Utility maximization in affine stochastic volatility models

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Abstract

We consider the classical problem of maximizing expected utility from terminal wealth. With the help of a martingale criterion explicit solutions are derived for power utility in a number of affine stochastic volatility models.

Key words: portfolio optimization, stochastic volatility, martingale method

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1 Introduction

A classical problem in Mathematical Finance is to maximize expected utility from terminal wealth in a securities market (cf. [24, 25] for an overview). This *Merton problem* is generally tackled either by methods from stochastic control theory, which lead to Hamilton-Jacobi-Bellmann equations, or by martingale methods, which appear in various forms. We use the second approach to derive explicit solutions for power utility in a number of stochastic volatility models. This extends earlier results for Lévy processes (cf. [3, 10, 18]), the Heston model (cf. [27]), and the Barndorff-Nielsen-Shephard model (cf. [4]). The key idea in the current paper is to represent the optimal strategy in terms of an *opportunity process* as it is used in [7] for quadratic hedging problems. In some asset price models this opportunity process can be computed explicitly, which in turn leads quickly to the solution of the utility maximization problem.

The goal of the paper is twofold. Firstly, we solve the portfolio selection problem in a rather complex setup allowing for some of the stylized facts observed in real data, namely jumps and stochastic volatility. Secondly, we indicate that the combination of a martingale approach, the notion of an opportunity process, and the calculus of semimartingale characteristics turns out to be very useful both for deriving candidate solutions and for verification.

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The paper is organized as follows. In the following section we derive a general sufficient condition for optimality of a given candidate strategy. This condition is applied in Section 3 to affine stochastic volatility models. The paper relies heavily on the calculus of semimartingale characteristics. Main results in this context are summarized in the appendix for the convenience of the reader.

For stochastic background and notation we refer to [17]. In particular, for a semimartingale X, we denote by $L(X)$ the set of X-integrable predictable processes and by $\varphi \bullet X$ the stochastic integral of $\varphi \in L(X)$ with respect to X. We write $\mathscr{E}(X)$ for the stochastic exponential of a semimartingale X and denote by $\mathscr{L}(Z) = \frac{1}{Z_-} \bullet Z$ the *stochastic logarithm* of a semimartingale Z satisfying $Z, Z_- \neq 0$ (cf. [17, II.8.3] for more details). When dealing with stochastic processes, superscripts usually refer to coordinates of a vector rather than powers. By I we denote the identity process, i.e. $I_t = t$.

2 The opportunity process in utility maximization

Our mathematical framework for a frictionless market model is as follows. Fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ in the sense of [17, I.1.2]. We consider traded securities whose price processes are expressed in terms of multiples of a numeraire security. More specifically, these securities are modelled by their discounted price process $S = (S^1, \dots, S^d)$ which is assumed to be an $\mathbb{R}^d$-valued semimartingale. We consider an investor who tries to maximize expected utility from terminal wealth. Her initial endowment is denoted by $v \in (0, \infty)$. *Trading strategies* are modelled by $\mathbb{R}^d$-valued predictable stochastic processes $\varphi = (\varphi^1, ..., \varphi^d) \in L(S)$, where φ^i_t denotes the number of shares of security i in the investor's portfolio at time t. A strategy φ is called *admissible* if its discounted *value process* $V(\varphi) := v + \varphi \bullet S$ is nonnegative (no debts allowed).

Definition 2.1 A *utility function* is a strictly increasing, strictly concave function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$, which is continuously differentiable on $(0, \infty)$.

In the following, u denotes a general utility function. Later we will only consider *power utility functions* of the form $u(x) = \frac{x^{1-p}}{1-p}$ for $p \in \mathbb{R}_+ \backslash \{0, 1\}$ or alternatively $u(x) = \log(x)$.

Definition 2.2 We say that an admissible strategy φ is *optimal for terminal wealth* if it maximizes $\psi \mapsto E(u(V_T(\psi)))$ over all admissible strategies ψ.

We now state a sufficient condition for optimality in terms of martingales.

Proposition 2.3 *Let φ be an admissible strategy and suppose there exists a positive semimartingale L with $L_T = 1$ such that both $Lu'(V(\varphi))$ and $Lu'(V(\varphi))S$ are σ-martingales. Then $Lu'(V(\varphi))V(\varphi)$ is a σ-martingale as well. If it is a martingale, φ is optimal for terminal wealth.*

PROOF. Set $Z := Lu'(V(\varphi))$ and denote by ψ any admissible strategy. Integration by parts yields

$$\begin{aligned} ZV(\psi) &= Z_0v + Z_- \bullet V(\psi) + V_-(\psi) \bullet Z + [Z, V(\psi)] \\ &= Z_0v + v \bullet Z + \psi \bullet (Z_- \bullet S + S_- \bullet Z + [Z, S]) - (V_-(\psi) - \psi S_-) \bullet Z \\ &= vZ + \psi \bullet (ZS) - (V_-(\psi) - \psi S_-) \bullet Z. \end{aligned}$$

Hence $ZV(\psi)$ is a σ-martingale by [20, Lemma 3.3].

Now suppose $ZV(\varphi)$ is a martingale. Observe that $ZV(\psi)$ is a nonnegative σ-martingale with $E(Z_0V_0(\psi)) = E(Z_0V_0(\varphi)) < \infty$. Hence it is a supermartingale by [20, Proposition 3.1]. Since u is concave, this implies

$$\begin{aligned} E(u(V_T(\psi))) &\leq E(u(V_T(\varphi))) + E(u'(V_T(\varphi))(V_T(\psi) - V_T(\varphi))) \\ &= E(u(V_T(\varphi))) + E(Z_TV_T(\psi)) - E(Z_TV_T(\varphi)) \\ &\leq E(u(V_T(\varphi))), \end{aligned}$$

which proves the claim. □

Remark 2.4 If we forget about the difference between martingales and σ-martingales, $Z := Lu'(V(\varphi))/(L_0u'(v))$ is the density process of an equivalent martingale measure. This measure appears frequently in papers that apply martingale or duality methods to tackle the utility maximization problem. In the general context of [28] the process Z above solves a dual minimization problem. The results of [28] actually yield some kind of converse to Proposition 2.3. For an optimal strategy φ there typically exists a positive semimartingale L with $L_T = 1$ such that $Lu'(V(\varphi))V(\varphi)$ is a martingale and $Lu'(V(\varphi))V(\psi)$ is a supermartingale for any admissible strategy ψ. An inspection of the proof of Proposition 2.3 reveals that these two conditions actually suffice for the optimality of φ.

For power utility Proposition 2.3 yields the maximal expected utility as well.

Corollary 2.5 *Let* $u(x) = \frac{x^{1-p}}{1-p}$ *for* $p \in \mathbb{R}_+\backslash\{0, 1\}$ *and fix an admissible strategy* φ. *Suppose there exists a positive semimartingale* L *with* $L_T = 1$ *such that* $LV(\varphi)^{-p}$ *and* $LV(\varphi)^{-p}S$ *are* σ*-martingales. Then* $LV(\varphi)^{1-p}$ *is a* σ*-martingale as well. If it is a martingale,* φ *is optimal for terminal wealth and the maximal expected utility is given by*

$$E(u(V_T(\varphi))) = \frac{v^{1-p}}{1-p}E(L_0). \tag{2.1}$$

PROOF. The first two assertions follow directly from Proposition 2.3. If $LV(\varphi)^{1-p}$ is a martingale, we have

$$E(u(V_T(\varphi))) = \frac{1}{1-p}E(V_T(\varphi)^{1-p}) = \frac{1}{1-p}E(L_TV_T(\varphi)^{1-p}) = \frac{v^{1-p}}{1-p}E(L_0)$$

as claimed. □

Remark 2.6 Since only the derivative u' appears in this criterion, the same result except for (2.1) can be applied to logarithmic utility by setting $p = 1$. Since in this case $LV(\varphi)^{1-p}$ is supposed to be a martingale, the only possible choice is $L = 1$. Hence the sufficient condition of Corollary 2.5 reduces to finding an admissible strategy φ such that $1/V(\varphi)$ and $S/V(\varphi)$ are σ-martingales. For related conditions cf. [11, 12].

In view of Corollary 2.5 our approach for finding optimal strategies consists of three steps. The first is to make an appropriate ansatz for L and φ up to some yet unknown parameters or deterministic functions. In view of Lemma A.4 the σ-martingale property can be viewed as a drift condition, which is used to determine the unknown parameters in a second step. Finally one verifies that the obtained candidate processes L, φ indeed meet all conditions of Corollary 2.5, in particular that the σ-martingale $LV(\varphi)^{1-p}$ is in fact a true martingale.

Remark 2.7 As discussed in Remark 2.4, one can replace the two σ-martingale conditions in Corollary 2.5 by the weaker requirement that $LV(\varphi)^{-p}V(\psi)$ is a supermartingale for any admissible strategy ψ. In addition, the martingale property of $LV(\varphi)^{1-p}$ is needed as in Corollary 2.5.

The idea to state optimality in terms of a process L as in Corollary 2.5 is inspired by a similar approach of [7] in the context of quadratic hedging, where L is called *opportunity process*. It makes sense to use the same terminology here. Indeed, we have

$$\begin{aligned} E(u(V_T(\varphi))|\mathscr{F}_t) &= \frac{1}{1-p}E(L_TV_T(\varphi)^{1-p}|\mathscr{F}_t) \\ &= \frac{1}{1-p}L_tV_t(\varphi)^{1-p} \end{aligned} \tag{2.2}$$

and hence

$$L_t = (1-p)E\left(u\left(\frac{V_T(\varphi)}{V_t(\varphi)}\right)\middle|\mathscr{F}_t\right). \tag{2.3}$$

The optimal strategy φ has value $V_t(\varphi)$ at time t. One easily verifies that on $[\![t,T]\!]$, φ is the $V_t(\varphi)$-fold of the investment strategy ψ which starts with initial endowment 1 at time t and maximizes the expected utility at T. In view of (2.3) this means that L_t stands — up to a factor $1-p$ — for the maximal utility from trading between t and T with inital endowment 1. The parallel statement for quadratic utility inspired the term *opportunity pocess* in [7]. Moreover, (2.2) means that $LV(\varphi)^{1-p}/(1-p)$ corresponds to the *value function* used in stochastic control theory.

Making an appropriate ansatz for L and φ is very similar to the usual approach of guessing the form of the value function and applying the dynamic programming principle. When it comes to verification, however, the present approach avoids some technical problems. Indeed, for $p \leq 1$ the value function becomes negative which complicates the proof of the relevant supermartingale property. By contrast, the processes in Remark 2.7 remain positive.

3 Solution in affine stochastic volatility models

To avoid integrability issues at $t = 0$ we assume from now on that $\mathscr{F}_0$-measurable random variables are almost surely constant. Instead one could require the existence of certain exponential moments. Moreover, we consider a single risky asset (i.e. $d = 1$) but the results extend to multiple stocks in a straightforward manner.

For the application of the optimality criterion in Corollary 2.5 two problems have to be solved. First one needs an appropriate ansatz for the optimal strategy φ and the process L. Having chosen parameters such that the drift rates of $LV(\varphi)^{-p}$ and $LV(\varphi)^{-p}S$ vanish, one must then establish that the σ-martingale LV^{1-p} is a true martingale. Both problems can be solved in a number of affine stochastic volatility models in the sense of [21].

In these models, the "volatility" y and the stochastic logarithm X of a discounted security price

$$S = S_0\mathscr{E}(X) \tag{3.1}$$

are modelled as a bivariate process with differential affine characteristics relative to y. Specifically, we assume that (y, X) is an $\mathbb{R}^2$-valued semimartingale such that its semimartingale characteristics (B, C, ν) (cf. [17]) are of the form

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0,t] \times G) = \int_0^t K_s(G) ds,$$

where the *differential characteristics* $\partial(y, X) := (b, c, K)$ (cf. the appendix for more details) are affine functions of y in the sense that

$$\begin{aligned} b_t &= \beta_0 + \beta_1 y_{t-}, \\ c_t &= \gamma_0 + \gamma_1 y_{t-}, \\ K_t(G) &= \kappa_0(G) + \kappa_1(G) y_{t-} \end{aligned} \tag{3.2}$$

for some Lévy-Khintchine triplets $(\beta_0, \gamma_0, \kappa_0)$, $(\beta_1, \gamma_1, \kappa_1)$ on $\mathbb{R}^2$ (cf. [21] for details).

In the case of Lévy processes (i.e. for $(\beta_1, \gamma_1, \kappa_1) = (0, 0, 0)$), the optimal strategy is known to invest a constant fraction of current wealth in the risky security, i.e. $\varphi_t = \alpha_1 \frac{V_{t-}(\varphi)}{S_{t-}}$ for some constant $\alpha_1 \in \mathbb{R}$ (cf. [18]). We replace the constant α_1 by some deterministic function $\alpha_1 \in L(X)$ for the more general class of models considered here. This leads to

$$V(\varphi) = v + \left(\alpha_1 \frac{V_-(\varphi)}{S_-}\right) \bullet S = v + V_-(\varphi) \bullet (\alpha_1 \bullet X). \tag{3.3}$$

Since α_1 is considered to be deterministic, the processes $(y, \mathscr{L}(V(\varphi)))$, $(y, \mathscr{L}(V(\varphi)^{-p}))$ turn out to be time-inhomogeneous affine processes in the sense of [9]. We guess that the opportunity process L is of exponentially affine form as well, more specifically

$$L_t = \exp(\alpha_2(t) + \alpha_3(t) y_t)$$

with deterministic functions $\alpha_2, \alpha_3 : [0, T] \to \mathbb{R}$. In order to have $L_T = 1$ we need $\alpha_2(T) = \alpha_3(T) = 0$. Up to the concrete form of α_1, α_2, α_3, we have specified candidate processes φ,

L. The functions are chosen such that the required σ-martingale property holds (cf. the proof of Theorems 3.2 and 3.3). In order to show the true martingale property of $LV(\varphi)^{1-p}$ we use results of [23] which state that exponentially affine σ-martingales are martingales under weak assumptions. The above ansatz works for important subclasses of affine processes but not for all, cf. the discussion at the end of Section 3.2.

Remark 3.1 In the literature, the asset price is sometimes modelled as ordinary exponential $S_t = S_0 \exp(X_t)$ with some bivariate affine process (y, X). In this case we have $S_t = S_0 \mathscr{E}(\widetilde{X}_t)$ with some bivariate affine process $(y, \widetilde{X})$ (cf. [23, Lemma 2.7]). Hence we are in the setup considered above.

3.1 Heston (1993)

We first consider the model of [14], given by (3.1) and the following stochastic differential equations (SDE's):

$$\begin{aligned} dX_t &= \mu y_t dt + \sqrt{y_t} dW_t, \\ dy_t &= (\vartheta - \lambda y_t) dt + \sigma \sqrt{y_t} dZ_t. \end{aligned} \tag{3.4}$$

Here, $\vartheta \geq 0$, $\lambda > 0$, μ, $\sigma \neq 0$ denote constants and W, Z Wiener processes with constant correlation ϱ. Calculation of the differential characteristics as in [21] yields that (y, X) is an affine process with triplets

$$\begin{aligned} (\beta_0, \gamma_0, \kappa_0) &= \left(\begin{pmatrix} \vartheta \\ 0 \end{pmatrix}, 0, 0 \right), \\ (\beta_1, \gamma_1, \kappa_1) &= \left(\begin{pmatrix} -\lambda \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & \sigma\varrho \\ \sigma\varrho & 1 \end{pmatrix}, 0 \right). \end{aligned}$$

With Corollary 2.5 and the approach outlined above, we obtain optimal strategies for power utility functions.

Theorem 3.2 *Let* $v > 0$, $u(x) = \frac{x^{1-p}}{1-p}$ *for some* $p \in \mathbb{R}_+ \backslash \{0, 1\}$ *and*

$$\begin{aligned} a &:= -\frac{\sigma^2}{2} - \frac{1-p}{2p}\sigma^2\varrho^2, \quad b := \lambda - \frac{1-p}{p}\sigma\varrho\mu, \quad c := -\frac{1-p}{2p}\mu^2, \\ D &:= b^2 - 4ac = \lambda^2 - \frac{1-p}{p}\left(2\lambda\sigma\varrho\mu + \sigma^2\mu^2\right). \end{aligned}$$

Case 1: If $D > 0$, *define*

$$\alpha_3(t) := -2c \frac{e^{\sqrt{D}(T-t)} - 1}{e^{\sqrt{D}(T-t)}(b + \sqrt{D}) - b + \sqrt{D}}.$$

Case 2: If $D = 0$ *and either* $b > 0$ *or* $b < 0$, $T < -2/b$, *define*

$$\alpha_3(t) := \frac{1}{a(T - t + 2/b)} - \frac{b}{2a}.$$

Case 3: If $D < 0$ and either $b > 0$, $T < \frac{2}{\sqrt{-D}}(\pi - \arctan(\frac{\sqrt{-D}}{b}))$, or $b = 0$, $T < \frac{\pi}{\sqrt{-D}}$, or $b < 0$, $T < \frac{2}{\sqrt{-D}} \arctan(\frac{\sqrt{-D}}{-b})$, define

$$\alpha_3(t) := -2c \frac{\sin(\frac{\sqrt{-D}}{2}(T-t))}{\sqrt{-D}\cos(\frac{\sqrt{-D}}{2}(T-t)) + b\sin(\frac{\sqrt{-D}}{2}(T-t))}.$$

Then in each case, the optimal strategy in the Heston model given initial endowment v is

$$\varphi_t := \alpha_1(t)\frac{v\mathscr{E}(\alpha_1 \bullet X)_t}{S_t},$$

where

$$\alpha_1(t) := \frac{\mu + \sigma\varrho\alpha_3(t)}{p}.$$

The corresponding discounted value process is $V(\varphi) = v\mathscr{E}(\alpha_1 \bullet X)$ and the maximal expected utility is given by

$$E(u(V_T(\varphi))) = \exp\left(\vartheta \int_0^T \alpha_3(s)ds + \alpha_3(0)y_0\right)\frac{v^{1-p}}{1-p}. \tag{3.5}$$

PROOF. Set $V := v\mathscr{E}(\alpha_1 \bullet X)$. Then

$$V(\varphi) = v + \varphi \bullet S = v + \left(\alpha_1 \frac{V}{S}\right) \bullet S = v + V \bullet (\alpha_1 \bullet X) = V,$$

i.e. V is the value process of φ. Since X is continuous, φ is admissible by [17, I.4.61]. Now we define $\alpha_2(t) := \vartheta \int_t^T \alpha_3(s)ds$. Note that the denominator does not vanish on $[0,T]$ in all three cases. Thus α_2 and α_3 belong to $C^\infty([0,T],\mathbb{R})$. In view of [5, 21.5.1.2] and $|b| > \sqrt{D}$ for $D > 0$ or by direct calculation, they solve the following terminal value problems:

$$\alpha_2'(t) = -\vartheta\alpha_3(t), \qquad \alpha_2(T) = 0, \tag{3.6}$$
$$\alpha_3'(t) = a\alpha_3^2(t) + b\alpha_3(t) + c, \quad \alpha_3(T) = 0. \tag{3.7}$$

Set $L_t := \exp(\alpha_2(t) + \alpha_3(t)y_t)$. Integration by parts yields

$$\alpha_2(I) + \alpha_3(I)y - \alpha_2(0) - \alpha_3(0)y_0 = (\alpha_2'(I) + \alpha_3'(I)y) \bullet I + \alpha_3(I) \bullet y,$$

where $I_t = t$ denotes the identity process. We can calculate the differential characteristics $(b, c, K) = \partial(y, L, S, V)$ of (y, L, S, V) in the following steps:

$$\partial\begin{pmatrix} y \\ I \\ X \end{pmatrix} \overset{\text{Prop. A.2}}{\longrightarrow} \partial\begin{pmatrix} y \\ \alpha_2(I) + \alpha_3(I)y \\ X \end{pmatrix} \overset{\text{Prop. A.3}}{\longrightarrow} \partial\begin{pmatrix} y \\ L \\ X \end{pmatrix} \overset{\text{Prop. A.2}}{\longrightarrow} \partial\begin{pmatrix} y \\ L \\ S \\ V \end{pmatrix}.$$

Inserting the definition of α_2 we obtain

$$b_t = \begin{pmatrix} \vartheta - \lambda y_t \\ L_t(\alpha_3'(t) - \lambda\alpha_3(t) + \frac{\sigma^2}{2}\alpha_3^2(t))y_t \\ S_t\mu y_t \\ V_t\alpha_1(t)\mu y_t \end{pmatrix},$$

$$c_t = \begin{pmatrix} \sigma^2 & L_t\alpha_3(t)\sigma^2 & S_t\sigma\varrho & V_t\sigma\varrho\alpha_1(t) \\ L_t\alpha_3(t)\sigma^2 & L_t^2\alpha_3^2(t)\sigma^2 & L_tS_t\alpha_3(t)\sigma\varrho & L_tV_t\sigma\varrho\alpha_1(t)\alpha_3(t) \\ S_t\sigma\varrho & L_tS_t\alpha_3(t)\sigma\varrho & S_t^2 & S_tV_t\alpha_1(t) \\ V_t\sigma\varrho\alpha_1(t) & L_tV_t\sigma\varrho\alpha_1(t)\alpha_3(t) & S_tV_t\alpha_1(t) & V_t^2\alpha_1^2(t) \end{pmatrix} y_t, \tag{3.8}$$

$$K_t = 0.$$

Since $V > 0$, we can then calculate the first differential characteristic of LV^{-p}, $LV^{-p}S$ by applying Proposition A.3. If we denote them by $b^{LV^{-p}}$ and $b^{LV^{-p}S}$, respectively, and insert $\alpha_1(t) = \frac{\sigma\varrho}{p}\alpha_3(t) + \frac{\mu}{p}$, we have

$$\begin{aligned} b_t^{LV^{-p}} &= L_tV_t^{-p}(\alpha_3'(t) - a\alpha_3^2(t) - b\alpha_3(t) - c)y_t = 0, \\ b_t^{LV^{-p}S} &= L_tV_t^{-p}S_t(\alpha_3'(t) - a\alpha_3^2(t) - b\alpha_3(t) - c)y_t = 0. \end{aligned}$$

In view of Lemma A.4, this implies that both processes are σ-martingales. Hence LV^{1-p} is a σ-martingale as well by Corollary 2.5.

In the present Itô process setup we could have avoided the use of semimartingale characteristics by working with the usual Itô process representation instead. Standard calculations yield that the drift part of LV^{-p} and $LV^{-p}S$ vanishes, which means that they are local and hence in particular σ-martingales. We work here with the less common notion of semimartingale characteristics because it allows for a more or less unified treatment of processes with and without jumps. This will become more apparent in the following section.

From Equations (3.8) and Propositions A.2, A.3 it follows that the differential characteristics $(b^*, c^*, F^*) = \partial(y, \mathscr{L}(LV^{1-p}))$ of $(y, \mathscr{L}(LV^{1-p}))$ are given by

$$b_t^* = \begin{pmatrix} \vartheta - \lambda y_t \\ 0 \end{pmatrix},$$

$$c_t^* = \begin{pmatrix} \sigma^2 & \sigma^2\alpha_3(t) + (1-p)\sigma\varrho\alpha_1(t) \\ \sigma^2\alpha_3(t) + (1-p)\sigma\varrho\alpha_1(t) & \sigma^2\alpha_3^2(t) + 2(1-p)\sigma\varrho\alpha_1(t)\alpha_3(t) + (1-p)^2\alpha_1^2(t) \end{pmatrix} y_t,$$

$$K_t^* = 0.$$

Since α_1 and α_3 are continuous, $\partial(y, \mathscr{L}(LV^{-p}))$ are affine relative to strongly admissible Lévy-Khintchine triplets in the sense of [23]. The conditions of [23, Theorem 3.1] are obviously satisfied, hence $v^{1-p}L_0\mathscr{E}(\mathscr{L}(LV^{1-p})) = LV^{1-p}$ is a martingale. In view of Corollary 2.5 we are done. □

Remarks.

1. For $p \geq 1$, the solution to Case 1 is derived by stochastic control methods in [27]. Case 3 appears on an informal level in [31]. Observe that Theorem 3.2 does not provide a solution beyond some critical time horizon T_∞, which may be finite for $p < 1$ in Cases 2 and 3. A straightforward analysis of (3.5) shows that the maximal expected utility increases to ∞ as T tends to T_∞ if the latter is finite. On the other hand, the optimal expected utility is generally an increasing function of the time horizon because one can always stop investing in the risky asset. Consequently, we have

$$\sup\{E(u(V_T(\varphi))) : \varphi \text{ admissible strategy}\} = \infty$$

   for $T_\infty \leq T < \infty$, which means that no optimal strategy with finite expected utility exists in this case. This complements related discussions in [15, 26].

2. Setting $p = 1$ in the proof of Theorem 3.2 yields the optimal portfolio for logarithmic utility. In this case $\log(V(\varphi))$ has a Heston-type dynamics similar to the process X in (3.4). Therefore an explicit formula for $E(u(V_T(\varphi)))$ could be stated for this case as well. We leave its derivation to the reader because it does not convey additional insight.

   As is well known, the optimal portfolio for logarithmic utility is myopic and can be computed in closed form for virtually any semimartingale model, cf. e.g. [12] and the references therein. In particular, an affine structure is not required.

3. In contrast to the next section, using the more general alternative approach of Remarks 2.4 resp. 2.7 does not lead to a more general statement in Theorem 3.2.

4. It has been observed repeatedly that portfolio selection problems are linked to some kind of distance minimization in the set of equivalent martingale measures (cf. e.g. [2, 13, 28, 19]). Let us briefly discuss how this is reflected in the present setup.

   Along the same lines as for LV^{1-p} one shows that LV^{-p} and $LV^{-p}S$ are martingales. This implies that $Z := LV^{-p}/(L_0 v^{-p})$ is the density process of an equivalent martingale measure. This measure minimizes the "L^q-distance" $E(-\operatorname{sgn}(q)(dQ/dP)^q)$ for $q := 1 - \frac{1}{p} \in (-\infty, 1)$ among all equivalent martingale measures Q, i.e. all probability measures $Q \sim P$ such that S is a Q-σ-martingale.

   Indeed, $L_T = 1$ implies that $LV^{-p}S$ is a martingale with terminal value $u'(V_T)S_T$. By [28, Theorem 2.2] the same holds for the solution process $\hat{Y}$ to the dual problem

$$\inf_{Y \in \mathscr{Y}(y)} E\left(\frac{p}{1-p} Y_T^{\frac{p-1}{p}}\right), \tag{3.9}$$

   where

$$\mathscr{Y}(y) := \{Y \geq 0 : Y_0 = y \text{ and } YV(\psi) \text{ is a supermartingale for all admissible } \psi\}$$

and the constant $y > 0$ is appropriately chosen. In view of $L_0 V_0^{-p} S_0 = \hat{Y}_0 S_0$, this implies $y = L_0 v^{-p}$. Moreover, $Z = \hat{Y}/\hat{Y}_0$ minimizes $Y \mapsto E(-\mathrm{sgn}(q) Y_T^q)$ among all $Y \in \mathscr{Y}(1)$ and hence *a fortiori* in the set of density processes of equivalent martingale measures.

The q-*optimal* equivalent martingale measure from the previous discussion is needed as a first step in the derivation of asymptotic utility-based option prices and hedging strategies according to [29, 30]. As a side remark, the corresponding measure in Heston's model for $q > 1$ is computed in [15].

3.2 Carr et al. (2003)

We now turn to a stochastic volatility model with jumps in y and $X = \mathscr{L}(S)$ as introduced by [6]:

$$\begin{aligned} X_t &= B_{Y_t}, \\ dY_t &= y_{t-} dt, \\ dy_t &= -\lambda y_{t-} dt + dZ_t. \end{aligned} \tag{3.10}$$

Here, $\lambda \neq 0$ is a constant and B, Z denote independent Lévy processes with Lévy-Khintchine triplets (b^B, c^B, K^B) and $(b^Z, 0, K^Z)$, respectively. Z is supposed to be increasing. Since we want to apply the results of [23], all triplets on $\mathbb{R}^n$ (with n depending on the process under consideration) are stated relative to the componentwise truncation function $h(x) = (h_1(x), \dots, h_n(x))$ with

$$h_i(x) = \begin{cases} 0, & \text{if } x = 0, \\ (1 \wedge |x_i|) \frac{x_i}{|x_i|}, & \text{otherwise.} \end{cases}$$

As shown in [21], the process (y, X) is affine in the sense of (3.2) with triplets $(\beta_j, \gamma_j, \kappa_j)$, $j = 0, 1$ given by

$$\beta_0 = \begin{pmatrix} b^Z \\ 0 \end{pmatrix}, \quad \gamma_0 = 0, \quad \kappa_0(G) = \int 1_G(z, 0) K^Z(dz) \quad \forall G \in \mathscr{B}^2,$$
$$\beta_1 = \begin{pmatrix} -\lambda \\ b^B \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & c^B \end{pmatrix}, \quad \kappa_1(G) = \int 1_G(0, x) K^B(dx) \quad \forall G \in \mathscr{B}^2.$$

As above, we can determine optimal strategies for power utility functions.

Theorem 3.3 *Let $v > 0$ and $u(x) = \frac{x^{1-p}}{1-p}$ for some $p \in \mathbb{R}_+ \backslash \{0, 1\}$. Assume there exists some $\alpha_1 \in \mathbb{R}$ such that the following conditions hold.*

1. $K^B(\{x \in \mathbb{R} : 1 + \alpha_1 x \leq 0\}) = 0$
2. $\int |x(1 + \alpha_1 x)^{-p} - h(x)| \, K^B(dx) < \infty$

3.

$$b^B - pc^B\alpha_1 + \int \left(\frac{x}{(1+\alpha_1 x)^p} - h(x) \right) K^B(dx) = 0 \tag{3.11}$$

4. If $p \in (0,1)$*, then* $\int_1^\infty e^{\alpha_3(0)z} K^Z(dz) < \infty$*, where*

$$\alpha_3(t) := \frac{e^{-\lambda(T-t)} - 1}{\lambda} \left(\frac{p(p-1)}{2} c^B \alpha_1^2 - \int \left(\frac{1 + p\alpha_1 x}{(1+\alpha_1 x)^p} - 1 \right) K^B(dx) \right). \tag{3.12}$$

Then

$$\varphi_t = \alpha_1 \frac{v\mathscr{E}(\alpha_1 X)_{t-}}{S_{t-}}$$

is optimal for initial endowment v *in model* (3.10) *with value process* $V(\varphi) = v\mathscr{E}(\alpha_1 X)$ *and maximal expected utility*

$$\begin{aligned} &E(u(V_T(\varphi))) \\ &= \exp\left(\int_0^T \left(b^Z \alpha_3(s) + \int \left(e^{\alpha_3(s)z} - 1 - \alpha_3(s)h(z) \right) K^Z(dz) \right) ds + \alpha_3(0) y_0 \right) \frac{v^{1-p}}{1-p}. \end{aligned}$$

PROOF. The general approach is the same as in the proof of Theorem 3.2, apart from the fact that we have to deal with jumps in y and X. Set $V := v\mathscr{E}(\alpha_1 X)$. As in the proof of Theorem 3.2 it follows that V is the value process of φ. We have

$$\begin{aligned} E\left(\sum_{t \le T} 1_{(-\infty,0]}(1 + \alpha_1 \Delta X_t) \right) &= E\left(1_{(-\infty,0]}(1 + \alpha_1 x) * \mu_T^X \right) \\ &= E\left(1_{(-\infty,0]}(1 + \alpha_1 x) * \nu_T^X \right) = 0 \end{aligned}$$

by [17, II.1.8] and Condition 1. Hence $P(\exists t \in [0,T] : \alpha_1 \Delta X_t \le -1) = 0$. By [17, I.4.61] this implies that V is positive. Therefore φ is admissible.

A second order Taylor expansion yields that $\frac{1+p\alpha_1 x}{(1+\alpha_1 x)^p} - 1 = O(x^2)$ for $x \to 0$. Together with Condition 2 this implies that α_3 is well defined because K^B is a Lévy measure. Set

$$\alpha_2(t) := \int_t^T \left(b^Z \alpha_3(s) + \int (e^{\alpha_3(s)z} - 1 - \alpha_3(s)h(z)) K^Z(dz) \right) ds.$$

If $p \in (0,1)$, then α_3 is positive and decreasing. Hence Condition 4 ensures that α_2 is finite valued. If $p \in (1,\infty)$, Condition 4 is automatically satisfied: indeed, Condition 1 and the Bernoulli inequality imply that α_3 is negative in this case, which in turn yields that α_2 is finite because K^Z is concentrated on $\mathbb{R}_+$ (cf. [33, Theorem 21.5]). The functions $\alpha_2, \alpha_3 \in C^1([0,T], \mathbb{R})$ solve the following terminal value problems:

$$\alpha_3'(t) = \lambda\alpha_3(t) + \frac{p(p-1)}{2} c^B \alpha_1^2 - \int \left(\frac{1 + p\alpha_1 x}{(1+\alpha_1 x)^p} - 1 \right) K^B(dx), \quad \alpha_3(T) = 0, \tag{3.13}$$

$$\alpha_2'(t) = -b^Z \alpha_3(s) - \int (e^{\alpha_3(s)z} - 1 - \alpha_3(s)h(z)) K^Z(dz), \qquad \alpha_2(T) = 0.$$

We set $L_t := \exp(\alpha_2(t) + \alpha_3(t) y_t)$ and calculate the differential characteristics $(b, c, K) := \partial(y, L, S, V)$ along the same lines as in the proof of Theorem 3.2. A straightforward but tedious calculation and inserting the definition of α_2 yields

$$\begin{aligned}
b_t^1 &= b^Z - \lambda y_{t-}, \\
b_t^2 &= L_{t-}(\alpha_3'(t) - \lambda\alpha_3(t)) y_{t-} + \int \left(h(L_{t-}(e^{\alpha_3(t)z} - 1)) - L_{t-}(e^{\alpha_3(t)z} - 1) \right) K^Z(dz), \\
b_t^3 &= S_{t-} b^B y_{t-} + \int (h(S_{t-}x) - S_{t-}h(x)) K^B(dx) y_{t-}, \\
b_t^4 &= V_{t-}\alpha_1 b^B y_{t-} + \int (h(V_{t-}\alpha_1 x) - V_{t-}\alpha_1 h(x)) K^B(dx) y_{t-},
\end{aligned}$$

as well as

$$c_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_{t-}^2 c^B & S_{t-}V_{t-}\alpha_1 c^B \\ 0 & 0 & S_{t-}V_{t-}\alpha_1 c^B & V_{t-}^2 \alpha_1^2 c^B \end{pmatrix} y_{t-},$$

and, for all $G \in \mathscr{B}^4$,

$$K_t(G) = \int 1_G(z, L_{t-}(e^{\alpha_3(t)z} - 1), 0, 0) K^Z(dz) + \int 1_G(0, 0, S_{t-}x, V_{t-}\alpha_1 x) K^B(dx) y_{t-}.$$

An application of Proposition A.3 yields the differential characteristics $\partial LV^{-p} = (b^{LV^{-p}}, c^{LV^{-p}}, K^{LV^{-p}})$ and $\partial LV^{-p}S = (b^{LV^{-p}S}, c^{LV^{-p}S}, K^{LV^{-p}S})$ of LV^{-p} and $LV^{-p}S$, respectively. We obtain

$$\begin{aligned}
b_t^{LV^{-p}} \quad = \quad & L_{t-}V_{t-}^{-p} y_{t-} \left(\alpha_3'(t) - \lambda\alpha_3(t) - p\alpha_1 b^B + \frac{1}{2} p(1+p)\alpha_1^2 c^B \right) \\
& + \int \Big(h\big(L_{t-}V_{t-}^{-p}((1+\alpha_1 x)^{-p} - 1)\big) + L_{t-}V_{t-}^{-p} p\alpha_1 h(x) \Big) K^B(dx) y_{t-} \\
& + \int \Big(h\big(L_{t-}V_{t-}^{-p}(e^{\alpha_3(t)z} - 1)\big) - L_{t-}V_{t-}^{-p}(e^{\alpha_3(t)z} - 1) \Big) K^Z(dz)
\end{aligned}$$

and

$$\begin{aligned}
K_t^{LV^{-p}}(G) \quad = \quad & \int 1_G\big(L_{t-}V_{t-}^{-p}((1+\alpha_1 x)^{-p} - 1)\big) K^B(dx) y_{t-} \\
& + \int 1_G\big(L_{t-}V_{t-}^{-p}(e^{\alpha_3(t)z} - 1)\big) K^Z(dz)
\end{aligned}$$

for $G \in \mathscr{B}$. Inserting b^B from Condition 3 and α_3' from (3.13), we finally get

$$b_t^{LV^{-p}} = \int (h(x) - x) K_t^{LV^{-p}}(dx), \tag{3.14}$$

which means that LV^{-p} is a σ-martingale (cf. Lemma A.4). A similar calculation yields

$$b_t^{LV^{-p}S} = \int (h(x) - x) K_t^{LV^{-p}S}(dx), \tag{3.15}$$

which means that $LV^{-p}S$ is a σ-martingale as well. Hence LV^{1-p} is a σ-martingale by Corollary 2.5. Once more applying Propositions A.2 and A.3, we obtain that $(y, \mathscr{L}(LV^{1-p}))$ is a bivariate time-inhomogeneous affine semimartingale in the sense of (3.2) relative to time-dependent triplets

$$\beta_0(t) = \begin{pmatrix} b^Z \\ \int(h(e^{\alpha_3(t)z}-1) - (e^{\alpha_3(t)z}-1))K^Z(dz) \end{pmatrix}, \quad \gamma_0(t) = 0,$$

$$\kappa_0(t, G) = \int 1_G(z, e^{\alpha_3(t)z}-1)K^Z(dz), \quad \forall G \in \mathscr{B}^2,$$

$$\beta_1(t) = \begin{pmatrix} -\lambda \\ \int(h((1+\alpha_1 x)^{1-p}-1) - ((1+\alpha_1 x)^{1-p}-1))K^B(dx) \end{pmatrix},$$

$$\gamma_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & (1-p)^2\alpha_1^2 c^B \end{pmatrix},$$

$$\kappa_1(t, G) = \int 1_G(0, (1+\alpha_1 x)^{1-p}-1)K^B(dx), \quad \forall G \in \mathscr{B}^2.$$

The martingale property of LV^{1-p} can now be established by verifying the sufficient conditions of [23, Theorem 3.1]. It is easy to see that the triplets are strongly admissible in the sense of [23]. Indeed, the continuity conditions follow from the continuity of α_3 and dominated convergence. The remaining assumptions of [23, Theorem 3.1] are also satisfied as can be easily checked. Hence LV^{1-p} is a martingale and the assertion follows from Corollary 2.5. □

Remarks.

1. An inspection of the proof reveals that Condition 4 can be replaced with the slightly weaker asumption that $\alpha_2(0)$ is finite. But as with Heston's model it may happen that Theorem 3.3 does not provide a solution for $p < 1$ beyond some finite time horizon T_∞. With additional effort one can show that the optimal expected utility is infinite for time horizons $T > T_\infty$.

2. Remarks 2 and 4 after Theorem 3.2 hold accordingly in the present setup.

3. For $\lambda = 0$ the proof of Theorem 3.3 remains valid if α_3 is replaced with

$$\alpha_3(t) := (t-T)\left(\frac{p(p-1)}{2}c^B\alpha_1^2 - \int\left(\frac{1+p\alpha_1 x}{(1+\alpha_1 x)^p} - 1\right)K^B(dx)\right).$$

   This case occurs in particular if $Z = 0$ as well, in which case y is constant and X is a Lévy process. An analogous modification applies in Theorem 3.4 below.

4. Suppose that B is chosen to be a Brownian motion with drift, more specifically $B_t = \mu + W_t$ with a standard Wiener process W and triplet $(b^B, c^B, K^B) = (\mu, 1, 0)$. Then

we obtain the dynamics of the model proposed by Barndorff-Nielsen and Shephard [1], more precisely

$$\begin{aligned} dX_t &= \mu y_{t-}dt + \sqrt{y_{t-}}dW_t, \\ dy_t &= -\lambda y_{t-}dt + dZ_t. \end{aligned}$$

In this case the asset price process is continuous and the first two conditions of Theorem 3.3 are automatically satisfied. The third then yields the optimal fraction of wealth in stock

$$\alpha_1 = \frac{\mu}{p}.$$

As for the integrability conditions on F^Z, we have

$$\alpha_3(0) = \frac{1-p}{2p}\mu^2\frac{1-e^{-\lambda T}}{\lambda}.$$

Portfolio selection in the Barndorff-Nielsen and Shephard model is studied using stochastic control methods by Benth et al. [4]. They allow for an additional constant drift term in the equation for X. On the other hand, they do not obtain closed-form expressions for the expected utility and for the density process of the corresponding q-optimal martingale measure.

Since the asset price now has jumps, the approach in Remarks 2.4 and 2.7 leads to a slightly more general result here.

Theorem 3.4 *In the setup of Theorem 3.3 replace Condition 3 by the following condition.*

3'. Suppose that

$$b^B - pc^B\alpha_1 + \int\left(\frac{x}{(1+\alpha_1 x)^p} - h(x)\right)K^B(dx) \geq 0$$

if there exists some $\gamma < \alpha_1$ such that $K^B(\{x \in \mathbb{R} : 1+\gamma x < 0\}) = 0$ and, moreover,

$$b^B - pc^B\alpha_1 + \int\left(\frac{x}{(1+\alpha_1 x)^p} - h(x)\right)K^B(dx) \leq 0$$

if there exists some $\gamma > \alpha_1$ such that $K^B(\{x \in \mathbb{R} : 1+\gamma x < 0\}) = 0$.

Moreover, let

$$\begin{aligned} \alpha_3(t) := {} & \frac{e^{-\lambda(T-t)}-1}{\lambda} \\ & \times\left((p-1)b^W\alpha_1 + \frac{p(1-p)}{2}c^B\alpha_1^2 - \int\left((1+\alpha_1 x)^{1-p} - 1 - (1-p)\alpha_1 h(x)\right)K^B(dx)\right). \end{aligned}$$

instead of (3.12). *Then the statement of Theorem 3.3 remains true in models satisfying NFLVR.*

PROOF. Observe that α_3 coincides with (3.12) if (3.11) holds. Up to (3.14) and (3.15) all statements in the previous proof still hold. Let ψ denote an admissible strategy. Since the market satisfies NFLVR, there exists an equivalent σ-martingale measure. Together with [17, I.2.27], admissibility therefore implies $\psi = 0$ on the set $\{V_-(\psi) = 0\}$. Consequently, we can write $\psi = \gamma V_-(\psi)/S_-$ for some predictable process γ. The nonegativity condition $V(\psi) \geq 0$ implies $\gamma_t \Delta X_t \geq -1$, which in turn means

$$K^B(\{x \in \mathbb{R} : 1 + \gamma_t x < 0\}) = 0 \tag{3.16}$$

outside some $dP \otimes dt$-null set. Similarly as in the previous proof, one computes the differential characteristics $\partial LV^{-p}V(\psi) = (b^{LV^{-p}V(\psi)}, c^{LV^{-p}V(\psi)}, K^{LV^{-p}V(\psi)})$ of $LV^{-p}V(\psi)$ and obtains

$$\begin{aligned} b_t^{LV^{-p}V(\psi)} &= \int (h(x) - x) K_t^{LV^{-p}V(\psi)}(dx) \\ &\quad + (\gamma_t - \alpha_1) \left(b^B - pc^B \alpha_1 + \int \Big(\frac{x}{(1 + \alpha_1 x)^p} - h(x) \Big) K^B(dx) \right) L_{t-} V_{t-}^{-p} V_{t-}(\psi) y_{t-}. \end{aligned}$$

Condition 3' and (3.16) yield

$$b_t^{LV^{-p}V(\psi)} + \int (x - h(x)) K_t^{LV^{-p}V(\psi)}(dx) \leq 0,$$

which means that $LV^{-p}V(\psi)$ is a σ-supermartingale by Lemma A.4. Since positive σ-supermartingales are supermartingales by [20, Proposition 3.1], the assertion follows now from Remark 2.7. □

Note that unlike in Theorem 3.2 above the dual minimizer $LV(\varphi)^{-p}/(L_0 v^{-p})$ is no longer guaranteed to be the density of an equivalent martingale measure. A related discussion for exponential Lévy models can be found in [16].

Since the q-optimal martingale measure does not have to exist, NFLVR is no longer satisfied automatically. However, by [8] combined with Lemma A.4 and [21, Proposition 2.7] the time-changed Lévy models considered here always admit an equivalent σ-martingale measure unless the Lévy process B is either a.s. increasing or decreasing.

The approach in this paper is not limited to the models presented here. It can be extended to other — but not all — affine stochastic volatility models. In order for the opportunity process L to be of exponentially affine form, one seems to need that the differential characteristics of X are a linear function of y with no additional constant part. For more details the interested reader is referred to [32]. Other rather straightforward extensions concern a superposition of Lévy-driven Ornstein-Uhlenbeck processes as in [1] instead of y in (3.10) or, alternatively, multivariate versions of the models in Sections 3.1, 3.2 with common volatility process y.

A Differential characteristics

This paper relies heavily on the calculus of semimartingale characteristics. For the convenience of the reader we summarize a few basic properties which can be found in [17] or [21], respectively.

To any $\mathbb{R}^d$-valued semimartingale X there is associated a triplet (B, C, ν) of *characteristics*, where B resp. C denote $\mathbb{R}^d$- resp. $\mathbb{R}^{d\times d}$-valued predictable processes and ν a random measure on $\mathbb{R}_+ \times \mathbb{R}^d$. The first characteristic B depends on a *truncation function* as e.g. $h(x) = |x|1_{\{|x|\leq 1\}}$, which is chosen a priori. The characteristics of most processes in applications are absolutely continuous in time, i.e. they can be written as

$$\begin{aligned} B_t &= \int_0^t b_s ds, \\ C_t &= \int_0^t c_s ds, \\ \nu([0,t]\times G) &= \int_0^t K_s(G) ds \quad \forall G \in \mathscr{B}^d, \end{aligned}$$

with predictable processes b, c and a transition kernel K from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$. In this case we call (b, c, K) the *differential characteristics* of X and denote them by ∂X. We implicitly assume that (b, c, K) is a good version in the sense that the values of c are non-negative symmetric matrices, $K_s(\{0\}) = 0$ and $\int (1 \wedge |x|^2) K_s(dx) < \infty$.

Proposition A.1 *An $\mathbb{R}^d$-valued semimartingale X with $X_0 = 0$ is a Lévy process if and only if it has a version (b, c, K) of the differential characteristics which does not depend on (ω, t). In this case (b, c, K) is equal to the Lévy-Khintchine triplet.*

From an intuitive viewpoint one can interpret differential characteristics as a local Lévy-Khintchine triplet. Very loosely speaking, a semimartingale with differential characteristics (b, c, K) resembles locally after t a Lévy process with triplet $(b, c, K)(\omega, t)$, i.e. with drift rate b, diffusion matrix c, and jump measure K.

Proposition A.2 *Let X be an $\mathbb{R}^d$-valued semimartingale and H an $\mathbb{R}^{n\times d}$-valued predictable process with $H^{j\cdot} \in L(X), j = 1, \dots, n$ (i.e. integrable with respect to X). If $\partial X = (b, c, K)$, then the differential characteristics of the $\mathbb{R}^n$-valued integral process*

$$H \bullet X := (H^{j\cdot} \bullet X)_{j=1,\dots,n}$$

are given by $\partial(H \bullet X) = (\widetilde{b}, \widetilde{c}, \widetilde{K})$, where

$$\begin{aligned} \widetilde{b}_t &= H_t b_t + \int (\widetilde{h}(H_t x) - H_t h(x)) K_t(dx), \\ \widetilde{c}_t &= H_t c_t H_t^\top, \\ \widetilde{K}_t(G) &= \int 1_G(H_t x) K_t(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G. \end{aligned}$$

Here $\widetilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the truncation function which is used on $\mathbb{R}^n$.

The combination of the previous two rules yields that we have

$$b_t = \mu_t, \quad c_t = \sigma_t^2, \quad K_t = 0$$

for the differential characteristics $\partial X = (b, c, K)$ of an Itô process X of the form

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

Itô's formula for differential characteristics reads as follows:

Proposition A.3 *Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics $\partial X = (b, c, K)$. Suppose that $f : U \to \mathbb{R}^n$ is twice continuously differentiable on some open subset $U \subset \mathbb{R}^d$ such that X, X_- are U-valued. Then the $\mathbb{R}^n$-valued semimartingale $f(X)$ has differential characteristics $\partial(f(X)) = (\widetilde{b}, \widetilde{c}, \widetilde{K})$, where*

$$\begin{aligned}
\widetilde{b}_t^i &= \sum_{k=1}^d D_k f^i(X_{t-}) b_t^k + \frac{1}{2} \sum_{k,l=1}^d D_{kl} f^i(X_{t-}) c_l^{kl} \\
&\quad + \int \left(\widetilde{h}^i(f(X_{t-} + x) - f(X_{t-})) - \sum_{k=1}^d D_k f^i(X_{t-}) h^k(x) \right) K_t(dx), \\
\widetilde{c}_t^{ij} &= \sum_{k,l=1}^d D_k f^i(X_{t-}) c_t^{kl} D_l f^j(X_{t-}), \\
\widetilde{K}_t(G) &= \int 1_G(f(X_{t-} + x) - f(X_{t-})) K_t(dx) \quad \forall G \in \mathscr{B}^n \text{ with } 0 \notin G.
\end{aligned}$$

The σ-martingale property can be directly read from the triplet (cf. [20] for further background).

Lemma A.4 *Let X be a semimartingale with differential characteristics (b, c, K). Then X is a σ-martingale (resp. σ-supermartingale) if and only if $\int_{\{|x|>1\}} |x| K(dx) < \infty$ and*

$$b + \int (x - h(x)) K(dx) = 0 \quad \text{(resp. } \leq 0\text{)}$$

hold outside some $dP \otimes dt$-null set.

PROOF. Cf. e.g. [22, Lemma A.2]. □

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