{t_{n-1}} = E(V(z)_{t_n} | \mathcal{F}_{t_{n-1}}).$$

Together with [7, Theorem 2.16(ii)], this establishes the semiflow property

$$\begin{aligned} &e^{\Psi_0(T-t_{n-1}, 0, z) + \Psi_1(T-t_{n-1}, 0, z) y_{t_{n-1}}} \\ &= e^{\Psi_0(T-t_n, 0, z) + \Psi_0(T/N, \Psi_1(T-t_n, 0, z), z) + \Psi_1(T/N, \Psi_1(T-t_n, 0, z), z) y_{t_{n-1}}}, \end{aligned}$$

for  $t_n \in \mathcal{T}$  and  $z \in R + i\mathbb{R}$ . Insertion into (4.5) yields the assertion.  $\square$

We now consider the expression for the minimal expected squared hedging error in Corollary 3.4. The first term  $J_1$  represents the variance of an unhedged exposure to the option. In view of Assumption 4.3, it can be computed by evaluating a double integral with the following integrand.

**Lemma 4.8** *For  $z_1, z_2 \in R + i\mathbb{R}$ , we have*

$$E(J_1(z_1, z_2)) = V(z_1 + z_2)_0 - V(z_1)_0 V(z_2)_0.$$

PROOF. This is due to the martingale property of  $V(z_1 + z_2)$ .  $\square$

We now turn to the second term in the formula for the hedging error in Corollary 3.4. Suppose for the moment that (3.1) holds. By Equations (4.4) and (4.5), we have

$$J_2(t_n, z_1, z_2) = V(z_1)_{t_{n-1}} V(z_2)_{t_{n-1}} \frac{(e^{\kappa_0(t_n, z_1) + \kappa_1(t_n, z_1) y_{t_{n-1}}} - 1)(e^{\kappa_0(t_n, z_2) + \kappa_1(t_n, z_2) y_{t_{n-1}}} - 1)}{e^{\delta_0 + \delta_1 y_{t_{n-1}}} - 1}.$$

In view of Corollary 3.4, it therefore remains to compute

$$\begin{aligned} &E(J_2(t_n, z_1, z_2)) \\ &= \left( I(\kappa_1(t_n, z_1) + \kappa_1(t_n, z_2), t_n, z_1, z_2) e^{\kappa_0(t_n, z_1) + \kappa_0(t_n, z_2)} - I(\kappa_1(t_n, z_2), t_n, z_1, z_2) e^{\kappa_0(t_n, z_2)} \right. \\ &\quad \left. - I(\kappa_1(t_n, z_1), t_n, z_1, z_2) e^{\kappa_0(t_n, z_1)} + I(0, t_n, z_1, z_2) \right) S_0^{z_1 + z_2} e^{\Psi_0(T-t_{n-1}, 0, z_1) + \Psi_0(T-t_{n-1}, 0, z_2)}, \end{aligned}$$

where

$$I(u, t_n, z_1, z_2) := E \left( \frac{\exp((u + \Psi_1(T - t_{n-1}, 0, z_1) + \Psi_1(T - t_{n-1}, 0, z_2))y_{t_{n-1}} + (z_1 + z_2)X_{t_{n-1}})}{\exp(\delta_0 + \delta_1 y_{t_{n-1}})} - 1 \right).$$

Unfortunately,  $I(u, t_n, z_1, z_2)$  can only be computed explicitly in some very special cases, unlike for continuous-time variance-optimal hedging. E.g. if  $N = 1$ , i.e. for *static hedging*, the sum in Corollary 3.4 only consists of the term  $J_2(T, z_1, z_2)$ , which is also deterministic in this case. Hence, we obtain

$$E(J_2(T, z_1, z_2)) = V(z_1)_0 V(z_2)_0 \frac{(e^{\nu_0(z_1) + \nu_1(z_1)y_0} - 1)(e^{\nu_0(z_2) + \nu_1(z_2)y_0} - 1)}{e^{\Psi_0(T, 0, 2) + \Psi_1(T, 0, 2)y_0} - 1},$$

for  $\nu_j(z) = \Psi_j(T, 0, z + 1) - \Psi_j(T, 0, z)$ ,  $j = 0, 1$ , which allows to compute the static hedging error by evaluating a double integral.

For Lévy processes, we have  $\delta_1 = 0$ . Hence the denominator in the expression for  $I$  reduces to a constant in this case and the expectations can be computed using (4.1). This leads to the formula obtained in [10].

For affine models, one can verify that as the number of trading times  $N$  tends to infinity, the argument of the expectation in the expression for  $I$  converges to an expression of the form

$$\frac{a + by_{t_{n-1}}}{c + dy_{t_{n-1}}} \exp(uy_{t_{n-1}} + (z_1 + z_2)X_{t_{n-1}}).$$

The expectation of this term can then be calculated, cf. the proof of [13, Theorem 4.2] for more details.

If the set of trading times is finite, it does not seem possible to calculate  $I$  in closed form. We discuss two ways to circumvent this problem and tackle the computation of the hedging error. The first is to use the approximation

$$\exp(\delta_0 + \delta_1 y_t) - 1 \approx \delta_0 + \delta_1 y_t,$$

which seems reasonable as the number  $N$  of trading dates tends to infinity, because both  $\delta_0 = \Psi_0(T/N, 0, 2)$  and  $\delta_1 = \Psi_1(T/N, 0, 2)$  converge to zero in this case. We obtain the following *first-order approximation*.

**Theorem 4.9** *Suppose that for any  $t \in \mathcal{T}_0$ , the following holds.*

1. *The mappings  $(u_1, u_2) \mapsto \Psi_j(T - t, u_1, u_2)$ ,  $j = 0, 1$  admit analytic extensions to*

$$\mathcal{S}' := \{z \in \mathbb{C}^2 : \operatorname{Re}(z) \in (-\infty, (M' \vee 0) + \varepsilon) \times ((2R \wedge 0) - \varepsilon, (2R \vee 2) + \varepsilon)\},$$

*for some  $\varepsilon > 0$  and  $M' := M \vee 2\Psi_1(T/N, M/2, R + 1)$ .*

- 2.

$$E \left( \frac{\exp(M' y_t + 2R X_t)}{\delta_0 + \delta_1 y_t} \right) < \infty. \quad (4.6)$$

Then (3.1) is satisfied and for  $n = 1, \dots, N$ ,  $z_1, z_2 \in R + i\mathbb{R}$ ,

$$u \in \Psi_1(T - t_{n-1}, 0, z_1) + \Psi_1(T - t_{n-1}, 0, z_2) \\ + \{\kappa_1(t_n, z_1) + \kappa_1(t_n, z_2), \kappa_1(t_n, z_1), \kappa_1(t_n, z_2), 0\},$$

we have

$$E \left( \frac{\exp(uy_{t_{n-1}} + (z_1 + z_2)X_{t_{n-1}})}{\delta_0 + \delta_1 y_{t_{n-1}}} \right) \\ = \begin{cases} \frac{e^{-u\delta_0/\delta_1}}{\delta_1} \int_0^1 \left( \frac{\delta_1}{\delta_0} + us \right) \exp \left( \frac{\delta_0}{\delta_1} us + \chi_0(s) + \chi_1(s)y_0 \right) ds & \text{if } \delta_j \neq 0, j = 0, 1, \\ \frac{1}{\delta_1} \int_0^1 \frac{1+us}{s} \exp(\vartheta_0(s) + \vartheta_1(s)y_0) ds & \text{if } \delta_0 = 0, \\ \frac{1}{\delta_0} \exp(\Psi_0(t_{n-1}, u, z_1 + z_2) + \Psi_1(t_{n-1}, u, z_1 + z_2)y_0) & \text{if } \delta_1 = 0, \end{cases}$$

where, for  $j = 0, 1$ ,

$$\delta_j = \Psi_j \left( \frac{T}{N}, 0, 2 \right), \quad \chi_j(s) := \Psi_j \left( t_{n-1}, \frac{\delta_1}{\delta_0} \log(s) + us, z_1 + z_2 \right), \\ \vartheta_j(s) := \Psi_j(t_{n-1}, \log(s) + us, z_1 + z_2).$$

PROOF. In view of (4.3), we have  $\exp(\delta_0 + \delta_1 y_t) - 1 > \delta_0 + \delta_1 y_t$ , which combined with (4.4) yields

$$E(|J_2(t_n, z_1, z_2)|) < \frac{1}{\delta_0 + \delta_1 y_{t_{n-1}}} \left| \frac{E(V(z_1)_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V(z_1)_{t_{n-1}} S_{t_{n-1}}}{S_{t_{n-1}}} \right| \times \\ \times \left| \frac{E(V(z_2)_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}}) - V(z_2)_{t_{n-1}} S_{t_{n-1}}}{S_{t_{n-1}}} \right|. \quad (4.7)$$

By Assumption 4.3, [7, Theorem 2.16(ii)] and Jensen's inequality, we have

$$\left| \frac{V(z_j)_{t_n} S_{t_n}}{S_{t_{n-1}}} \right| \leq S_0^R e^{\Psi_0(T-t_{n-1}, 0, R) + \Psi_1(T-t_{n-1}, 0, R)y_{t_{n-1}} + RX_{t_{n-1}}} \\ \leq S_0^R e^{\Psi_0(T-t_{n-1}, 0, R)} e^{\frac{1}{2}M'y_{t_{n-1}} + RX_{t_{n-1}}},$$

for  $j = 1, 2$ . Likewise, it follows from Jensen's inequality that

$$\left| \frac{E(V(z_j)_{t_n} S_{t_n} | \mathcal{F}_{t_{n-1}})}{S_{t_{n-1}}} \right| \leq E \left( |V(z_j)_{t_n}| S_{t_n}^{-1} e^{-X_{t_{n-1}}} \right) \\ \leq S_0^R E \left( e^{\Psi_0(T-t_n, 0, R) + \Psi_1(T-t_n, 0, R)y_{t_n} + (R+1)X_{t_n}} | \mathcal{F}_{t_{n-1}} \right) e^{-X_{t_{n-1}}} \\ \leq S_0^R e^{\Psi_0(T-t_n, 0, R)} E \left( e^{\frac{1}{2}M'y_{t_n} + (R+1)X_{t_n}} | \mathcal{F}_{t_{n-1}} \right) e^{-X_{t_{n-1}}} \\ = S_0^R e^{\Psi_0(T-t_n, 0, R)} e^{\Psi_0(T/N, M/2, R+1) + \Psi_1(T/N, M/2, R+1)y_{t_{n-1}} + RX_{t_{n-1}}} \\ \leq S_0^R e^{\Psi_0(T-t_n, 0, R) + \Psi_0(T/N, M/2, R+1)} e^{\frac{1}{2}M'y_{t_{n-1}} + RX_{t_{n-1}}},$$

for  $j = 1, 2$ , where we have used Assumption 4.3 and [7, Theorem 2.16(ii)] for the equality. Together with (4.7), this implies

$$|J_2(t_n, z_1, z_2)| \leq C \frac{\exp(M'y_{t_{n-1}} + 2RX_{t_{n-1}})}{\delta_0 + \delta_1 y_{t_{n-1}}},$$

for some constant  $C > 0$  which does not depend on  $\omega$  and  $z_1, z_2$ . Consequently, (4.6) and Assumption 3.1 yield (3.1). The second part of the assertion now follows along the lines of the proof of [13, Theorem 4.2] under the stated assumptions.  $\square$

**Example 4.10** In most applications the denominator in (4.6) is actually bounded away from zero (cf. Example 4.6). In this situation, (4.6) follows immediately from Condition 1 of Theorem 4.9 and [7, Theorem 2.16(ii)].

In view of Theorem 4.9, the hedging error can be approximated by a sum of triple integrals with known integrands. Notice that because

$$\frac{1}{\delta_0 + \delta_1 y_t + \frac{1}{2}(\delta_0 + \delta_1 y_t)^2} = \frac{1}{\delta_0 + \delta_1 y_t} - \frac{1}{\delta_0 + 2 + \delta_1 y_t},$$

a *second-order approximation* based on

$$\exp(\delta_0 + \delta_1 y_t) - 1 \approx \delta_0 + \delta_1 y_t + \frac{1}{2}(\delta_0 + \delta_1 y_t)^2$$

follows directly from Theorem 4.9.

Instead of using the closed-form approximation proposed above, one can eschew semi-explicit computations and instead calculate the hedging error using a Monte-Carlo simulation as in [6]:

1. Simulate  $K \in \mathbb{N}$  independent trajectories  $(y(\omega_k), X(\omega_k))$ ,  $k = 1, \dots, K$  of  $(y, X)$  and compute the realizations  $S(\omega_k) = S_0 \exp(X(\omega_k))$  and  $H = f(S_T(\omega_k))$  of  $S$  and  $H$ .
2. Calculate the values of  $v_0^*$  and  $\varphi_{t_n}^*(\omega_k)$ ,  $t_n \in \mathcal{T}$  using numerical integration to evaluate the formulas from Theorem 4.7.
3. Compute the realized squared hedging errors

$$J_0(\omega_k) = \left( v_0^* + \sum_{t_n \in \mathcal{T}} \varphi_{t_n}^*(\omega_k) \Delta S_{t_n}(\omega_k) - f(S_T(\omega_k)) \right)^2.$$

4. Use the empirical mean  $\frac{1}{K} \sum_{k=1}^K J_0(\omega_k)$  as an estimator for  $J_0$ .

In addition to its simplicity, this approach has the advantage of approximating the entire distribution of the hedging error, rather than just its mean. On the other hand, computation time is increased.

## 5 Numerical illustration

In order to illustrate the applicability of our formulas and examine the effect of discrete trading, we now investigate a numerical example. More specifically, we consider the NIG-Gamma-OU model from Examples 4.2 and 4.4. As for parameters, we use the values estimated in [16] using the generalized method of moments, adjusting the drift rate  $\mu$  of  $L$  in order to ensure the martingale property of  $S$ :

$$\beta = -16.0, \quad \alpha = 90.1, \quad \delta = 85.9, \quad \mu = 15.0, \quad \lambda = 2.54, \quad a = 0.847, \quad b = 17.5.$$

By Example 4.4, Assumption 4.3 and the prerequisites of Theorem 4.9 are satisfied for European call options and e.g.  $R = 1.1$ . Henceforth, we consider a European call with discounted strike  $K = 100$  and maturity  $T = 0.25$  years. The results for the variance-optimal initial hedge ratio  $\varphi_0^*$  for  $N = 1$  (static hedging) and  $N = 12$  (weekly rebalancing) are shown in Figures 1 and 2.

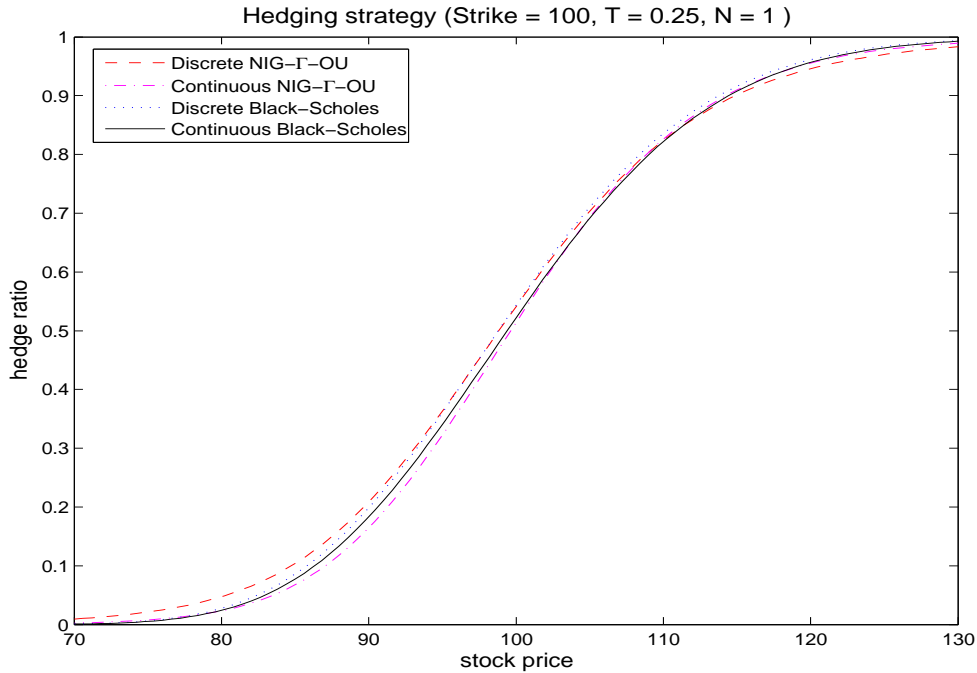


Figure 1: Variance-optimal initial hedge ratios for  $N = 1$ .

For static hedging, the impact of discretization seems to be quite pronounced, in particular for out-of-the-money options. Also notice that this effect turns out to be substantially bigger for the NIG-Gamma-OU than for the Black-Scholes model. For weekly rebalancing, the effect of discretization on the initial hedge ratio already becomes marginal. More specifically, the difference between the discrete- and continuous-time variance-optimal hedging strategies is barely visible in Figure 2. Figure 3 shows a simulated path of the discrete variance-optimal hedges for  $N = 1, 3, 12, 60$ .

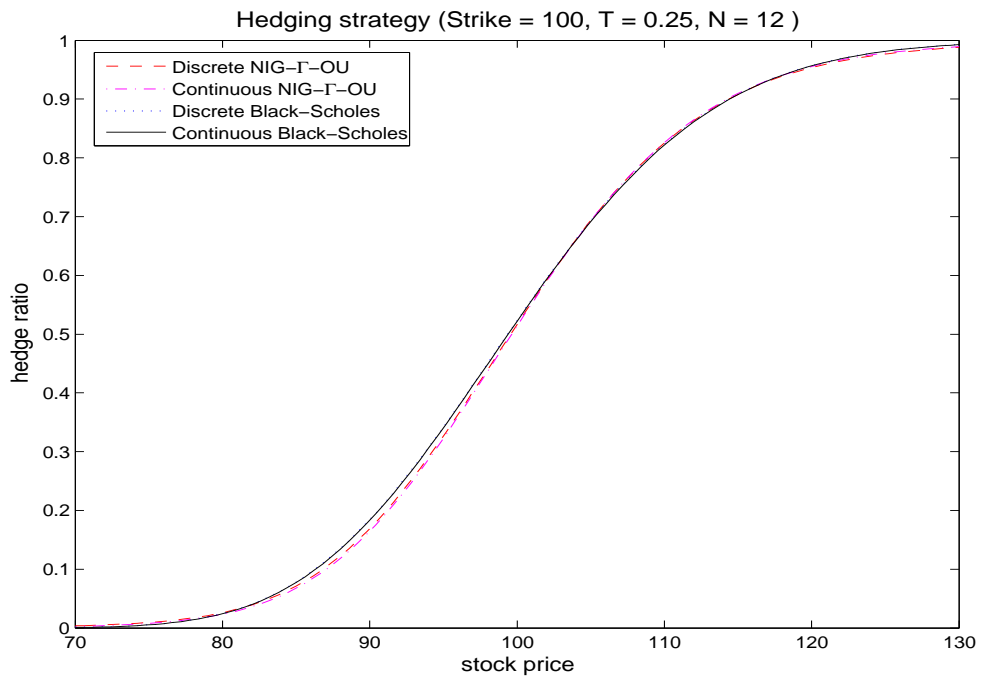


Figure 2: Variance-optimal initial hedge ratios for  $N = 12$ .

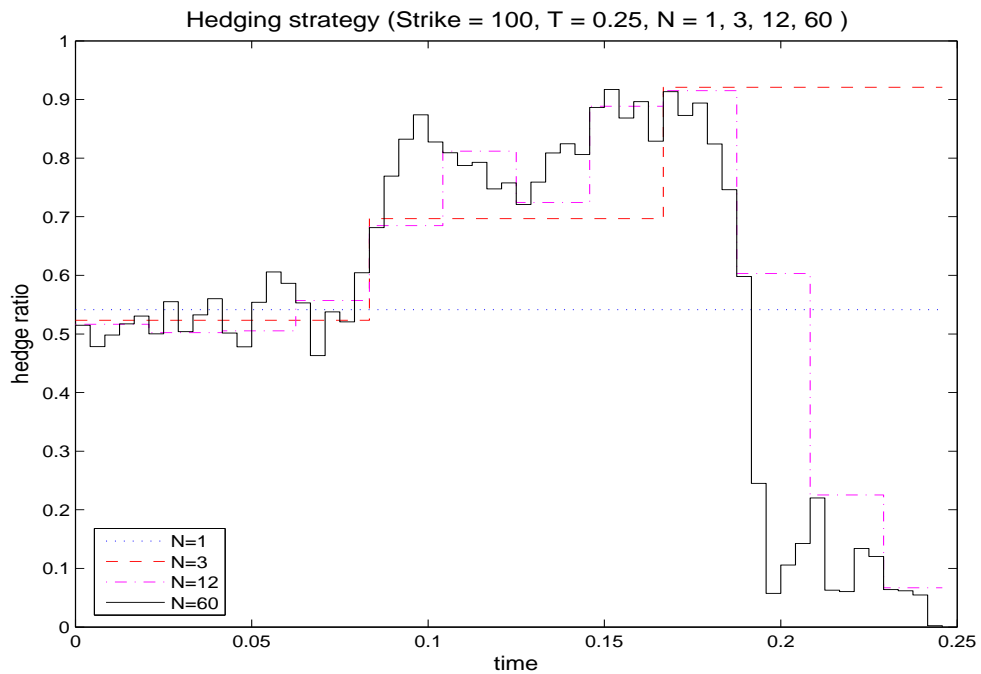


Figure 3: A simulated path of optimal hedge ratios for  $N = 1, 3, 12, 60$ .

We now turn to the minimal expected squared hedging error, which is depicted for  $N = 1$  (static hedging) to  $N = 60$  (daily rebalancing) in Figure 4.

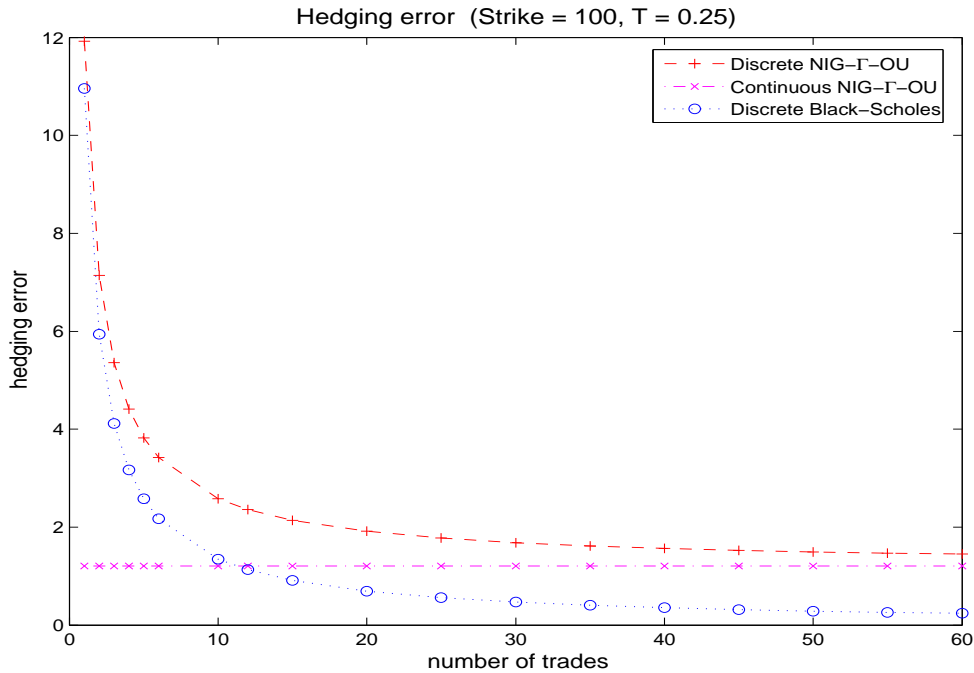


Figure 4: Minimal expected squared hedging errors.

As the number  $N$  of trading dates tends to infinity, the discrete hedging errors approach the respective continuous-time limits both in the Black-Scholes model and in the NIG-Gamma-OU model. Naturally, this limit vanishes in the complete Black-Scholes model. As noticed above, the static hedging error for  $N = 1$  can be computed without using any approximations. For  $N \geq 2$ , the discrete-time hedging error in the given NIG-Gamma-OU model is approximated surprisingly well by the sum of the respective continuous-time hedging error and the corresponding discrete-time hedging error in the Black-Scholes model. In fact, the maximal absolute difference is smaller than 0.045. If such an approximation can be used for the specific model at hand, computation time can often be drastically reduced by evaluating the formulas from [10, 12] instead of Theorem 4.9.

Note that the discrete hedging errors in the NIG-Gamma-OU model have been approximated using Theorem 4.9. Since the corresponding results for a simulation study using one million Monte-Carlo runs differ by less than 2.5% for  $N = 1, \dots, 60$ , we do not show them here. However, in Figure 5, we use the results of the Monte-Carlo study to depict an approximation of the *distribution* of the hedging error for  $N = 1$ ,  $N = 12$  and  $N = 60$ . Apparently, not only the variance of the hedging error but also its law depend crucially on the rebalancing frequency.

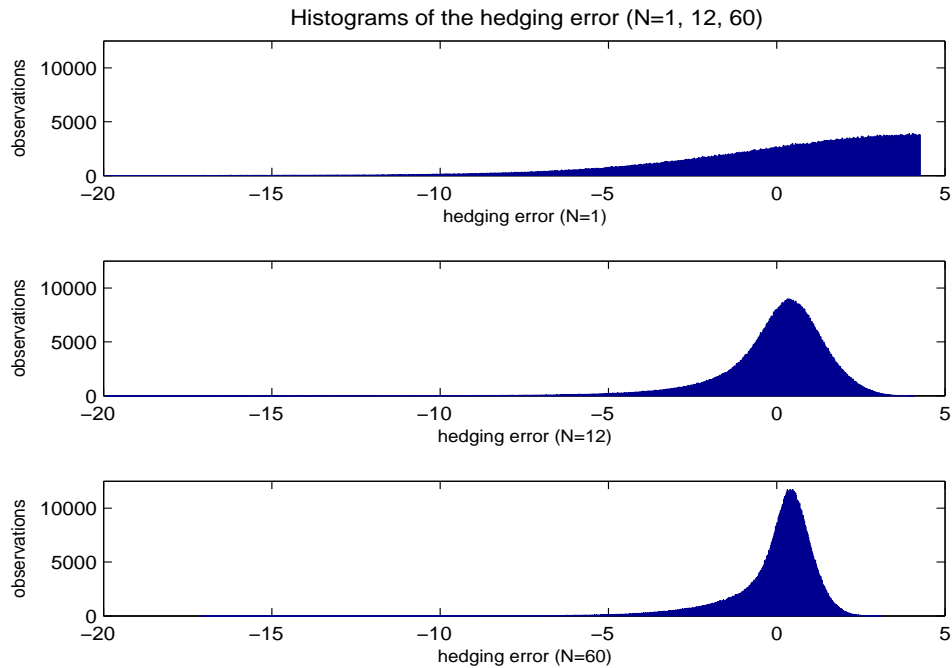


Figure 5: Approximated distribution of the hedging error for  $N = 1, 12, 60$ .

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