

## ON USING SHADOW PRICES IN PORTFOLIO OPTIMIZATION WITH TRANSACTION COSTS

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In frictionless markets, utility maximization problems are typically solved either by stochastic control or by martingale methods. Beginning with the seminal paper of Davis and Norman [*Math. Oper. Res.* **15** (1990) 676–713], stochastic control theory has also been used to solve various problems of this type in the presence of proportional transaction costs. Martingale methods, on the other hand, have so far only been used to derive general structural results. These apply the duality theory for frictionless markets typically to a fictitious *shadow price process* lying within the bid-ask bounds of the real price process.

In this paper, we show that this dual approach can actually be used for both deriving a candidate solution and verification in Merton's problem with logarithmic utility and proportional transaction costs. In particular, we determine the shadow price process.

**1. Introduction.** A basic question in mathematical finance is how to choose an optimal investment strategy in a securities market or, more specifically, how to maximize utility from consumption. This is often called the *Merton problem* because it was solved by Merton [16, 17] for power and logarithmic utility functions in a Markovian Itô process model. In a market with a riskless bank account and one risky asset following a geometric Brownian motion, the optimal strategy turns out to invest a *constant fraction*  $\pi^*$  of wealth in the risky asset and to consume at a rate proportional to current wealth. This means that it is optimal for the investor to keep her portfolio holdings in bank and stock on the so-called *Merton line* with slope  $\pi^*/(1 - \pi^*)$ .

In a continuous time setting, proportional transaction costs were introduced to the Merton problem by Magill and Constantinides [15]. Their paper contains the fundamental insight that it is optimal to refrain from transacting while the portfolio holdings remain in a wedge-shaped *no-transaction region*, that is, while the fraction of wealth held in stock lies inside some interval  $[\pi_1^*, \pi_2^*]$ . However, their solution is derived in a somewhat heuristic way and also did not show how to compute the location of the boundaries  $\pi_1^*, \pi_2^*$ .

Mathematically rigorous results were first obtained in the seminal paper of Davis and Norman [6]. They show that it is indeed optimal to keep the proportion of total wealth held in stock between fractions  $\pi_1^*, \pi_2^*$  and they also prove that

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these two numbers can be determined as the solution to a free boundary value problem. The theory of viscosity solutions to Hamilton–Jacobi–Bellman equations was introduced to this problem by Shreve and Soner [19] who succeeded in removing several assumptions needed in [6].

These articles aiming for the computation of the optimal portfolio employ tools from stochastic control. It seems that unlike for frictionless markets, martingale methods have so far only been used to obtain structural existence results in the presence of transaction costs. In this context, the martingale and duality theory for frictionless markets is often applied to a *shadow price process*  $\tilde{S}$  lying within the bid-ask bounds of the real price process  $S$ . Economically speaking, the frictionless price process  $\tilde{S}$  and the original price process  $S$  with transaction costs lead to identical decisions and gains for the investor under consideration. This concept has been used in the context of the Fundamental theorem of Asset Pricing (cf. [11] and recently [8, 9]), local risk minimization [13], super-replication [2, 4] and utility maximization [3, 5, 14].

In the present study, we reconsider Merton’s problem for logarithmic utility and under proportional transaction costs as in [6]. Our goal is threefold. Most importantly, we show that the shadow price approach can be used to come up with a candidate solution to the utility maximization problem under transaction costs. Moreover, the ensuing verification procedure appears—at least for the problem at hand—to be relatively simple compared to the very impressive and nontrivial reasoning in [6] and [19]. Finally, we also construct the shadow price as part of the solution. For a recent application of the approach of the current paper, we refer to [12].

The more involved case of power utility is treated in [6, 19] as well. The application of the present approach to this case is subject of current research. While it is still possible to come up with a candidate for the shadow price, the corresponding free boundary problem appears to be more difficult than its counterpart in [6]. This stems from the fact that it may be more difficult to determine the shadow price than the optimal strategy for power utility (cf. Remark 4.7 for more details).

The remainder of the paper is organized as follows. The setup is introduced in Section 2. Subsequently, we heuristically derive the free-boundary problem that characterizes the solution. Verification is done in Section 4.

**2. The Merton problem with transaction costs.** We study the problem of maximizing expected logarithmic utility from consumption over an infinite horizon in the presence of proportional transaction costs. Except for a slightly larger class of admissible strategies, we work in the setup of [6].

The mathematical framework is as follows: fix a complete, filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$  supporting a standard Brownian motion  $(W_t)_{t \in \mathbb{R}_+}$ . Our market consists of two investment opportunities: a bank account or bond with constant value 1 and a risky asset (“stock”) whose discounted price process  $S$  is

modelled as a geometric Brownian motion, that is,

$$(2.1) \quad S_t := S_0 e^{(\mu I + \sigma W)_t} = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

with  $I_t := t$  and constants  $S_0, \sigma > 0, \mu \in \mathbb{R}$ . We consider an investor who disposes of an *initial endowment*  $(\eta_B, \eta_S) \in \mathbb{R}_+^2$ , referring to the number of bonds and stocks, respectively. Whenever stock is purchased or sold, transaction costs are imposed equal to a constant fraction of the amount transacted, the fractions being  $\bar{\lambda} \in [0, \infty)$  on purchase and  $\underline{\lambda} \in [0, 1)$  on sale, not both being equal to zero. Since transactions of infinite variation lead to instantaneous ruin, we limit ourselves to the following set of strategies.

DEFINITION 2.1. A *trading strategy* is an  $\mathbb{R}^2$ -valued predictable process  $\varphi = (\varphi^0, \varphi^1)$  of finite variation, where  $\varphi_t^0$  and  $\varphi_t^1$  denote the number of shares held in the bank account and in stock at time  $t$ , respectively. A (discounted) *consumption rate* is an  $\mathbb{R}_+$ -valued, adapted stochastic process  $c$  satisfying  $\int_0^t c_s ds < \infty$  a.s. for all  $t \geq 0$ . A pair  $(\varphi, c)$  of a trading strategy  $\varphi$  and a consumption rate  $c$  is called *portfolio/consumption pair*.

To capture the notion of a self-financing strategy, we use the intuition that no funds are added or withdrawn. To this end, we write the second component  $\varphi^1$  of any strategy  $\varphi$  as difference  $\varphi^1 = \varphi^\uparrow - \varphi^\downarrow$  of two increasing processes  $\varphi^\uparrow$  and  $\varphi^\downarrow$  which do not grow at the same time. Moreover, we denote by

$$(2.2) \quad \underline{S} := (1 - \underline{\lambda})S, \quad \bar{S} := (1 + \bar{\lambda})S,$$

the bid and ask price of the stock, respectively. The proceeds of selling stock must be added to the bank account while the expenses from consumption and the purchase of stock have to be deducted from the bank account in any infinitesimal period  $(t - dt, t]$ , that is, we require

$$(2.3) \quad d\varphi_t^0 = \underline{S}_{t-} d\varphi_t^\downarrow - \bar{S}_{t-} d\varphi_t^\uparrow - c_t dt$$

for *self-financing strategies*. Written in integral terms, this amounts to the *self-financing condition*

$$(2.4) \quad \varphi^0 = \varphi_0^0 + \int_0^\cdot \underline{S}_{t-} d\varphi_t^\downarrow - \int_0^\cdot \bar{S}_{t-} d\varphi_t^\uparrow - \int_0^\cdot c_t dt.$$

In our setup (2.1, 2.2), we obviously have  $\underline{S}_- = \underline{S}$  and  $\bar{S}_- = \bar{S}$  but the above definition makes sense for discontinuous bid and ask price processes  $\underline{S}, \bar{S}$  as well. The second and third term on the right-hand side represent the cumulative amount of wealth gained selling, respectively, spent buying stock, while the last term represents cumulated consumption.

REMARK 2.2. Partial integration similarly as in [10], I.4.49b, shows that for  $\underline{S} = \overline{S}$ , we recover the usual self-financing condition in frictionless markets (cf. [18] for more details).

The value of a portfolio is not obvious either because securities have no unique price. As is common in the literature, we use the value that would be obtained if the portfolio were to be liquidated immediately.

DEFINITION 2.3. The (liquidation) value process of a trading strategy  $\varphi$  is defined as

$$V(\varphi) := \varphi^0 + (\varphi^1)^+ \underline{S} - (\varphi^1)^- \overline{S}.$$

A self-financing portfolio/consumption pair  $(\varphi, c)$  is called *admissible* if  $(\varphi_0^0, \varphi_0^1) = (\eta_B, \eta_S)$  and  $V(\varphi) \geq 0$ . An admissible pair  $(\varphi, c)$  is called *optimal* if it maximizes

$$(2.5) \quad \kappa \mapsto E\left(\int_0^\infty e^{-\delta t} \log(\kappa_t) dt\right)$$

over all admissible portfolio/consumption pairs  $(\psi, \kappa)$ , where  $\delta > 0$  denotes a fixed given *impatience rate*.

Note that the “true” price process  $S$  is irrelevant for the problem as it does not appear in the definitions; only the bid and ask prices  $\underline{S}, \overline{S}$  matter. Moreover, since  $\delta > 0$ , the value function of the Merton problem *without* transaction costs is finite by [6], Theorem 2.1. Hence, it follows that this holds in the present setup with transaction costs as well.

Our notion of admissible strategies is slightly more general than that in [6, 19]. However, it will turn out later on that the optimal strategies in both sets coincide.

LEMMA 2.4. For any admissible policy  $(c, L, U)$  in the sense of [6], there exists a trading strategy  $\varphi = (\varphi^0, \varphi^1)$  such that  $(\varphi, c)$  is an admissible portfolio/consumption pair.

PROOF. The initial endowment in [6] can be expressed in terms of wealth as  $(x, y) = (\eta_B, \eta_S S_0)$ . Define  $s_0, s_1$  as in [6], (3.1), and set  $\varphi_t^0 := s_0(t-)$ ,  $\varphi_t^1 := s_1(t-)/S_t$ . A simple calculation shows that  $((\varphi^0, \varphi^1), c)$  is an admissible portfolio/consumption pair.  $\square$

**3. Heuristic derivation of the solution.** As indicated in the Introduction, the martingale approach relies decisively on shadow price processes, which we define as follows.

DEFINITION 3.1. We call a semimartingale  $\tilde{S}$  shadow price process if

$$(3.1) \quad \underline{S} \leq \tilde{S} \leq \bar{S}$$

and if the maximal expected utilities for  $S, \underline{\lambda}, \bar{\lambda}$  and for the price process  $\tilde{S}$  without transaction costs coincide.

Obviously, the maximal expected utility for any frictionless price process  $\tilde{S}$  satisfying (3.1) is at least as high as for the original market with transaction costs, since the investor is always buying at  $\tilde{S}_t \leq \bar{S}_t$  and selling for  $\tilde{S}_t \geq \underline{S}_t$ .

A shadow price process can be interpreted as a kind of least favourable frictionless market extension. The corresponding optimal portfolio trades only when the shadow price happens to coincide with the bid or ask price, respectively. Otherwise, it would achieve higher profits with  $\tilde{S}$  than with  $S$  and transaction costs.

Let us assume that such a shadow price process  $\tilde{S}$  exists. If it were known in the first place, it would be of great help because portfolio selection problems without transaction costs are considerably easier to solve. But it is not known at this stage. Hence, we must solve the problems of determining  $\tilde{S}$  and of portfolio optimization relative to  $\tilde{S}$  simultaneously.

To this end, we parametrize the shadow price process in the following form:

$$(3.2) \quad \tilde{S} = S \exp(C)$$

with some  $[\underline{C}, \bar{C}]$ -valued process  $C$  where

$$\underline{C} := \log(1 - \underline{\lambda}) \quad \text{and} \quad \bar{C} := \log(1 + \bar{\lambda}).$$

Since  $S$  is an Itô process, we expect  $\tilde{S}$  and hence  $C$  to be Itô processes as well. We even guess that  $C$  is an Itô diffusion, that is,

$$(3.3) \quad dC_t = \tilde{\mu}(C_t) dt + \tilde{\sigma}(C_t) dW_t$$

with some deterministic functions  $\tilde{\mu}, \tilde{\sigma}$ . Any admissible portfolio/consumption pair  $(\varphi, c)$  is completely determined by  $c$  and the fraction of wealth invested in stocks

$$(3.4) \quad \tilde{\pi} := \frac{\varphi^1 \tilde{S}}{\varphi^0 + \varphi^1 \tilde{S}},$$

where bookkeeping is done here relative to shadow prices  $\tilde{S}$ . Hence, we must determine four unknown objects, namely the ansatz functions  $\tilde{\mu}, \tilde{\sigma}$  as well as the optimal consumption rate  $c$  and the optimal fraction  $\tilde{\pi}$  of wealth in stocks.

Standard results yield the optimal strategy for the frictionless price process  $\tilde{S}$ . For example, by [7], Theorem 3.1, we have

$$(3.5) \quad \tilde{\pi} = \frac{\mu - \sigma^2/2 + \tilde{\mu}(C)}{(\sigma + \tilde{\sigma}(C))^2} + \frac{1}{2}, \quad c = \delta \tilde{V}(\varphi),$$

where

$$(3.6) \quad \tilde{V}(\varphi) = \varphi^0 + \varphi^1 \tilde{S}$$

denotes the value process of  $\varphi$  in the frictionless market with price process  $\tilde{S}$ . This already determines the optimal consumption rate. To simplify the following calculations, we assume  $\tilde{\pi} > 0$  and work with

$$\beta := \log\left(\frac{\tilde{\pi}}{1 - \tilde{\pi}}\right)$$

instead of  $\tilde{\pi}$ . By (3.4) this implies  $\beta := \log(\varphi^1) + \log(\tilde{S}) - \log(\varphi^0)$ .

Since the optimal strategy trades the shadow price process only when it coincides with bid or ask price,  $\varphi^1$  must be constant on  $]0, T[$  with  $T := \inf\{t > 0 : C_t \in \{\underline{C}, \bar{C}\}\}$ . By (2.4) and Itô's formula, we have

$$d \log(\varphi_t^0) = \frac{-c_t}{\varphi_t^0} dt = \frac{-\delta \tilde{V}_t(\varphi)}{\tilde{V}_t(\varphi) - \tilde{\pi}_t \tilde{V}_t(\varphi)} dt = \frac{-\delta}{1 - \tilde{\pi}_t} dt$$

on  $]0, T[$ , hence insertion of (3.5) yields

$$(3.7) \quad \begin{aligned} d\beta_t &= d \log(\varphi_t^1) + d \log(\tilde{S}_t) - d \log(\varphi_t^0) \\ &= \left( \mu - \frac{\sigma^2}{2} + \tilde{\mu}(C_t) + \frac{\delta(\sigma + \tilde{\sigma}(C_t))^2}{1/2(\sigma + \tilde{\sigma}(C_t))^2 - (\mu - \sigma^2/2 + \tilde{\mu}(C_t))} \right) dt \\ &\quad + (\sigma + \tilde{\sigma}(C_t)) dW_t. \end{aligned}$$

On the other hand, we know from (3.5) that  $\tilde{\pi}$  is a function of  $C$ , which in turn yields  $\beta = f(C)$  for some function  $f$ . By Itô's formula, this implies

$$(3.8) \quad d\beta_t = \left( f'(C_t) \tilde{\mu}(C_t) + f''(C_t) \frac{\tilde{\sigma}(C_t)^2}{2} \right) dt + f'(C_t) \tilde{\sigma}(C_t) dW_t.$$

From (3.7), (3.8) and (3.5), we now obtain three conditions for the three functions  $\tilde{\mu}, \tilde{\sigma}, f$ :

$$(3.9) \quad \frac{1}{1 + e^{-f}} = \frac{\mu - \sigma^2/2 + \tilde{\mu}}{(\sigma + \tilde{\sigma})^2} + \frac{1}{2},$$

$$(3.10) \quad \begin{aligned} \mu - \frac{\sigma^2}{2} + \tilde{\mu} + \frac{\delta(\sigma + \tilde{\sigma})^2}{1/2(\sigma + \tilde{\sigma})^2 - (\mu - \sigma^2/2 + \tilde{\mu})} \\ = f' \tilde{\mu} + f'' \frac{\tilde{\sigma}^2}{2}, \end{aligned}$$

$$(3.11) \quad \sigma + \tilde{\sigma} = f' \tilde{\sigma}.$$

Equations (3.11) and (3.9) yield

$$(3.12) \quad \tilde{\sigma} = \frac{\sigma}{f' - 1}, \quad \tilde{\mu} = -\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2} \left(\frac{f'}{f' - 1}\right)^2 \frac{1 - e^{-f}}{1 + e^{-f}}.$$

By inserting into (3.10), we obtain the following ordinary differential equation (ODE) for  $f$ :

$$\begin{aligned}
 (3.13) \quad f''(x) &= \frac{2\delta}{\sigma^2}(1 + e^{f(x)}) + \left(\frac{2\mu}{\sigma^2} - 1 - \frac{4\delta}{\sigma^2}(1 + e^{f(x)})\right)f'(x) \\
 &+ \left(-\frac{4\mu}{\sigma^2} + \frac{2}{1 + e^{-f(x)}} + 1 + \frac{2\delta}{\sigma^2}(1 + e^{f(x)})\right)(f'(x))^2 \\
 &+ \left(\frac{2\mu}{\sigma^2} - \frac{2}{1 + e^{-f(x)}}\right)(f'(x))^3.
 \end{aligned}$$

Because of missing boundary conditions, (3.13) does not yet yield the solution. We obtain such conditions heuristically as follows. In order to lead to finite maximal expected utility, the shadow price process should be arbitrage-free and hence allow for an equivalent martingale measure. This in turn means that  $\tilde{S}$  and hence also  $C$  should not have any singular part in their semimartingale decomposition. Put differently, we expect the Itô process representation (3.3) to hold even when  $C$  reaches the boundary points  $\underline{C}, \bar{C}$ .

The number of shares of stock  $\varphi^1$ , on the other hand, changes only when  $C$  hits the boundary. As this is likely to happen only on a Lebesgue-null set of times,  $\varphi^1$  must have a singular part in order to move at all. In view of the connection between  $\varphi^1$  and  $\beta$ , this suggests that  $\beta$  has a singular part as well. This means that  $f$  cannot be a  $C^2$  function on the closed interval  $[\underline{C}, \bar{C}]$  because otherwise  $\beta = f(C)$  would be an Itô process, too. A natural way out is the ansatz  $f'(\underline{C}) = -\infty = f'(\bar{C})$  in order for  $\beta$  to have a singular part at the boundary. Hence, we complement ODE (3.13) by boundary conditions

$$(3.14) \quad \lim_{x \downarrow \underline{C}} f'(x) = -\infty = \lim_{x \uparrow \bar{C}} f'(x).$$

In order to avoid infinite derivatives, we consider instead the inverse function  $g := f^{-1}$ . Equation (3.13) turns into

$$\begin{aligned}
 (3.15) \quad g''(y) &= \left(-\frac{2\mu}{\sigma^2} + \frac{2}{1 + e^{-y}}\right) \\
 &+ \left(\frac{4\mu}{\sigma^2} - \frac{2}{1 + e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1 + e^y)\right)g'(y) \\
 &+ \left(-\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1 + e^y)\right)(g'(y))^2 - \frac{2\delta}{\sigma^2}(1 + e^y)(g'(y))^3
 \end{aligned}$$

on the a priori unknown interval  $[\underline{\beta}, \bar{\beta}] := [f(\bar{C}), f(\underline{C})]$  and (3.14) translates into free boundary conditions

$$(3.16) \quad g(\underline{\beta}) = \bar{C}, \quad g(\bar{\beta}) = \underline{C}, \quad g'(\underline{\beta}) = 0, \quad g'(\bar{\beta}) = 0.$$

Equations (3.15), (3.16) together with (3.2)–(3.6) and  $f = g^{-1}$  constitute our ansatz for the portfolio optimization problem.

In summary, the solution to the free boundary problem (3.15), (3.16)—or equivalently (3.13), (3.14)—leads to the optimal strategy. The ODE itself is derived based on the optimality of  $\tilde{\pi}$  for  $\tilde{S}$  and the constancy of  $\varphi^1$  on  $]0, T[$ . In the next section, we show that this ansatz indeed yields the true solution.

Our result resembles [6] in that the solution is expressed in terms of a free boundary problem. However, both the ODE and the boundary conditions are different, since the function  $g$  refers to the shadow price process from the present dual approach and therefore does not appear explicitly in the framework of [6] (but cf. Remark 4.7).

**4. Construction of the shadow price process.** We turn now to verification of the candidate solution from the previous section. The idea is rather simple. Using (3.2), (3.3), we define a candidate shadow price process  $\tilde{S}$ . In order to prove that it is indeed a shadow price process, we show that the optimal portfolio relative to  $\tilde{S}$  trades only at the boundaries  $\underline{S}, \bar{S}$ . However, existence of a solution to stochastic differential equation (SDE) (3.3) is not immediately obvious. Therefore, we consider instead the corresponding Skorokhod SDE for  $\beta = f(C)$  with instantaneous reflection at some boundaries  $\underline{\beta} < \bar{\beta}$ . The process  $C = g(\beta)$  is then defined in a second step.

We begin with an existence result for the free boundary value problem derived above. We make the following assumption which guarantees that the fraction of wealth held in stock remains positive and which is needed in [6] as well [(5.1) in that paper].

*Standing assumption.*

$$(4.1) \quad 0 < \mu < \sigma^2.$$

REMARK 4.1. It is shown in [19] that this condition is not needed to ensure the existence of an optimal strategy characterized by a wedge-shaped no-transaction region. If the transformation  $\beta = \log(\tilde{\pi}/(1 - \tilde{\pi}))$  was not used in our approach, we would still obtain a free boundary problem, but as in [6] it is less obvious whether or not it admits a solution.

PROPOSITION 4.2. *There exist  $\underline{\beta} < \bar{\beta}$  and a strictly decreasing mapping  $g : [\underline{\beta}, \bar{\beta}] \rightarrow [\underline{C}, \bar{C}]$  satisfying the free boundary problem (3.15), (3.16).*

PROOF. Since we have assumed  $0 < \frac{\mu}{\sigma^2} < 1$ , there is a unique solution  $y$  to  $\frac{2}{1+e^{-y}} - \frac{2\mu}{\sigma^2} = 0$ , namely  $y_0 = -\log(\frac{\sigma^2}{\mu} - 1)$ . For any  $\underline{\beta}_\Delta := y_0 - \Delta$  with  $\Delta > 0$ , there exists a local solution  $g_\Delta$  of the initial value problem corresponding to (3.15) and initial values  $g_\Delta(\underline{\beta}_\Delta) = \bar{C}$  and  $g'_\Delta(\underline{\beta}_\Delta) = 0$ . Set

$$M' := \max \left\{ \sqrt[3]{\frac{4(\mu + \sigma^2)}{\delta}}, \sqrt{\frac{8\mu}{\delta}}, 8 + \frac{4\mu + 2\sigma^2}{\delta} \right\}.$$



Then we have  $g''_{\Delta}(y) > 0$  for  $g'_{\Delta}(y) < -M'$  and  $g''_{\Delta}(y) < 0$  for  $g'_{\Delta} > M'$  by (3.15). Therefore,  $g'_{\Delta}$  only takes values in  $[-M', M']$ , which implies that  $g_{\Delta}$  does not explode.

From (3.15) and  $\Delta > 0$ , it follows that  $g''_{\Delta}(y) < 0$  in a neighborhood  $\mathcal{U}$  of  $\underline{\beta}_{\Delta}$  and hence  $g'_{\Delta}(y) < 0$  in  $\mathcal{U}$ . For sufficiently large  $y$  and  $g'_{\Delta}(y) < 0$ , the right-hand side of (3.15) is positive and bounded away from zero by a positive constant. Hence, a comparison argument shows that there exist further zeros of  $g'_{\Delta}$ , the first of which we denote by  $\bar{\beta}_{\Delta}$ . Note that by definition  $g_{\Delta}$  is strictly decreasing on  $[\underline{\beta}_{\Delta}, \bar{\beta}_{\Delta}]$ . It remains to show that for properly chosen  $\Delta$ , we can achieve  $g(\bar{\beta}_{\Delta}) = \underline{C}$  for any  $\underline{C} < \bar{C}$ .

STEP 1. We first show  $g_{\Delta}(\bar{\beta}_{\Delta}) \rightarrow \bar{C}$  as  $\Delta \rightarrow 0$ . This can be seen as follows. Observe that for  $|y - y_0| < 1$ , (3.15) and  $g'_{\Delta}(y) \in [-M', M']$  yield

$$\begin{aligned}
 |g''_{\Delta}(y)| < M'' &:= \frac{2\mu}{\sigma^2} + 2 + \left( \frac{4\mu}{\sigma^2} + 3 + \frac{2\delta}{\sigma^2}(1 + e^{y_0+1}) \right) M' \\
 &\quad + \left( \frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1 + e^{y_0+1}) \right) (M')^2 \\
 &\quad + \frac{2\delta}{\sigma^2}(1 + e^{y_0+1})(M')^3.
 \end{aligned}$$

Hence,  $|g'_{\Delta}(y)| \leq 2M''\Delta$  for  $y \in [y_0 - \Delta, y_0 + \Delta]$  and  $\Delta < 1$ . Combined with (3.15), this yields

$$(4.2) \quad \sup_{y \in [y_0 - \Delta, y_0 + \Delta]} |g''_{\Delta}(y)| \rightarrow 0 \quad \text{for } \Delta \rightarrow 0.$$

For  $\Delta$  sufficiently small,  $y \in [y_0 + \Delta, y_0 + 1]$ , and

$$\begin{aligned}
 |g'_{\Delta}(y)| < m_{\Delta} &:= \max \left\{ \frac{1/3(-\mu/\sigma^2 + 1/(1 + e^{-(y_0+\Delta)}))}{(4\mu)/\sigma^2 + 3 + (2\delta)/\sigma^2(1 + e^{y_0+1})}, \right. \\
 &\quad \sqrt{\frac{1/3(-\mu/\sigma^2 + 1/(1 + e^{-(y_0+\Delta)}))}{(2\mu)/\sigma^2 + 1 + (4\delta)/\sigma^2(1 + e^{y_0+1})}}, \\
 &\quad \left. \sqrt[3]{\frac{1/3(-\mu/\sigma^2 + 1/(1 + e^{-(y_0+\Delta)}))}{(2\delta)/\sigma^2(1 + e^{y_0+1})}} \right\},
 \end{aligned}$$

(3.15) and a first-order Taylor expansion imply

$$(4.3) \quad g''_{\Delta}(y) > \frac{\mu}{\sigma^2} + \frac{1}{1 + e^{-(y_0+\Delta)}} > \frac{e^{-y_0}}{2(1 + e^{-y_0})^2} \Delta > 0.$$

Equation (4.2) yields

$$|g'_{\Delta}(y_0 + \Delta)| \leq 2\Delta \sup_{y \in [y_0 - \Delta, y_0 + \Delta]} |g''_{\Delta}(y)| < m_{\Delta}$$

for sufficiently small  $\Delta$ . By (4.3) we have that if  $g'_\Delta$  does not have a zero on  $[y_0 - \Delta, y_0 + \Delta]$ , that is,  $g'(y_0 + \Delta) < 0$ , then  $g''_\Delta(y) > \frac{e^{-y_0}}{2(1+e^{-y_0})^2}\Delta$  on  $[y_0 + \Delta, \min\{\bar{\beta}_\Delta, y_0 + 1\}]$ . Using (4.2), this yields

$$\bar{\beta}_\Delta - \underline{\beta}_\Delta < 2\Delta + \frac{2\Delta \sup_{y \in [y_0 - \Delta, y_0 + \Delta]} |g''(y)|}{e^{-y_0}/(2(1+e^{-y_0})^2)\Delta} \rightarrow 0$$

for  $\Delta \rightarrow 0$ . Since  $|g'_\Delta(y)| < M'$ , an application of the mean value theorem completes the first step.

STEP 2. We now establish  $\bar{\beta}_\Delta \geq y_0$  and  $g(\bar{\beta}_\Delta) \rightarrow -\infty$  as  $\Delta \rightarrow \infty$ . To this end, let  $y^* < y_0$ . Then we have  $g''_\Delta(y) < -\frac{\mu}{\sigma^2} + \frac{1}{1+e^{y^*}} < 0$  if  $y \leq y^*$  and

$$|g'_\Delta(y)| < m' := \max \left\{ \frac{1/3|-\mu/\sigma^2 + 1/(1+e^{-y^*})|}{(4\mu)/\sigma^2 + 3 + (2\delta)/\sigma^2(1+e^{y^*})}, \right. \\ \sqrt{\frac{1/3|-\mu/\sigma^2 + 1/(1+e^{-y^*})|}{(2\mu)/\sigma^2 + 1 + (4\delta)/\sigma^2(1+e^{y^*})}}, \\ \left. \sqrt[3]{\frac{1/3|-\mu/\sigma^2 + 1/(1+e^{-y^*})|}{(2\delta)/\sigma^2(1+e^{y^*})}} \right\}.$$

Since  $g''_\Delta(\underline{\beta}_\Delta) < 0$ , this implies  $g'_\Delta(y) < 0$  for  $y \leq y^*$  as well as  $|g'_\Delta(y)| \geq m'$  for  $y \in [y_0 - \Delta + \frac{m'}{\mu/\sigma^2 - (1+e^{-y^*})^{-1}}, y^*]$ . By the first statement and since  $y^* < y_0$  was chosen arbitrarily, we have  $\bar{\beta}_\Delta \geq y_0$ . In addition, the second statement and the mean value theorem show that  $g_\Delta(\bar{\beta}_\Delta) \rightarrow -\infty$  as  $\Delta \rightarrow \infty$ .

STEP 3. We now establish  $\bar{\beta}_\Delta > y_0$ . By Step 2 it remains to show that  $\bar{\beta}_\Delta \neq y_0$ . Suppose that  $\bar{\beta}_\Delta = y_0$ . Then  $g'_\Delta(y_0) = 0 = g''_\Delta(y_0)$  and it follows from a Taylor expansion around  $y_0$  that

$$g''_\Delta(y) = \frac{2e^{-y_0}}{(1+e^{-y_0})^2}(y - y_0) + O((y - y_0)^2) < 0$$

for  $y \in (y_0 - \varepsilon, y_0)$  and sufficiently small  $\varepsilon > 0$ , hence  $g'(y) > 0$  for some  $y < y_0$ . By the intermediate value theorem, there exists a zero of  $g'$  on  $(\underline{\beta}_\Delta, y_0)$ , in contradiction to the definition of  $\bar{\beta}_\Delta$ . Therefore, we have  $\bar{\beta}_\Delta > y_0$  as claimed.

STEP 4. Next, we prove that  $(g_\Delta, g'_\Delta)$  converges toward  $(g_{\Delta_0}, g'_{\Delta_0})$  uniformly on compacts as  $\Delta \rightarrow \Delta_0$ . To this end, we consider the solution  $f^\Delta : \mathbb{R}_+ \rightarrow \mathbb{R}^3$  to the initial value problem

$$\frac{d}{dy}(f_1^\Delta, f_2^\Delta, f_3^\Delta)(y) = (1, f_3^\Delta(y), h(f_1^\Delta(y), f_3^\Delta(y)))$$

with

$$h(y, z) := \left( -\frac{2\mu}{\sigma^2} + \frac{2}{1 + e^{-y}} \right) + \left( \frac{4\mu}{\sigma^2} - \frac{2}{1 + e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1 + e^y) \right) z$$

$$+ \left( -\frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1 + e^y) \right) z^2 - \frac{2\delta}{\sigma^2}(1 + e^y) z^3$$

and initial values  $(f_1^\Delta, f_2^\Delta, f_3^\Delta)(0) = (y_0 - \Delta, \bar{C}, 0)$ . The solution to this problem is

$$(f_1^\Delta, f_2^\Delta, f_3^\Delta)(y) = (y + y_0 - \Delta, g_\Delta(y + y_0 - \Delta), g'_\Delta(y + y_0 - \Delta)).$$

Note that

$$|g_\Delta(y) - g_{\Delta_0}(y)| = |f_2^\Delta(y - y_0 + \Delta) - f_2^{\Delta_0}(y - y_0 + \Delta_0)|$$

$$\leq |f_2^\Delta(y - y_0 + \Delta) - f_2^{\Delta_0}(y - y_0 + \Delta)| + M'|\Delta - \Delta_0|$$

and similarly for  $g''$ . Hence, it suffices to show that  $f^\Delta$  depends uniformly on compacts on its initial value  $f^\Delta(0)$ .

$h$  is locally Lipschitz and hence globally Lipschitz in  $z$  on  $[-M', M']$  and in  $y$  on compacts. The desired uniform convergence follows now from the corollary to [1], Theorem V.3.2.

STEP 5. In view of Steps 1 and 2 as well as the intermediate value theorem, it remains to show that  $g(\bar{\beta}_\Delta)$  depends continuously on  $\Delta$ . Fix  $\Delta_0 > 0$ . Since  $\bar{\beta}_{\Delta_0} > y_0$  by Step 3, a Taylor expansion around  $\bar{\beta}_{\Delta_0}$  yields that  $g'_{\Delta_0}$  is strictly increasing in a sufficiently small neighborhood  $\mathcal{W}$  of  $\bar{\beta}_{\Delta_0}$ . Now consider  $\Delta$  sufficiently close to  $\Delta_0$ . Recall that  $g'_\Delta(y)$  does not vanish for  $\underline{\beta}_\Delta < y \leq y_0$ . By the uniform convergence from Step 4, the first zero  $\bar{\beta}_\Delta$  of  $g'_\Delta$  after  $\underline{\beta}_\Delta$  is close to the first zero  $\bar{\beta}_{\Delta_0}$  of  $g'_{\Delta_0}$  after  $\underline{\beta}_{\Delta_0}$ . In view of

$$|g_\Delta(\bar{\beta}_\Delta) - g_{\Delta_0}(\bar{\beta}_{\Delta_0})| \leq |g_\Delta(\bar{\beta}_\Delta) - g_{\Delta_0}(\bar{\beta}_\Delta)| + |g_{\Delta_0}(\bar{\beta}_\Delta) - g_{\Delta_0}(\bar{\beta}_{\Delta_0})|$$

and Step 4, this completes the proof.  $\square$

We now construct the process  $\beta$  as the solution to an SDE with instantaneous reflection. The coefficients  $a$  and  $b$  in (4.4) below are chosen in line with (3.7) and (3.12).

LEMMA 4.3. *Let  $\beta_0 \in [\underline{\beta}, \bar{\beta}]$  and*

$$a(y) := \frac{\sigma^2}{2} \left( \frac{1 - e^{-y}}{1 + e^{-y}} \right) \left( \frac{1}{1 - g'(y)} \right)^2 + \delta(1 + e^y), \quad b(y) := \frac{\sigma}{1 - g'(y)}$$

for  $\beta \in [\underline{\beta}, \bar{\beta}]$ . Then there exists a solution to the Skorokhod SDE

$$d\beta_t = a(\beta_t) dt + b(\beta_t) dW_t$$

with instantaneous reflection at  $\underline{\beta}, \bar{\beta}$ , that is, a continuous, adapted,  $[\underline{\beta}, \bar{\beta}]$ -valued process  $\beta$  and nondecreasing adapted processes  $\Phi, \Psi$  such that  $\Phi$  and  $\Psi$  increase only on the sets  $\{\beta = \underline{\beta}\}$  and  $\{\beta = \bar{\beta}\}$ , respectively, and

$$(4.4) \quad \beta_t = \beta_0 + \int_0^t a(\beta_s) ds + \int_0^t b(\beta_s) dW_s + \Phi_t - \Psi_t$$

holds for all  $t \in \mathbb{R}_+$ .

PROOF. In view of [20], it suffices to prove that the coefficients  $a(\cdot)$  and  $b(\cdot)$  are globally Lipschitz on  $[\underline{\beta}, \bar{\beta}]$ . By the mean value theorem it is enough to show that their derivatives are bounded on  $(\underline{\beta}, \bar{\beta})$ . Let  $y \in (\underline{\beta}, \bar{\beta})$  be fixed. Then we have

$$(4.5) \quad b'(y) = \sigma \frac{g''(y)}{(1 - g'(y))^2},$$

$g'(y) \leq 0$  implies  $|1 - g'(y)| \geq \max\{1, |g'(y)|\}$ . Moreover,  $g'$  is bounded on  $[\underline{\beta}, \bar{\beta}]$  by the proof of Proposition 4.2. Boundedness of  $b'$  now follows from (3.15) and (4.5). Boundedness of  $a'$  is shown along the same lines.  $\square$

We now define  $C$  and the shadow price process  $\tilde{S}$  as motivated in Section 3.

LEMMA 4.4. For  $\beta_0 \in [\underline{\beta}, \bar{\beta}]$  let  $\beta$  be the process from Lemma 4.3. Then  $C := g(\beta)$  is a  $[\underline{C}, \bar{C}]$ -valued Itô process of the form

$$C_t = g(\beta_0) + \int_0^t \left( -\mu + \frac{\sigma^2}{2} + \frac{\sigma^2}{2} \left( \frac{1 - e^{-\beta_s}}{1 + e^{-\beta_s}} \right) \left( \frac{1}{1 - g'(\beta_s)} \right)^2 \right) ds + \int_0^t \frac{\sigma g'(\beta_s)}{1 - g'(\beta_s)} dW_s$$

and the Itô process  $\tilde{S} := S \exp(C)$  satisfies

$$\tilde{S}_t = S_0 e^{C_0} \exp \left( \int_0^t \frac{\sigma^2}{2} \left( \frac{1 - e^{-\beta_s}}{1 + e^{-\beta_s}} \right) \left( \frac{1}{1 - g'(\beta_s)} \right)^2 ds + \int_0^t \frac{\sigma}{1 - g'(\beta_s)} dW_s \right).$$

PROOF.  $g$  can be extended to a  $C^2$ -function on an open set containing  $[\underline{\beta}, \bar{\beta}]$ , e.g., by attaching suitable parabolas at  $\underline{\beta}, \bar{\beta}$ . Since  $\Phi$  and  $\Psi$  are of finite variation and  $g'$  vanishes on the support of the Stieltjes measures corresponding to  $\Phi$  and  $\Psi$ , Itô's formula yields

$$dC_t = (g'(\beta_t)a(\beta_t) + \frac{1}{2}g''(\beta_t)b(\beta_t)^2) dt + g'(\beta_t)b(\beta_t) dW_t.$$

The claims follow by inserting the definitions of  $a$  and  $b$ , (3.15), and the definition of  $S$ .  $\square$

Next, we show that  $\tilde{S}$  is indeed a shadow price process, i.e., the same portfolio/consumption pair  $(\varphi, c)$  is optimal with the same expected utility both in the frictionless market with price process  $\tilde{S}$  and in the market with price process  $S$  and proportional transaction costs  $\underline{\lambda}, \bar{\lambda}$ . In the frictionless market with price process  $\tilde{S}$ , standard results yields the optimal strategy and consumption rate.

LEMMA 4.5. *Set*

$$(4.6) \quad \beta_0 := \begin{cases} \underline{\beta}, & \text{if } \frac{\eta_S \bar{S}_0}{\eta_B + \eta_S \bar{S}_0} < \frac{1}{1 + e^{-\underline{\beta}}}, \\ \bar{\beta}, & \text{if } \frac{\eta_S \underline{S}_0}{\eta_B + \eta_S \underline{S}_0} > \frac{1}{1 + e^{-\bar{\beta}}}. \end{cases}$$

Otherwise, let  $\beta_0$  denote the  $[\underline{\beta}, \bar{\beta}]$ -valued solution  $y$  to

$$\frac{\eta_S S_0 e^{g(y)}}{\eta_B + \eta_S S_0 e^{g(y)}} = \frac{1}{1 + e^{-y}}.$$

For processes  $\beta$  and  $\tilde{S}$  as in Lemma 4.4, define

$$\begin{aligned} \tilde{V}_t &:= (\eta_B + \eta_S \tilde{S}_0) \mathcal{E} \left( \int_0^t \frac{1}{(1 + e^{-\beta_s}) \tilde{S}_s} d\tilde{S}_s - \int_0^t \delta ds \right), \\ c_t &:= -\delta \tilde{V}_t, \\ \varphi_t^1 &:= \frac{1}{1 + e^{-\beta_t}} \frac{\tilde{V}_t}{\tilde{S}_t}, \quad \varphi_t^0 := \tilde{V}_t - \varphi_t^1 \tilde{S}_t. \end{aligned}$$

Then

$$(4.7) \quad \begin{aligned} \varphi_t^0 &= \varphi_0^0 - \int_0^t c_s ds - \int_0^t \frac{\tilde{V}_s e^{-\beta_s}}{(1 + e^{-\beta_s})^2} d\Phi_s + \int_0^t \frac{\tilde{V}_s e^{-\beta_s}}{(1 + e^{-\beta_s})^2} d\Psi_s, \\ \varphi_t^1 &= \varphi_0^1 + \int_0^t \frac{\varphi_s^1 e^{-\beta_s}}{1 + e^{-\beta_s}} d\Phi_s - \int_0^t \frac{\varphi_s^1 e^{-\beta_s}}{1 + e^{-\beta_s}} d\Psi_s \end{aligned}$$

and  $(\varphi, c)$  is an optimal portfolio/consumption pair with value process  $\tilde{V}$  for initial wealth  $\eta_B + \eta_S \tilde{S}_0$  in the frictionless market with price process  $\tilde{S}$ .

PROOF. One easily verifies that  $\beta_0$  is well defined. Moreover, we have

$$(4.8) \quad \log(\varphi_t^1) = \log(\tilde{V}_t) - \left( \mu - \frac{\sigma^2}{2} \right) t - \sigma W_t - C_t - \log(1 + e^{-\beta_t}).$$

By [10], Theorem I.4.61,

$$d \log(\tilde{V}_t) = \left( \frac{\sigma^2}{2(1 + e^{-\beta_t})^2} \left( \frac{1}{1 - g'(\beta_t)} \right)^2 - \delta \right) dt + \frac{\sigma}{1 + e^{-\beta_t}} \left( \frac{1}{1 - g'(\beta_t)} \right) dW_t.$$

$C$  is given in Lemma 4.4 and for the last term in (4.8), Itô's formula yields

$$-d \log(1 + e^{-\beta_t}) = \frac{e^{-\beta_t}}{1 + e^{-\beta_t}} d\beta_t - \frac{1}{2} \frac{e^{-\beta_t}}{(1 + e^{-\beta_t})^2} d[\beta, \beta]_t.$$

Summing up terms, we have

$$d \log(\varphi_t^1) = \frac{e^{-\beta_t}}{1 + e^{-\beta_t}} d\Phi_t - \frac{e^{-\beta_t}}{1 + e^{-\beta_t}} d\Psi_t.$$

Hence,  $\log(\varphi^1)$  is of finite variation and another application of Itô's formula yields the claimed representation for  $\varphi^1$ . Obviously,  $\tilde{V}$  is the value process of  $\varphi$  relative to  $\tilde{S}$ . By definition, we have

$$(4.9) \quad d\tilde{V}_t = \varphi_t^1 d\tilde{S}_t - c_t dt,$$

which means that  $(\varphi, c)$  is a self-financing portfolio/consumption pair for price process  $\tilde{S}$ . The integral representation of  $\varphi^0$  now follows from

$$d\varphi_t^0 = d(\tilde{V}_t - \varphi_t^1 \tilde{S}_t) = -c_t dt - \tilde{S}_t d\varphi_t^1,$$

where we used integration by parts in the sense of [10], I.4.49b. For  $t \in \mathbb{R}_+$  set

$$K_t := \int_0^t e^{-\delta s} ds, \quad \kappa_t := e^{\delta t} c_t, \quad \psi_t^0 := \varphi_t^0 + \int_0^t c_s ds, \quad \psi_t^1 := \varphi_t^1.$$

Then  $(\varphi, c)$  is optimal in the sense of Definition 2.3 (adapted to frictionless markets where the restriction to strategies of finite variation is dropped) if and only if  $(\psi, \kappa)$  is optimal in the sense of [7], Definition 2.2. The differential characteristics  $(\tilde{b}, \tilde{c}, \tilde{F})$  of  $\tilde{S}$  are given by  $\tilde{F} = 0$  and

$$\tilde{b}_t = \tilde{S}_t \sigma^2 \frac{1}{1 + e^{-\beta_t}} \left( \frac{1}{1 - g'(\beta_t)} \right)^2, \quad \tilde{c}_t = \tilde{S}_t^2 \sigma^2 \left( \frac{1}{1 - g'(\beta_t)} \right)^2.$$

Hence, [7], Theorem 3.1, with  $H_t = \tilde{b}_t / \tilde{c}_t$ ,  $K_\infty = 1/\delta$  and  $K_\infty - K_t = \frac{1}{\delta} e^{-\delta t}$  yields the optimality of  $(\varphi, c)$ .  $\square$

If (4.6) holds, then  $(\varphi_0^0, \varphi_0^1) \neq (\eta_B, \eta_S)$ . In this case, we can and do modify the initial portfolio to

$$(4.10) \quad (\varphi_0^0, \varphi_0^1) := (\eta_B, \eta_S)$$

without affecting the initial wealth, gains, or optimality. From now on,  $\varphi$  refers to this slightly changed strategy. The case (4.6) happens if the initial portfolio is not situated in the no-trade region of the transaction costs model, which makes an initial bulk trade necessary.

(4.7) implies that the optimal strategy  $\varphi$  is of finite variation and constant until  $\tilde{S}$  visits the boundary  $\{\underline{S}, \bar{S}\}$  the next time. Since sales and purchases take place at the same prices as in the market with transaction costs  $\underline{\lambda}, \bar{\lambda}$  and price process  $S$ , the portfolio/consumption pair  $(\varphi, c)$  is admissible in this market as well. Conversely, since shares can be bought at least as cheaply and sold at least as expensively, any admissible consumption rate in the market with price process  $S$  and transaction costs is admissible in the frictionless market with price process  $\tilde{S}$ , too. Hence,  $(\varphi, c)$  is optimal in the market with transaction costs as well. Made precise, this is stated in the following theorem.

**THEOREM 4.6.** *The portfolio/consumption pair  $(\varphi, c)$  defined in Lemma 4.5 and (4.10) is also optimal in the market with price process  $S$  and proportional transaction costs  $\bar{\lambda}, \underline{\lambda}$ . In particular,  $\tilde{S}$  is a shadow price process in this market.*

**PROOF.** Let  $((\psi^0, \psi^\uparrow - \psi^\downarrow), \kappa)$  be an admissible portfolio/consumption pair in the market with price process  $S$  and transaction costs  $\underline{\lambda}, \bar{\lambda}$ . By  $\underline{S} \leq \tilde{S} \leq \bar{S}$  and the self-financing condition (2.4),

$$\tilde{\psi}^0 := \psi_0^0 + \int_0^\cdot \tilde{S}_t d\psi_t^\downarrow - \int_0^\cdot \tilde{S}_t d\psi_t^\uparrow - \int_0^\cdot c_t dt \geq \psi^0.$$

Together with  $\underline{S} \leq \tilde{S} \leq \bar{S}$  it follows that  $((\tilde{\psi}^0, \psi^1), \kappa)$  is an admissible portfolio/consumption pair in the frictionless market with price process  $\tilde{S}$ . By optimality of  $(\varphi, c)$  defined in Lemma 4.5, this implies

$$E\left(\int_0^\infty e^{-\delta t} \log(c_t) dt\right) \geq E\left(\int_0^\infty e^{-\delta t} \log(\kappa_t) dt\right).$$

Therefore, it remains to prove that  $(\varphi, c)$  is admissible in the market with price process  $S$  and proportional transaction costs  $\bar{\lambda}, \underline{\lambda}$ . Let us begin with  $\varphi$  as in Lemma 4.6, that is, without the modification from (4.10). Since  $\Phi$  and  $\Psi$  increase only on the sets  $\{\tilde{S} = \bar{S}\}$  and  $\{\tilde{S} = \underline{S}\}$ , respectively, the self-financing condition for  $(\varphi, c)$  and (4.7) yield

$$\begin{aligned} \varphi^0 &= \varphi_0^0 + \int_0^\cdot \tilde{S}_t d\varphi_t^\downarrow - \int_0^\cdot \tilde{S}_t d\varphi_t^\uparrow - \int_0^\cdot c_t dt \\ &= \varphi_0^0 + \int_0^\cdot \underline{S}_t d\varphi_t^\downarrow - \int_0^\cdot \bar{S}_t d\varphi_t^\uparrow - \int_0^\cdot c_t dt. \end{aligned}$$

This shows that  $(\varphi, c)$  is self-financing in the market with price process  $S$  and transaction costs  $\bar{\lambda}, \underline{\lambda}$ . We now turn back to  $\varphi$  as in (4.10). By definition of  $\tilde{S}_0$ , both sides of (2.4) are unaffected by this modification, at least if the initial values

of  $\varphi^\uparrow, \varphi^\downarrow$  are chosen accordingly. This implies that the slightly changed  $(\varphi, c)$  is self-financing for  $S, \bar{\lambda}, \underline{\lambda}$  as well. By  $\varphi^0, \varphi^1 \geq 0$ , it is also admissible. This completes the proof.  $\square$

In the language of [6], the optimal policy is  $(c, L, U)$  with

$$L_t = (\varphi_0^1 - \eta_S)^+ S_0 + \int_0^t \frac{\varphi_s^1 S_s e^{-\beta_s}}{1 + e^{-\beta_s}} d\Phi_s,$$

$$U_t = (\varphi_0^1 - \eta_S)^- S_0 + \int_0^t \frac{\varphi_s^1 S_s e^{-\beta_s}}{1 + e^{-\beta_s}} d\Psi_s.$$

In particular, it belongs to the slightly smaller set of admissible controls in [6, 19], where the cumulative values  $L, U$  of purchases and sales are supposed to be right continuous. Therefore, the optimal strategies in our and their setup coincide.

REMARK 4.7. In the case of logarithmic utility, it is possible to recover the shadow price  $\tilde{S}$  from the results of [6]. General results on logarithmic utility maximization in frictionless markets show that the optimal consumption rate  $c$  equals the  $1/\delta$ -fold of the investor’s current wealth measured in terms of the shadow price. Hence, the consumption rate calculated in [6] determines the shadow value process  $\tilde{V}$ , which in turn allows to back out the shadow price  $\tilde{S}$ . More precisely, the shadow price can be constructed in a very subtle way using the results of [6], as was pointed out to us by the very insightful comments of an anonymous referee: in the proof of [6], Theorem 5.1, it is shown that the value function is of the form

$$(4.11) \quad v(x, y) = \frac{1}{\delta} \log\left(p\left(\frac{x}{y}\right)\left(x + q\left(\frac{x}{y}\right)y\right)\right)$$

with functions  $p, q$  related through the identity

$$(4.12) \quad p'(x) = -p(x)q'(x)/(x + q(x)).$$

Differentiating (4.11) and inserting (4.12) leads to

$$\frac{1}{v_x(x, y)} = \delta\left(x + q\left(\frac{x}{y}\right)y\right).$$

In view of [6], Theorem 4.3, this shows that the optimal consumption policy is given by  $c = \delta(s_0 + q(\frac{s_0}{s_1})s_1)$ . By [7], Theorem 3.1, this implies that the optimal value process w.r.t. the shadow price is given by

$$(4.13) \quad \tilde{V} = s_0 + q\left(\frac{s_0}{s_1}\right)s_1.$$

A close look at the construction of the function  $q$  in the proof of [6], Theorem 5.1, reveals that  $q$  is increasing with  $q(\frac{s_0}{s_1}) = 1 - \underline{\lambda}$  when  $\frac{s_0}{s_1}$  hits the lower boundary,



resp.,  $q(\frac{s_0}{s_1}) = 1 + \bar{\lambda}$  for the upper boundary of the no-trade region. Therefore, it follows from [6], (3.1), that

$$\tilde{V} = (\varphi^0 + \Delta s_0) + q\left(\frac{s_0}{s_1}\right)(\varphi^1 S + \Delta s_1) = \varphi^0 + \varphi^1 q\left(\frac{s_0}{s_1}\right)S$$

for the optimal trading strategy

$$\varphi_t^0 = s_0(t-), \quad \varphi_t^1 := \frac{s_1(t-)}{S},$$

corresponding to the optimal policy  $(L, U)$  of [6]. This shows that  $q(\frac{s_0}{s_1})S$  coincides with the shadow price process  $\tilde{S}$  constructed above.

However, if one wants to verify that  $q(\frac{s_0}{s_1})S$  indeed is a shadow price without using the results provided here, the ensuing verification procedure appears to be as involved as our approach of dealing with the utility optimization problem and the computation of the shadow price process simultaneously. More specifically, one knows by construction that  $q(\frac{s_0}{s_1})S$  is  $[(1 - \underline{\lambda})S, (1 + \bar{\lambda})S]$ -valued and positioned at the respective boundary whenever the strategy  $\varphi$  trades. By the proof of Theorem 4.6, it therefore suffices to show that  $(\varphi, c)$  is optimal w.r.t.  $\tilde{S} = q(\frac{s_0}{s_1})S$  in order for  $q(\frac{s_0}{s_1})S$  to be a shadow price. In view of [7], Theorem 3.1, this amounts to verifying that

$$(4.14) \quad \frac{\varphi^1 q(s_0/s_1)S}{s_0 + q(s_0/s_1)s_1} = \frac{b}{c}$$

for the differential semimartingale characteristics  $(b, c, 0)$  of the continuous process  $q(\frac{s_0}{s_1})S$ . In particular, one has to prove that the properties of the function  $q$  ensure that  $S$  is an Itô process and calculate its Itô decomposition. The optimality condition (4.14) then has to be verified using [6], (5.7), which leads to rather tedious computations. Moreover, the analysis of [6] requires the technical Condition B, which is not needed for our approach.

As a side remark, it is interesting to note that this link between optimal policy and shadow price is only apparent for logarithmic utility. Therefore, it is not possible to extract the shadow price from the results of [6] for power utility functions  $u(x) = x^{1-p}/(1-p)$ . Using the present approach of solving for the optimal strategy and the shadow price simultaneously still leads to equations for the optimal strategy and the shadow price. However, the corresponding free boundary problem appears to be more complicated than its counterpart in [6]. At this stage it is not clear whether this additional complexity can be removed through suitable transformations as in the proof of [6], Theorem 5.1, or whether the shadow price is indeed more difficult to obtain than the optimal policy for power utility.

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