

Option Pricing in Multivariate Stochastic Volatility Models of OU Type*

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Abstract. We present a multivariate stochastic volatility model with leverage, which is flexible enough to recapture the individual dynamics as well as the interdependencies between several assets, while still being highly analytically tractable. First, we derive the characteristic function and give conditions that ensure its analyticity and absolute integrability in some open complex strip around zero. Therefore we can use Fourier methods to compute the prices of multiasset options efficiently. To show the applicability of our results, we propose a concrete specification, the Ornstein–Uhlenbeck (OU)–Wishart model, where the dynamics of each individual asset coincide with the popular Γ -OU Barndorff-Nielsen–Shepard model. This model can be well calibrated to market prices, which we illustrate with an example using options on the exchange rates of some major currencies. Finally, we show that covariance swaps can also be priced in closed form.

Key words. multivariate stochastic volatility models, OU-type processes, option pricing

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1. Introduction. This paper deals with the pricing of options depending on several underlying assets. While there is a vast amount of literature on the pricing of single-asset options (see, e.g., [9, 42] for an overview), the amount of literature considering the multiasset case is rather limited. This is most likely due to the fact that the trade-off between *flexibility* and *tractability* is particularly delicate in a multivariate setting. On the one hand, the model under consideration should be flexible enough to recapture stylized facts observed in real option prices. When dealing with multiple underlyings, this becomes challenging, since not only the individual assets but also their joint behavior has to be taken into account. On the other hand, one needs enough mathematical structure to calculate option prices in the first place and to be able to calibrate the model to market prices. Due to an increasing number of state variables and parameters, this is also not an easy task in a multidimensional framework. In this article we propose the multivariate Ornstein–Uhlenbeck (OU)-type stochastic volatility model of Pigorsch and Stelzer [38] in the generalized form introduced by Barndorff-Nielsen

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and Stelzer [4], which seems to present a reasonable compromise between these competing requirements.

The log-price processes $Y = (Y^1, \dots, Y^d)$ of d financial assets are modeled as

$$(1.1) \quad dY_t = (\mu + \beta(\Sigma_t)) dt + \Sigma_t^{\frac{1}{2}} dW_t + \rho(dL_t),$$

$$(1.2) \quad d\Sigma_t = (A\Sigma_t + \Sigma_t A^\top) dt + dL_t,$$

where $\mu \in \mathbb{R}^d$, A is a real $d \times d$ matrix, and β, ρ are linear operators from the real $d \times d$ matrices to \mathbb{R}^d . Moreover, W is an \mathbb{R}^d -valued Wiener process and L is an independent matrix subordinator, i.e., a Lévy process which has only positive semidefinite increments. Hence, the covariance process Σ is an OU-type process with values in the positive semidefinite matrices; cf. Barndorff-Nielsen and Stelzer [3]. Thus we call (1.1), (1.2) the *multivariate stochastic volatility model of OU-type*. The positive semidefinite OU-type process Σ introduces a stochastic volatility and, which is difficult to achieve using several univariate models, a stochastic correlation between the assets. Moreover, Σ is mean reverting and increases only by jumps. The jumps represent the arrival of new information that results in positive shocks in the volatility and positive or negative shocks in the correlation of some assets. Due to the leverage term $\rho(dL_t)$ they are correlated with price jumps. The present model is a multivariate generalization of the non-Gaussian OU-type stochastic volatility model introduced by Barndorff-Nielsen and Shephard [2] (henceforth the Barndorff-Nielsen–Shephard (BNS) model). For one underlying, these models are found to be both flexible and tractable in Nicolato and Venardos [37]. The key reason is that the characteristic function of the return process can often be computed in closed form, which allows European options to be priced efficiently using the Fourier methods introduced by Carr and Madan [8] as well as Raible [39]. In the present study, we show that a similar approach is also applicable in the multivariate case. Recently, Benth and Vos [5] have discussed a somewhat similar model in the context of energy markets. However, they do not establish conditions for the applicability of Fourier pricing and, more importantly, do not calibrate their model to market prices.

Alternatively, the covariance process Σ can also be modeled by other processes taking values in the positive semidefinite matrices. In particular, several authors have advocated using a diffusion model based on the Wishart process; cf., e.g., Da Fonseca, Grasselli, and Tebaldi [13], Gouriéroux [20], Gouriéroux and Sufana [21], and Da Fonseca and Grasselli [11]. This leads to a multivariate generalization of the model of Heston [24]. However, there is empirical evidence suggesting that volatility jumps (together with the stock price) (cf. Jacod and Todorov [31]), which cannot be recaptured by a diffusion model. Moreover, the treatment of square-root processes on the cone of positive semidefinite matrices is mathematically quite involved; see Cuchiero et al. [10].¹ For example, whereas Da Fonseca and Grasselli [11] have very recently succeeded in calibrating their model to market prices, the resulting parameters do not satisfy the drift condition for the existence of the underlying square-root diffusion, suggesting that a more sophisticated optimization routine is necessary.

¹This study generalizes the theory of *affine* processes from the positive univariate factors treated in [16, 15] to factor processes taking values in the cone of symmetric positive semidefinite matrices. In particular, to ensure the existence of square-root processes, a quite intricate drift condition turns out to be necessary.

Another possible approach is to consider multivariate models based on a concatenation of univariate building blocks. This approach is taken, e.g., by Luciano and Schoutens [34] using Lévy processes, by Dimitroff, Lorenz, and Szimayer [14], who consider a multivariate Heston model, and by Hubalek and Nicolato [27], who put forward a multifactor BNS model. However, all these models either have a somewhat limited capability to catch complex dependence structures (compare section 4.2) or lead to tricky (factor) identification issues. Apart from models where all parameters are determined by single-asset options, we are not aware of successful calibrations of such models. The paper of Ma [35] proposes a two-dimensional Black–Scholes model where the correlation between the two Brownian motions is stochastic and given by a diffusion process with values in an interval contained in $[-1, 1]$. However, pricing can be done only via Monte Carlo simulation in this model. In addition, an extension to higher dimensions is not obvious, since the necessary positive semidefiniteness of the correlation matrix of the Brownian noise imposes additional constraints, which are hard to incorporate.

The remainder of this paper is organized as follows. Sections 2.1 and 2.2 introduce the multivariate stochastic volatility model of OU type. Afterwards, we derive the joint characteristic function of (Y_t, Σ_t) . We then show in section 2.4 that a simple moment condition on L implies analyticity and absolute integrability of the moment generating function of Y_t in some open complex strip around zero. Equivalent martingale measures are discussed in section 2.5, where we also present a subclass that preserves the structure of our model. In section 3, we recall how to use Fourier methods to compute prices of multiasset options efficiently. Subsequently, we propose the OU–Wishart model, where L is a compound Poisson process with Wishart distributed jumps. It turns out that the OU–Wishart model has margins which are in distribution equivalent to a Γ -OU BNS model, one of the tractable specifications commonly used in the univariate case. Moreover, the characteristic function can be computed in closed form, which makes option pricing and calibration particularly feasible. In an illustrative example we calibrate a bivariate OU–Wishart model to market prices, and we compare its performance to the multivariate variance gamma model of [34] and a multivariate extension with stochastic volatility. As a final application, we show in section 5 that covariance swaps can also be priced in closed form. Appendix A contains a result on multidimensional analytic functions which is needed to establish the regularity of the moment generating function in section 2.4.

Notation. $M_{d,n}(\mathbb{R})$ (resp., $M_{d,n}(\mathbb{C})$) represent the $d \times n$ matrices with real (resp., complex) entries. We abbreviate $M_d(\cdot) = M_{d,d}(\cdot)$. \mathbb{S}_d denotes the subspace of $M_d(\mathbb{R})$ of all symmetric matrices. We write \mathbb{S}_d^+ for the cone of all positive semidefinite matrices and \mathbb{S}_d^{++} for the open cone of all positive definite matrices. The identity matrix in $M_d(\mathbb{R})$ is denoted by I_d . $\sigma(A)$ denotes the set of all eigenvalues of $A \in M_d(\mathbb{C})$. We write $\operatorname{Re}(z)$ or $\operatorname{Im}(z)$ for the real or imaginary part of $z \in \mathbb{C}^d$ or $z \in M_d(\mathbb{C})$, which has to be understood componentwise. The components of a vector or matrix are denoted by subscripts; however, for stochastic processes we use superscripts to avoid double indices.

On \mathbb{R}^d , we typically use the Euclidean scalar product, $\langle x, y \rangle_{\mathbb{R}^d} := x^\top y$, and on $M_d(\mathbb{R})$ or \mathbb{S}_d the scalar products given by $\langle A, B \rangle_{M_d(\mathbb{R})} := \operatorname{tr}(A^\top B)$ or $\langle A, B \rangle_{\mathbb{S}_d} := \operatorname{tr}(AB)$, respectively. However, due to the equivalence of all norms on finite-dimensional vector spaces, most results hold independently of the norm. We also write $\langle x, y \rangle = x^\top y$ for $x, y \in \mathbb{C}^d$, although this is

only a bilinear form but not a scalar product on \mathbb{C}^d .

We denote by $\text{vec} : M_d(\mathbb{R}) \rightarrow \mathbb{R}^{d^2}$ the bijective linear operator that stacks the columns of a matrix below one other. With the above norms, vec is a Hilbert space isometry. Likewise, for a symmetric matrix $S \in \mathbb{S}_d$ we denote by $\text{vech}(S)$ the vector consisting of the columns of the upper-diagonal part including the diagonal.

Furthermore, we employ an intuitive notation concerning integration with respect to matrix-valued processes. For an $M_{m,n}(\mathbb{R})$ -valued Lévy process L , and $M_{d,m}(\mathbb{R})$ - (resp., $M_{n,p}(\mathbb{R})$ -) valued processes X, Y integrable with respect to L , the term $\int_0^t X_s dL_s Y_s$ is to be understood as the $d \times p$ (random) matrix with (i, j) th entry $\sum_{k=1}^m \sum_{l=1}^n \int_0^t X_s^{ik} dL_s^{kl} Y_s^{lj}$.

2. The multivariate stochastic volatility model of OU type. For the remainder of the paper, fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ in the sense of [30, Definition I.1.3], where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ is trivial and $T > 0$ is a fixed terminal time.

2.1. Positive semidefinite processes of OU type. To formulate our model, we need to introduce the concept of matrix subordinators as studied in [1].

Definition 2.1. An \mathbb{S}_d -valued Lévy process $L = (L_t)_{t \in \mathbb{R}_+}$ is called a matrix subordinator if $L_t - L_s \in \mathbb{S}_d^+$ for all $t > s$.

The characteristic function of a matrix subordinator L is given by $E(e^{i\text{tr}(ZL_1)}) = \exp(\psi_L(Z))$ for the characteristic exponent

$$\psi_L(Z) = i\text{tr}(\gamma_L Z) + \int_{\mathbb{S}_d^+} (e^{i\text{tr}(XZ)} - 1) \kappa_L(dX), \quad Z \in M_d(\mathbb{R}),$$

where $\gamma_L \in \mathbb{S}_d^+$ and κ_L is a Lévy measure on \mathbb{S}_d satisfying $\kappa_L(\mathbb{S}_d \setminus \mathbb{S}_d^+) = 0$ as well as $\int_{\{\|X\| \leq 1\}} \|X\| \kappa_L(dX) < \infty$.

Positive semidefinite processes of OU type are a generalization of nonnegative OU-type processes (cf. [3]). Let L be a matrix subordinator, and let $A \in M_d(\mathbb{R})$. The positive semidefinite OU-type process $\Sigma = (\Sigma_t)_{t \in \mathbb{R}_+}$ is defined as the unique strong solution to the stochastic differential equation

$$(2.1) \quad d\Sigma_t = (A\Sigma_t + \Sigma_t A^\top) dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^+.$$

It is given by

$$(2.2) \quad \Sigma_t = e^{At} \Sigma_0 e^{A^\top t} + \int_0^t e^{A(t-s)} dL_s e^{A^\top(t-s)}.$$

Since $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}_+$, this process can be used to model the stochastic evolution of a covariance matrix. As in the univariate case there exists a closed-form expression for the integrated volatility. Suppose

$$(2.3) \quad 0 \notin \sigma(A) + \sigma(A).$$

Then, the integrated OU-type process Σ^+ is given by

$$(2.4) \quad \Sigma_t^+ := \int_0^t \Sigma_s ds = \mathbf{A}^{-1}(\Sigma_t - \Sigma_0 - L_t),$$

where $\mathbf{A} : X \mapsto AX + XA^\top$. Note that condition (2.3) implies that the operator \mathbf{A} is invertible; cf. [26, Theorem 4.4.5]. In the case where Σ is *mean reverting*, i.e., A has only eigenvalues with strictly negative real part, condition (2.3) is trivially satisfied.

2.2. Definition and marginal dynamics of the model. The following model was introduced and studied in [38] from a statistical point of view in the no-leverage case and has also been considered in [4]. Here we discuss its applicability to option pricing.

Let L be a matrix subordinator with characteristic exponent ψ_L and let W be an independent \mathbb{R}^d -valued Wiener process. The *multivariate stochastic volatility model of OU type* is then given by

$$(2.5) \quad dY_t = (\mu + \beta(\Sigma_t)) dt + \Sigma_t^{\frac{1}{2}} dW_t + \rho(dL_t), \quad Y_0 \in \mathbb{R}^d,$$

$$(2.6) \quad d\Sigma_t = (A\Sigma_t + \Sigma_t A^\top) dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^+,$$

with linear operators $\beta, \rho : M_d(\mathbb{R}) \rightarrow \mathbb{R}^d$, $\mu \in \mathbb{R}^d$, and $A \in M_d(\mathbb{R})$ such that $0 \notin \sigma(A) + \sigma(A)$.

We have specified the *risk premium* β and the *leverage operator* ρ in a quite general form. The following specification turns out to be particularly tractable.

Definition 2.2. We call β and ρ diagonal if, for $\beta_1, \dots, \beta_d \in \mathbb{R}$ and $\rho_1, \dots, \rho_d \in \mathbb{R}$,

$$\beta(X) = \begin{pmatrix} \beta_1 X_{11} \\ \vdots \\ \beta_d X_{dd} \end{pmatrix}, \quad \rho(X) = \begin{pmatrix} \rho_1 X_{11} \\ \vdots \\ \rho_d X_{dd} \end{pmatrix}, \quad \forall X \in M_d(\mathbb{R}).$$

In the following, $\beta^i(X)$ or $\rho^i(X)$, $i \in \{1, \dots, d\}$, denote the i th component of the vector $\beta(X)$ or $\rho(X)$, respectively. The marginal dynamics of the individual assets have been derived in [4, Proposition 4.3].

Theorem 2.3. Let $i \in \{1, \dots, d\}$. Then we have

$$(Y_t^i)_{t \in \mathbb{R}_+} \stackrel{fidi}{=} \left(\mu_i t + \beta^i(\Sigma_t^+) + \int_0^t (\Sigma_s^{ii})^{\frac{1}{2}} dW_s^i + \rho^i(L_t) \right)_{t \in \mathbb{R}_+},$$

where $\stackrel{fidi}{=}$ denotes equality of all finite-dimensional distributions.

Let us now consider the case where A is a diagonal matrix,

$$A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix},$$

and β, ρ are diagonal as well. Then, for every $i \in \{1, \dots, d\}$, we have

$$(2.7) \quad dY_t^i \stackrel{fidi}{=} (\mu_i + \beta_i \Sigma_t^{ii}) dt + (\Sigma_t^{ii})^{\frac{1}{2}} dW_t^i + \rho_i dL_t^{ii},$$

$$(2.8) \quad d\Sigma_t^{ii} = 2a_i \Sigma_t^{ii} dt + dL_t^{ii}.$$

Evidently, every diagonal element L^{ii} , $i = 1, \dots, d$, of a matrix subordinator L is a univariate subordinator, and thus Σ^{ii} is a nonnegative OU-type process. Consequently, the model for the i th asset is equivalent in distribution to a univariate BNS model.

2.3. Characteristic function. Let $\langle \cdot, \cdot \rangle_V, \langle \cdot, \cdot \rangle_W$ be bilinear forms as introduced in the notation, where V, W may be one of $\mathbb{R}^d, \mathbb{C}^d$, or $M_d(\cdot)$. Given a linear operator $T : V \rightarrow W$, the *adjoint* $T^* : W \rightarrow V$ is the unique linear operator such that $\langle Tx, y \rangle_W = \langle x, T^*y \rangle_V$ for all $x \in V$ and $y \in W$. Directly by definition we obtain the following.

Lemma 2.4. *Let $y \in \mathbb{R}^d, z \in M_d(\mathbb{R})$, and $t \in \mathbb{R}_+$. Then the adjoints of the linear operators*

$$\begin{aligned} \mathbf{A} : X &\mapsto AX + XA^\top, & \mathcal{B}(t) : X &\mapsto e^{At}Xe^{A^\top t} - X, \\ \mathcal{C}(t) : X &\mapsto e^{At}Xe^{A^\top t}z + \beta(\mathbf{A}^{-1}(\mathcal{B}(t)X))y^\top + \rho(X)y^\top + \frac{i}{2}yy^\top\mathbf{A}^{-1}(\mathcal{B}(t)X) \end{aligned}$$

on $M_d(\mathbb{C})$ are given by

$$\begin{aligned} \mathbf{A}^* : X &\mapsto A^\top X + XA, & \mathcal{B}(t)^* : X &\mapsto e^{A^\top t}Xe^{At} - X, \\ \mathcal{C}(t)^* : X &\mapsto e^{A^\top t}Xz^\top e^{At} + \rho^*(Xy) + \mathcal{B}(t)^*\mathbf{A}^{-*} \left(\beta^*(Xy) + \frac{i}{2}Xyy^\top \right). \end{aligned}$$

Note that for diagonal ρ it holds that

$$\rho^*(X) = \begin{pmatrix} \rho_1 X_{11} & & 0 \\ & \ddots & \\ 0 & & \rho_d X_{dd} \end{pmatrix}$$

for all $X \in M_d(\mathbb{R})$.

Our main objective in this section is to compute the joint characteristic function of (Y_t, Σ_t) . This will pave the way for Fourier pricing of multiasset options later on. Note that we use the scalar product

$$\langle (x_1, y_1), (x_2, y_2) \rangle := x_1^\top x_2 + \text{tr}(y_1^\top y_2)$$

on $\mathbb{R}^d \times M_d(\mathbb{R})$.

Theorem 2.5 (joint characteristic function). *For every $(y, z) \in \mathbb{R}^d \times M_d(\mathbb{R})$ and $t \in \mathbb{R}_+$, the joint characteristic function of (Y_t, Σ_t) is given by*

$$\begin{aligned} &E[\exp(i \langle (y, z), (Y_t, \Sigma_t) \rangle)] \\ &= \exp \left\{ iy^\top(Y_0 + \mu t) + i \text{tr}(\Sigma_0 e^{A^\top t} z e^{At}) \right. \\ &\quad + i \text{tr} \left(\Sigma_0 \left(e^{A^\top t} \mathbf{A}^{-*} \left(\beta^*(y) + \frac{i}{2} yy^\top \right) e^{At} - \mathbf{A}^{-*} \left(\beta^*(y) + \frac{i}{2} yy^\top \right) \right) \right) \\ &\quad \left. + \int_0^t \psi_L \left(e^{A^\top s} z e^{As} + \rho^*(y) + e^{A^\top s} \mathbf{A}^{-*} \left(\beta^*(y) + \frac{i}{2} yy^\top \right) e^{As} - \mathbf{A}^{-*} \left(\beta^*(y) + \frac{i}{2} yy^\top \right) \right) ds \right\}, \end{aligned}$$

where $\mathbf{A}^{-*} := (\mathbf{A}^*)^{-1}$ denotes the inverse of the adjoint of $\mathbf{A} : X \mapsto AX + XA^\top$, that is, the inverse of $\mathbf{A}^* : X \mapsto A^\top X + XA$.

Note that for $z = 0$ we obtain the characteristic function of Y_t .

Proof. Since Σ is adapted to the filtration generated by L , and by the independence of L and W ,

$$\begin{aligned} & E[\exp(\langle (y, z), (Y_t, \Sigma_t) \rangle)] \\ &= e^{iy^\top(Y_0 + \mu t)} E \left[e^{i\text{tr}(z^\top \Sigma_t) + iy^\top(\beta(\Sigma_t^+) + \rho(L_t))} E \left(e^{iy^\top \int_0^t \Sigma_s^{\frac{1}{2}} dW_s} \middle| (L_s)_{s \in \mathbb{R}_+} \right) \right] \\ &= e^{iy^\top(Y_0 + \mu t)} E \left[e^{i\text{tr}(z^\top \Sigma_t) + iy^\top(\beta(\Sigma_t^+) + \rho(L_t))} \exp \left(-\frac{1}{2} y^\top \Sigma_t^+ y \right) \right]. \end{aligned}$$

By (2.4) and using the fact that the trace is invariant under cyclic permutations the last term equals

$$e^{iy^\top(Y_0 + \mu t)} E \left[e^{i\text{tr}(z^\top \Sigma_t + \beta(\mathbf{A}^{-1}(\Sigma_t - \Sigma_0 - L_t))y^\top + \rho(L_t)y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1}(\Sigma_t - \Sigma_0 - L_t))} \right].$$

In view of (2.2), we have

$$\Sigma_t - \Sigma_0 - L_t = \int_0^t \mathcal{B}(t-s) dL_s + \mathcal{B}(t)\Sigma_0$$

for the linear operator $\mathcal{B}(t)$ from Lemma 2.4. Therefore,

$$\begin{aligned} & E[\exp(i \langle (y, z), (Y_t, \Sigma_t) \rangle)] \\ &= \exp \left(iy^\top(Y_0 + \mu t) + i\text{tr} \left(z^\top e^{At} \Sigma_0 e^{A^\top t} + \beta(\mathbf{A}^{-1}(\mathcal{B}(t)\Sigma_0))y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1}(\mathcal{B}(t)\Sigma_0) \right) \right) \\ &\quad \times E \left[\exp \left(i\text{tr} \left(z^\top \int_0^t e^{A(t-s)} dL_s e^{A^\top(t-s)} + \beta \left(\mathbf{A}^{-1} \left(\int_0^t e^{A(t-s)} dL_s e^{A^\top(t-s)} - L_t \right) \right) y^\top \right. \right. \right. \\ &\quad \left. \left. \left. + \rho(L_t)y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1} \left(\int_0^t e^{A(t-s)} dL_s e^{A^\top(t-s)} - L_t \right) \right) \right) \right] \\ &= \exp \left(iy^\top(Y_0 + \mu t) + i\text{tr} \left(z^\top e^{At} \Sigma_0 e^{A^\top t} + \beta(\mathbf{A}^{-1}(\mathcal{B}(t)\Sigma_0))y^\top + \frac{i}{2}yy^\top \mathbf{A}^{-1}(\mathcal{B}(t)\Sigma_0) \right) \right) \\ &\quad \times E \left[\exp \left(i\text{tr} \left(\left(\int_0^t \mathcal{C}(t-s) dL_s \right)^\top I_d \right) \right) \right] \end{aligned}$$

with the linear operator $\mathcal{C}(t)$ from Lemma 2.4, since $\mathbf{A}^{-1}(\int_0^t e^{A(t-s)} dL_s e^{A^\top(t-s)} - L_t) \in \mathbb{S}_d$. An immediate multivariate generalization of results obtained in [40, Proposition 2.4] (see also [18, Lemma 3.1]) yields an explicit formula for the expectation above:

$$E \left[\exp \left(i\text{tr} \left(\left(\int_0^t \mathcal{C}(t-s) dL_s \right)^\top I_d \right) \right) \right] = \exp \left(\int_0^t \psi_L(\mathcal{C}(s)^* I_d) ds \right).$$

By Lemma 2.4 we have

$$e^{\int_0^t \psi_L(\mathcal{C}(s)^* I_d) ds} = e^{\int_0^t \psi_L \left(e^{A^\top s} z^\top e^{As} + \rho^*(y) + e^{A^\top s} \mathbf{A}^{-*} (\beta^*(y) + \frac{i}{2}yy^\top) e^{As} - \mathbf{A}^{-*} (\beta^*(y) + \frac{i}{2}yy^\top) \right) ds}.$$

This expression is well defined, because

$$e^{A^\top s} z^\top e^{As} + \rho^*(y) + e^{A^\top s} \mathbf{A}^{-*} \left(\beta^*(y) + \frac{i}{2} yy^\top \right) e^{As} - \mathbf{A}^{-*} \left(\beta^*(y) + \frac{i}{2} yy^\top \right) \in M_d(\mathbb{R}) + i\mathbb{S}_d^+$$

for all $s \in [0, t]$. Indeed, this follows from

$$(2.9) \quad e^{A^\top s} \mathbf{A}^{-*} (yy^\top) e^{As} - \mathbf{A}^{-*} (yy^\top) = \int_0^s e^{A^\top u} yy^\top e^{Au} du \in \mathbb{S}_d^+.$$

Finally, we infer from Lemma 2.4 that

$$\operatorname{tr} \left(\beta(\mathbf{A}^{-1}(\mathcal{B}(t)\Sigma_0))y^\top + \frac{i}{2} yy^\top \mathbf{A}^{-1}(\mathcal{B}(t)\Sigma_0) \right) = \operatorname{tr} \left(\Sigma_0 \left(\mathcal{B}(t)^* \mathbf{A}^{-*} \left(\beta^*(y) + \frac{i}{2} yy^\top \right) \right) \right),$$

which gives the desired result by noting that $\operatorname{tr}(z\Sigma_t) = \operatorname{tr}(z^\top \Sigma_t)$. \blacksquare

2.4. Regularity of the moment generating function. In this section we provide conditions ensuring that the characteristic function of Y_t admits an analytic extension Φ_{Y_t} to some open convex neighborhood of 0 in \mathbb{C}^d . Afterwards, we show absolute integrability. The regularity results obtained in this section will allow us to apply Fourier methods in section 3 to compute option prices efficiently.

Definition 2.6. For any $t \in [0, T]$, the moment generating function of Y_t is defined as

$$\Phi_{Y_t}(y) := E[\exp(y^\top Y_t)]$$

for all $y \in \mathbb{C}^d$ such that the expectation exists.

Note that Φ_{Y_t} may not exist anywhere but on $i\mathbb{R}^d$, where it coincides with the characteristic function of Y_t . The next lemma is a first step towards conditions for the existence and analyticity of the moment generating function Φ_{Y_t} in a complex neighborhood of zero.

Lemma 2.7. Let L be a matrix subordinator with cumulant transform Θ_L , that is,

$$\Theta_L(Z) = \psi_L(-iZ) = \operatorname{tr}(\gamma_L Z) + \int_{\mathbb{S}_d^+} (e^{\operatorname{tr}(XZ)} - 1) \kappa_L(dX), \quad Z \in M_d(\mathbb{C}),$$

and let $\epsilon > 0$. Then Θ_L is analytic on the open convex set

$$(2.10) \quad S_\epsilon := \{Z \in M_d(\mathbb{C}) : \|\operatorname{Re}(Z)\| < \epsilon\} - \mathbb{S}_d^+$$

if and only if

$$(2.11) \quad \int_{\{\|X\| \geq 1\}} e^{\operatorname{tr}(RX)} \kappa_L(dX) < \infty \quad \forall R \in M_d(\mathbb{R}) \text{ with } \|R\| < \epsilon.$$

Proof. If (2.11) holds, [15, Lemma A.2] implies that $Z \mapsto E(e^{\operatorname{tr}(ZL_1)}) = e^{\Theta_L(Z)}$ is analytic on S_ϵ . Due to assumption (2.11), dominated convergence yields that Θ_L is continuous on S_ϵ . The claim now follows from Lemma A.1. Conversely, if Θ_L is analytic on S_ϵ , then [15,

Lemma A.4] implies that $E(\text{tr}(ZL_1)) = e^{\Theta_L(Z)}$ for all $Z \in S_\epsilon$. Thus, by [41, Theorem 25.17], condition (2.11) holds. \blacksquare

The next theorem is a nontrivial (especially due to the involved heavy matrix calculus) generalization of [37, Theorem 2.2] to the multivariate case. It holds for all submultiplicative matrix norms on $M_d(\mathbb{R})$ that satisfy $\|yy^\top\| = \|y\|^2$ for all $y \in \mathbb{R}^d$, where we use the Euclidean norm on \mathbb{R}^d . For example, this holds true for the Frobenius norm and the spectral norm (the operator norm associated with the Euclidean norm).

Theorem 2.8 (strip of analyticity). *Suppose the matrix subordinator L satisfies*

$$(2.12) \quad \int_{\{\|X\| \geq 1\}} e^{\text{tr}(RX)} \kappa_L(dX) < \infty \quad \forall R \in M_d(\mathbb{R}) \text{ with } \|R\| < \epsilon$$

for some $\epsilon > 0$. Then the moment generating function Φ_{Y_t} of Y_t is analytic on the open convex set

$$S_\theta := \{y \in \mathbb{C}^d : \|\text{Re}(y)\| < \theta\},$$

where

$$(2.13) \quad \theta := -\frac{\|\rho\|}{(e^{2\|A\|t} + 1)\|\mathbf{A}^{-1}\|} - \|\beta\| + \sqrt{\Delta} > 0$$

with

$$\Delta := \left(\frac{\|\rho\|}{(e^{2\|A\|t} + 1)\|\mathbf{A}^{-1}\|} + \|\beta\| \right)^2 + \frac{2\epsilon}{(e^{2\|A\|t} + 1)\|\mathbf{A}^{-1}\|}.$$

Moreover,

$$(2.14) \quad \Phi_{Y_t}(y) = \exp \left(y^\top(Y_0 + \mu t) + \text{tr}(\Sigma_0 H_y(t)) + \int_0^t \Theta_L(H_y(s) + \rho^*(y)) ds \right)$$

for all $y \in S_\theta$, where

$$(2.15) \quad H_y(s) := e^{A^\top s} \mathbf{A}^{-*} \left(\beta^*(y) + \frac{1}{2}yy^\top \right) e^{As} - \mathbf{A}^{-*} \left(\beta^*(y) + \frac{1}{2}yy^\top \right).$$

Proof. The main part of the proof is to show that the function

$$G(y) := \exp \left(y^\top(Y_0 + \mu t) + \text{tr}(\Sigma_0 H_y(t)) + \int_0^t \Theta_L(H_y(s) + \rho^*(y)) ds \right)$$

is analytic on S_θ . First we want to find a θ such that for all $u \in \mathbb{R}^d$ with $\|u\| < \theta$, it holds that $\|H_u(s) + \rho^*(u)\| < \epsilon$ for all $s \in [0, t]$. Since

$$\begin{aligned} \|H_u(s) + \rho^*(u)\| &= \left\| e^{A^\top s} \mathbf{A}^{-*} \left(\beta^*(u) + \frac{1}{2}uu^\top \right) e^{As} - \mathbf{A}^{-*} \left(\beta^*(u) + \frac{1}{2}uu^\top \right) + \rho^*(u) \right\| \\ &\leq \frac{1}{2}(e^{2\|A\|t} + 1)\|\mathbf{A}^{-1}\|\|u\|^2 + \left(\|\rho\| + (e^{2\|A\|t} + 1)\|\mathbf{A}^{-1}\|\|\beta\| \right)\|u\|, \end{aligned}$$

we have to find the roots of the polynomial

$$p(x) := \frac{1}{2}(e^{2\|A\|t} + 1) \|\mathbf{A}^{-1}\| x^2 + \left(\|\rho\| + (e^{2\|A\|t} + 1) \|\mathbf{A}^{-1}\| \|\beta\| \right) x - \epsilon.$$

The positive one is given by θ as stated in (2.13). Note that $\theta > 0$, because p is a cup-shaped parabola with $p(0) = -\epsilon < 0$.

Now let $y \in S_\theta$, i.e., $y = u + iv$ with $\|u\| < \theta$. Using $\operatorname{Re}(yy^\top) = uu^\top - vv^\top$ and (2.9) we get

$$\begin{aligned} \operatorname{Re}(H_y(s) + \rho^*(y)) &= H_u(s) + \rho^*(u) - \frac{1}{2} \left(e^{A^\top s} \mathbf{A}^{-*} (vv^\top) e^{As} - \mathbf{A}^{-*} (vv^\top) \right) \\ &= H_u(s) + \rho^*(u) - \frac{1}{2} \int_0^s e^{A^\top r} vv^\top e^{Ar} dr. \end{aligned}$$

Because of $\int_0^s e^{A^\top r} vv^\top e^{Ar} dr \in \mathbb{S}_d^+$, we have

$$\begin{aligned} &\int_{\{\|X\| \geq 1\}} e^{\operatorname{tr}(\operatorname{Re}(H_y(s) + \rho^*(y))X)} \kappa_L(dX) \\ &= \int_{\{\|X\| \geq 1\}} e^{\operatorname{tr}((H_u(s) + \rho^*(u))X)} e^{-\frac{1}{2} \operatorname{tr} \left(\left(\int_0^s e^{A^\top r} vv^\top e^{Ar} dr \right) X \right)} \kappa_L(dX) < \infty \end{aligned}$$

by assumption (2.12), since $\|H_u(s) + \rho^*(u)\| < \epsilon$. Thus, by Lemma 2.7 the function

$$S_\theta \in y \mapsto \Theta_L(H_y(s) + \rho^*(y))$$

is analytic on S_θ for every $s \in [0, t]$. An application of Fubini's and Morera's theorems shows that integration over $[0, t]$ preserves analyticity (cf. [33, p. 228]); hence G is analytic on S_θ .

Obviously, we have $\Phi_{Y_t}(iy) = G(iy)$ for all $y \in \mathbb{R}^d$ by Theorem 2.5 and the definition of G . Thus, [15, Lemma A.4] finally implies $\Phi_{Y_t} \equiv G$ on S_θ . ■

With Theorem 2.8 at hand, we can establish the following result.

Theorem 2.9 (absolute integrability). *If (2.12) holds for some $\epsilon > 0$, then $w \mapsto \Phi_{Y_t}(y + iw)$ is absolutely integrable, for all $y \in \mathbb{R}^d$ with $\|y\| < \theta$, where θ is given as in Theorem 2.8.*

Proof. As in the proof of Theorem 2.8, we obtain from

$$\operatorname{Re}(H_{y+iw}(s)) = H_y(s) - \frac{1}{2} \int_0^s e^{A^\top r} ww^\top e^{Ar} dr$$

and $\operatorname{Re}(e^{\operatorname{tr}(Z)}) \leq |e^{\operatorname{tr}(Z)}| = e^{\operatorname{Re}(\operatorname{tr}(Z))} = e^{\operatorname{tr}(\operatorname{Re}(Z))}$ for $Z \in M_d(\mathbb{C})$ that

$$\begin{aligned} &\operatorname{Re} \left(\int_0^t \int_{\mathbb{S}_d^+} \left(e^{\operatorname{tr}((H_{y+iw}(s) + \rho^*(y+iw))X)} - 1 \right) \kappa_L(dX) ds \right) \\ &\leq \int_0^t \int_{\mathbb{S}_d^+} \left(e^{\operatorname{tr}((H_y(s) + \rho^*(y))X)} - 1 \right) \kappa_L(dX) ds. \end{aligned}$$

Using this inequality yields

$$\begin{aligned} & |\Phi_{Y_t}(y + iw)| \\ & \leq \Phi_{Y_t}(y) e^{-\frac{1}{2} \text{tr}(\Sigma_0 (e^{A^\top t} \mathbf{A}^{-*} (ww^\top) e^{At} - \mathbf{A}^{-*} (ww^\top))) - \frac{1}{2} \int_0^t \text{tr}(\gamma_L (e^{A^\top s} \mathbf{A}^{-*} (ww^\top) e^{As} - \mathbf{A}^{-*} (ww^\top))) ds} \\ & = \Phi_{Y_t}(y) e^{-\frac{1}{2} \langle (\mathbf{A}^{-1} \mathcal{B}(t)(\Sigma_0) + \int_0^t \mathbf{A}^{-1} \mathcal{B}(s)(\gamma_L) ds) w, w \rangle} \end{aligned}$$

with $\mathcal{B}(t)$ as in Lemma 2.4. Note that $\mathbf{A}^{-1} \mathcal{B}(t)(\Sigma_0) + \int_0^t \mathbf{A}^{-1} \mathcal{B}(s)(\gamma_L) ds \in \mathbb{S}_d^+$, and hence

$$\int_{\mathbb{R}^d} |\Phi_{Y_t}(y + iw)| dw \leq \Phi_{Y_t}(y) \int_{\mathbb{R}^d} e^{-\frac{1}{2} \langle (\mathbf{A}^{-1} \mathcal{B}(t)(\Sigma_0) + \int_0^t \mathbf{A}^{-1} \mathcal{B}(s)(\gamma_L) ds) w, w \rangle} dw < \infty,$$

by Theorem 2.8, and because the integrand is proportional to the density of a multivariate normal distribution. ■

2.5. Martingale conditions and equivalent Martingale measures. For notational convenience, we work in this section with the model

$$(2.16) \quad dY_t = (\mu + \beta(\Sigma_t)) dt + \Sigma_t^{\frac{1}{2}} dW_t + \rho(dL_t), \quad Y_0 \in \mathbb{R}^d,$$

$$(2.17) \quad d\Sigma_t = (\gamma_L + A\Sigma_t + \Sigma_t A^\top) dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^{++},$$

where L is a driftless matrix subordinator with Lévy measure κ_L . Clearly, this is our multivariate stochastic volatility model of OU type (2.5), (2.6), except that μ in (2.5) is replaced by $\mu - \rho(\gamma_L)$, such that there is no deterministic drift from the leverage term $\rho(dL_t)$.

In mathematical finance, Y is used to model the joint dynamics of the log-returns of d assets with price processes $S_t^i = S_0^i e^{Y_t^i}$, where we set $Y_0^i = 0$ from now on and, hence, S_0 denotes the vector of initial prices.

The martingale property of the *discounted stock prices* $(e^{-rt} S_t)_{t \in [0, T]}$ for a constant interest rate $r > 0$ can be characterized as follows.

Theorem 2.10. *The discounted price process $(e^{-rt} S_t)_{t \in [0, T]}$ is a martingale if and only if, for $i = 1, \dots, d$,*

$$(2.18) \quad \int_{\{\|X\| > 1\}} e^{\rho^i(X)} \kappa_L(dX) < \infty$$

and

$$(2.19) \quad \beta^i(X) = -\frac{1}{2} X_{ii}, \quad X \in \mathbb{S}_d^+,$$

$$(2.20) \quad \mu_i = r - \int_{\mathbb{S}_d^+} (e^{\rho^i(X)} - 1) \kappa_L(dX).$$

Proof. Define $\widehat{S}_t := e^{-rt} S_t$ for all $t \in [0, T]$, and let $i \in \{1, \dots, d\}$. By Itô's formula and [30, Proposition III.6.35], \widehat{S}^i is a local martingale if and only if (2.18), (2.19), and (2.20) hold. Thus it remains to show that it is actually a true martingale under the stated assumptions. Since \widehat{S} is a positive local martingale, it is a supermartingale and hence a martingale if and

only if $E(\widehat{S}_T^i) = \widehat{S}_0^i$ for all $i \in \{1, \dots, d\}$. This can be seen as follows. By Theorem 2.3, (2.19), and (2.20) we have

$$\begin{aligned} E(\widehat{S}_T^i) &= \widehat{S}_0^i E \left(\exp \left((\mu^i - r)T + \beta^i(\Sigma_T^+) + \int_0^T (\Sigma_s^{ii})^{\frac{1}{2}} dW_s^i + \rho^i(L_T) \right) \right) \\ &= \widehat{S}_0^i e^{-T \int_{\mathbb{S}_d^+} (e^{\rho^i(X)} - 1) \kappa_L(dX)} E \left(e^{-\frac{1}{2}(\Sigma_T^+)^{ii} + \rho^i(L_T)} E \left(e^{\int_0^T (\Sigma_s^{ii})^{\frac{1}{2}} dW_s^i} \middle| (L_s)_{s \in [0, T]} \right) \right) \\ &= \widehat{S}_0^i e^{-T \int_{\mathbb{S}_d^+} (e^{\rho^i(X)} - 1) \kappa_L(dX)} E \left(e^{\rho^i(L_T)} \right) \\ &= \widehat{S}_0^i. \end{aligned}$$

This proves the assertion. \blacksquare

As in [37, Theorem 3.1], it is possible to characterize the set of all equivalent martingale measures (henceforth EMMs) if the underlying filtration is generated by W and L . More specifically, it follows from the martingale representation theorem (cf. [30, Theorem III.4.34]) that the density process $Z_t = E(\frac{dQ}{dP} | \mathcal{F}_t)$ of any equivalent martingale measure Q can be written as

$$(2.21) \quad Z = \mathcal{E} \left(\int_0^\cdot \psi_s dW_s + (Y - 1) * (\mu^L - \nu^L) \right)$$

for suitable processes ψ and Y in this case. Here μ^L (resp., ν^L) denote the random measure of jumps (resp., its compensator) (cf. [30, section II.1] for more details). Under an arbitrary EMM, L may not be a Lévy process, and W and L may not be independent. However, there is a subclass of *structure preserving* EMMs under which L remains a Lévy process independent of W . This translates into the following specifications of ψ and Y (cf. [37, Theorem 3.2] for the univariate case).

Theorem 2.11 (structure preserving EMMs). *Let $y : \mathbb{S}_d^+ \rightarrow (0, \infty)$ such that*

- (i) $\int_{\mathbb{S}_d^+} (\sqrt{y(X)} - 1)^2 \kappa_L(dX) < \infty$,
- (ii) $\int_{\{|X| > 1\}} e^{\rho^i(X)} \kappa_L^y(dX) < \infty$, $i = 1, \dots, d$,

where $\kappa_L^y(B) := \int_B y(X) \kappa_L(dX)$ for $B \in \mathcal{B}(\mathbb{S}_d^+)$. Define the \mathbb{R}^d -valued process $(\psi_t)_{t \in [0, T]}$ as

$$\psi_t = -\Sigma_t^{-\frac{1}{2}} \left(\mu + \beta(\Sigma_t) + \frac{1}{2} \begin{pmatrix} \Sigma_t^{11} \\ \vdots \\ \Sigma_t^{dd} \end{pmatrix} + \begin{pmatrix} \int_{\mathbb{S}_d^+} (e^{\rho^1(X)} - 1) \kappa_L^y(dX) \\ \vdots \\ \int_{\mathbb{S}_d^+} (e^{\rho^d(X)} - 1) \kappa_L^y(dX) \end{pmatrix} - \mathbf{1}r \right),$$

where $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^d$. Then $Z = \mathcal{E}(\int_0^\cdot \psi_s dW_s + (y - 1) * (\mu^L - \nu^L))$ is a density process, and the probability measure Q defined by $\frac{dQ}{dP} = Z_T$ is an EMM. Moreover, $W^Q := W - \int_0^\cdot \psi_s ds$ is a Q -standard Brownian motion, and L is an independent driftless Q -matrix subordinator with Lévy measure κ_L^y . The Q -dynamics of (Y, Σ) are given by

$$\begin{aligned} dY_t^i &= \left(r - \int_{\mathbb{S}_d^+} (e^{\rho^i(X)} - 1) \kappa_L^y(dX) - \frac{1}{2} \Sigma_t^{ii} \right) dt + \left(\Sigma_t^{\frac{1}{2}} dW_t^Q \right)^i + \rho^i(dL_t), \quad i = 1, \dots, d, \\ d\Sigma_t &= (\gamma_L + A\Sigma_t + \Sigma_t A^\top) dt + dL_t. \end{aligned}$$

Proof. Since $y - 1 > -1$, Z is strictly positive by [30, Theorem I.4.61]. The martingale property of Z follows along the lines of the proof of [37, Theorem 3.2]. The remaining assertions follow from [32, Proposition 1] and the Lévy–Khintchine formula by applying the Girsanov–Jacod–Mémmin theorem as in [32, Proposition 4] to the $\mathbb{R}^{\frac{1}{2}d(d+1)}$ -valued process

$$\tilde{L} = \begin{pmatrix} W^Q \\ 0 \end{pmatrix} + \text{vech}(L),$$

where $W^Q := W - \int_0^\cdot \psi_s ds$. \blacksquare

The previous theorem shows that it is possible to use a model of the same type under the real-world probability measure P and some EMM Q , e.g., to do option pricing and risk management within the same model class. The model parameters under Q can be determined by calibration and the model parameters under P by statistical methods.

3. Option pricing using integral transform methods. In this section we first recall results of [17] on Fourier pricing in general multivariate semimartingale models. To this end, let $S = (S_0^1 e^{Y^1}, \dots, S_0^d e^{Y^d})$ be a d -dimensional semimartingale such that the discounted price process $(e^{-rt} S_t)_{t \in [0, T]}$ is a martingale under some pricing measure Q for some constant instantaneous interest rate $r > 0$.

We want to determine the price $E_Q(e^{-rT} f(Y_T - s))$ of a European option with payoff $f(Y_T - s)$ at maturity T , where $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a measurable function and $s := (-\log(S_0^1), \dots, -\log(S_0^d))$. Denote by \hat{f} the *Fourier transform* of f . The following theorem is from [17, Theorem 3.2] and represents a multivariate generalization of integral transform methods first introduced in the context of option pricing by Carr and Madan [8] and Raible [39].

Theorem 3.1 (Fourier pricing). *Fix $R \in \mathbb{R}^d$, let $g(x) := e^{-\langle R, x \rangle} f(x)$ for $x \in \mathbb{R}^d$, and assume that*

- (i) $g \in L^1 \cap L^\infty$, (ii) $\Phi_{Y_T}(R) < \infty$, (iii) $w \mapsto \Phi_{Y_T}(R + iw)$ belongs to L^1 .

Then,

$$(3.1) \quad E_Q(e^{-rT} f(Y_T - s)) = \frac{e^{-\langle R, s \rangle - rT}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u, s \rangle} \Phi_{Y_T}(R + iu) \hat{f}(iR - u) du.$$

Observe that Theorems 2.8 and 2.9 show that conditions (ii) and (iii) are satisfied for our multivariate stochastic volatility model of OU type (2.5), (2.6) if condition (2.12) holds, i.e., if L has enough exponential moments. More specifically, the vector R has to lie in the intersection of the domains of Φ_{Y_T} and \hat{f} .

We now present some examples. As is well known, the Fourier transform of the payoff function of a *plain vanilla call option* with strike $K > 0$, $f(x) = (e^x - K)^+$, is given by

$$(3.2) \quad \hat{f}(z) = \frac{K^{1+iz}}{iz(1+iz)}$$

for $z \in \mathbb{C}$ with $\text{Im}(z) > 1$. The Fourier transforms of many other single-asset options like barrier, self-quanto, and power options as well as multiasset options like worst-of and best-of

options can be found, e.g., in the survey [17]. From the unpublished paper of [27] we have the following formulae for basket and spread options.

Example 3.1.

1. The Fourier transform of $f(x) = (K - \sum_{j=1}^d e^{x_j})^+$, $K > 0$, that is, the payoff function of a *basket* put option, is given by

$$\widehat{f}(z) = K^{1+i\sum_{j=1}^d z_j} \frac{\prod_{j=1}^d \Gamma(iz_j)}{\Gamma(2 + i\sum_{j=1}^d z_j)}$$

for all $z \in \mathbb{C}^d$ with $\text{Im}(z_j) < 0$, $j = 1, \dots, d$. The price of the corresponding call can easily be derived using the put-call-parity $(K - x)^+ = (x - K)^+ - x + K$. Since we have separated the initial values s in (3.1), we can use FFT methods to compute the prices of weighted baskets for several weights efficiently.

2. The Fourier transform of the payoff function of a *spread* call option, $f(x) = (e^{x_1} - e^{x_2} - K)^+$, $K > 0$, is given by

$$\widehat{f}(z) = \frac{K^{1+iz_1+iz_2}}{iz_1(1+iz_1)} \frac{\Gamma(iz_2)\Gamma(-iz_1-iz_2-1)}{\Gamma(-iz_1-1)}$$

for all $z \in \mathbb{C}^2$ with $\text{Im}(z_1) > 1$, $\text{Im}(z_2) < 0$, and $\text{Im}(z_1 + z_2) > 1$; see also [29].

Since the Fourier transform of $(e^{x_1} - e^{x_2})^+$ does not exist anywhere, we cannot use Theorem 3.1 to price zero-strike spread options. Nevertheless, we can derive a similar formula directly. Alternatively, one could use the change of numeraire technique of [36], which would lead to formulae of a similar complexity.

Proposition 3.2 (spread options with zero strike). *Suppose that*

$$\Phi_{(Y_T^1, Y_T^2)}(R, 1 - R) < \infty \quad \text{for some } R > 1.$$

Then the price of a zero-strike spread option with payoff $(S_0^1 e^{Y_T^1} - S_0^2 e^{Y_T^2})^+$ is given by

$$E_Q(e^{-rT}(S_T^1 - S_T^2)^+) = \frac{e^{R(s_2 - s_1) - s_2 - rT}}{2\pi} \int_{\mathbb{R}} e^{iu(s_2 - s_1)} \frac{\Phi_{(Y_T^1, Y_T^2)}(R + iu, 1 - R - iu)}{(R + iu)(R + iu - 1)} du,$$

where $s_1 = -\ln(S_0^1)$ and $s_2 = -\ln(S_0^2)$.

Observe that unlike for $K > 0$, one only needs to compute a one-dimensional integral to determine the price of a zero-strike spread option. This will be exploited in the calibration procedure in section 4.

Proof. Let $R > 1$, and define $f_K(x) = (e^x - K)^+$ for $K > 0$, and $g_K(x) = e^{-Rx} f_K(x)$. By Fourier inversion and (3.2), we have

$$f_{e^y}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{(R+iu)x} e^{(1-R-iu)y}}{(R+iu)(R+iu-1)} du$$

for all $y \in \mathbb{R}$. Hence, for the function $h_{e^y}(x) := (S_0^1 e^x - S_0^2 e^y)^+ = f_{e^{y-s_2}}(x - s_1)$ we get

$$h_{e^y}(x) = \frac{1}{2\pi} e^{R(s_2 - s_1) - s_2} \int_{\mathbb{R}} e^{iu(s_2 - s_1)} \frac{e^{(R+iu)x} e^{(1-R-iu)y}}{(R+iu)(R+iu-1)} du.$$

Finally, by Fubini's theorem

$$\begin{aligned} E_Q(h_{e_{Y_T^2}}(Y_T^1)) &= \frac{e^{R(s_2-s_1)-s_2}}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{iu(s_2-s_1)} \frac{e^{(R+iu)x} e^{(1-R-iu)y}}{(R+iu)(R+iu-1)} du P_{(Y_T^1, Y_T^2)}(dx, dy) \\ &= \frac{e^{R(s_2-s_1)-s_2}}{2\pi} \int_{\mathbb{R}} e^{iu(s_2-s_1)} \frac{\Phi_{(Y_T^1, Y_T^2)}(R+iu, 1-R-iu)}{(R+iu)(R+iu-1)} du, \end{aligned}$$

where the application of Fubini's theorem is justified by

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| \frac{e^{(R+iu)x} e^{(1-R-iu)y}}{(R+iu)(R+iu-1)} \right| du P_{(Y_T^1, Y_T^2)}(dx, dy) &= \int_{\mathbb{R}^2} e^{Rx} e^{(1-R)y} \int_{\mathbb{R}} |\widehat{g}_1(u)| du P_{(Y_T^1, Y_T^2)}(dx, dy) \\ &\leq \|\widehat{g}_1\|_{L^1} \Phi_{(Y_T^1, Y_T^2)}(R, 1-R) < \infty, \end{aligned}$$

since $\|\widehat{g}_1\|_{L^1} < \infty$ as shown in [17, Example 5.1]. \blacksquare

4. Calibration of the OU–Wishart model. We now put forward a specific parametric specification of the model discussed in section 2. To this end, let $n \in \mathbb{N}$, $\Theta \in \mathbb{S}_d^+$ and let X be a $d \times n$ random matrix with independent and identically distributed standard normal entries. Then, the matrix $M := \Theta^{\frac{1}{2}} X X^\top \Theta^{\frac{1}{2}}$ is said to be *Wishart distributed*, written as $M \sim \mathcal{W}_d(n, \Theta)$. Note that this definition can be extended to noninteger $n > d - 1$ using the characteristic function

$$(4.1) \quad Z \mapsto \det(I_d - 2iZ\Theta)^{-\frac{1}{2}n};$$

see [23, Theorem 3.3.7]. Since $M \in \mathbb{S}_d^+$ almost surely, we can define a compound Poisson matrix subordinator L with intensity λ and $\mathcal{W}_d(n, \Theta)$ distributed jumps. We call the resulting multivariate stochastic volatility model of OU type an *OU–Wishart model*.

Remark 4.1. There exists a subclass of structure preserving EMMs Q (cf. Theorem 2.11) such that we have an OU–Wishart model under both P and Q . This means that L is a compound Poisson process with $\mathcal{W}_d(n, \Theta)$ distributed jumps and intensity λ under P , and $\mathcal{W}_d(\tilde{n}, \tilde{\Theta})$ distributed jumps with intensity $\tilde{\lambda}$ under Q . We need only assume that the Wishart distribution under both P and Q has a Lebesgue density, i.e., $n, \tilde{n} > d - 1$ and $\Theta, \tilde{\Theta} \in \mathbb{S}_d^{++}$. Then, one simply has to take y as the quotient of the respective Lévy densities. Hence, by [23, Theorem 3.2.1], y has to be defined as

$$y(X) = \frac{\tilde{\lambda}}{\lambda} \left(\frac{2^{\frac{1}{2}(\tilde{n}-n)d} \Gamma_d(\frac{1}{2}\tilde{n}) \det(\tilde{\Theta})^{\frac{1}{2}\tilde{n}}}{\Gamma_d(\frac{1}{2}n) \det(\Theta)^{\frac{1}{2}n}} \right)^{-1} \det(X)^{\frac{1}{2}(\tilde{n}-n)} e^{-\frac{1}{2}\text{tr}((\tilde{\Theta}^{-1}-\Theta^{-1})X)}, \quad X \in \mathbb{S}_d^+.$$

Since we have $\int_{\mathbb{S}_d^+} e^{\text{tr}(RX)} \kappa_L(dX) = \lambda \det(I_d - 2R\Theta)^{-\frac{1}{2}n}$ by (4.1), we see that the compound Poisson process L has exponential moments as long as $\|R\| < \frac{1}{2\|\Theta\|}$, where $\|\cdot\|$ denotes the spectral norm. Consequently, (2.12) holds for $\epsilon := \frac{1}{2\|\Theta\|}$, and we can apply the integral transform methods from the previous section to compute prices of multiasset options.

Note that for the particularly simple special case of diagonal A , β , and ρ , each asset follows a BNS model at the margins by (2.7) and (2.8). In particular, for $n = 2$ we see that

L^{ii} , $i = 1, \dots, d$, is a compound Poisson subordinator with exponentially distributed jumps; thus we have in distribution the Γ -OU BNS model with a stationary gamma distribution at the margins; cf., e.g., [37, section 2.2]. Then, the characteristic functions of the single assets are known in closed form. Note that while the characteristic function of the stationary distribution of the marginal OU-type process is still known for $n \neq 2$, it no longer corresponds to a gamma distribution in this case.

4.1. The OU–Wishart model in dimension 2. We work directly under a pricing measure Q and consider the following specific two-dimensional case of our model, where we restrict ourselves in particular to a diagonal mean-reversion matrix A and a leverage term ρ such that both jumps of the respective variance and of the covariance enter the price. Our model is given by

$$\begin{aligned} \begin{pmatrix} dY_t^1 \\ dY_t^2 \end{pmatrix} &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \Sigma_t^{11} \\ \Sigma_t^{22} \end{pmatrix} dt + \begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{12} & \Sigma_t^{22} \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} + \begin{pmatrix} \rho_1 dL_t^{11} + \rho_{12} dL_t^{12} \\ \rho_2 dL_t^{22} + \rho_{21} dL_t^{12} \end{pmatrix}, \\ \begin{pmatrix} d\Sigma_t^{11} & d\Sigma_t^{12} \\ d\Sigma_t^{12} & d\Sigma_t^{22} \end{pmatrix} &= \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} + \begin{pmatrix} 2a_1 \Sigma_t^{11} & (a_1 + a_2) \Sigma_t^{12} \\ (a_1 + a_2) \Sigma_t^{12} & 2a_2 \Sigma_t^{22} \end{pmatrix} dt + \begin{pmatrix} dL_t^{11} & dL_t^{12} \\ dL_t^{12} & dL_t^{22} \end{pmatrix} \end{aligned}$$

with initial values

$$Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} \Sigma_0^{11} & \Sigma_0^{12} \\ \Sigma_0^{12} & \Sigma_0^{22} \end{pmatrix} \in \mathbb{S}_2^{++}$$

and parameters $\gamma_1, \gamma_2 \geq 0$, $a_1, a_2 < 0$, $\rho_1, \rho_2, \rho_{12}, \rho_{21} \in \mathbb{R}$. L is a compound Poisson process with intensity λ and $\mathcal{W}_2(n, \Theta)$ -jumps, where $n = 2$ and

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} \end{pmatrix} \in \mathbb{S}_2^+.$$

Therefore, all components of L jump at the same time. Since the second-order properties of the Wishart distribution are known explicitly (cf. [23, Theorem 3.3.15]), the covariances of the jumps are given by

$$\begin{aligned} \text{Cov}(\Delta L_t^{11}, \Delta L_t^{12} | \Delta L_t^{11} \neq 0) &= 4\Theta_{11}\Theta_{12}, \\ \text{Cov}(\Delta L_t^{22}, \Delta L_t^{12} | \Delta L_t^{11} \neq 0) &= 4\Theta_{22}\Theta_{12}, \\ \text{Cov}(\Delta L_t^{11}, \Delta L_t^{22} | \Delta L_t^{11} \neq 0) &= 4\Theta_{12}^2. \end{aligned}$$

This shows that even if ρ is diagonal, i.e., $\rho_{12} = 0 = \rho_{21}$, the leverage terms of both assets are correlated. If ρ is nondiagonal, then θ_{12} also influences the marginal distribution of each asset.

Multiasset option pricing. By (2.14) and (4.1), the joint moment generating function of (Y^1, Y^2) is given by

$$\begin{aligned} E[e^{y^\top Y_t}] &= \exp \left(y^\top \mu t + \text{tr}(\Sigma_0 H_y(t)) + \int_0^t \text{tr}(\gamma_L H_y(s)) ds + \lambda \int_0^t \frac{1}{\det(I_2 - 2(H_y(s) + \rho^*(y))\Theta)} ds - \lambda t \right) \end{aligned}$$

with H_y as in (2.15), $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, $\gamma_L = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$, and $\rho^*(y) = \begin{pmatrix} \rho_{1y_1} & \rho_{12y_1} \\ \rho_{21y_2} & \rho_{2y_2} \end{pmatrix}$. It does not seem to be possible to obtain a closed-form expression in terms of ordinary functions, unless one sets $a_1 = a_2 =: a$. In this case, if $\Delta = \sqrt{4b_0b_2 - b_1^2} \neq 0$, one has

$$\begin{aligned} E[e^{y_1 Y_t^1 + y_2 Y_t^2}] &= \exp \left\{ y_1 \mu_1 t + y_2 \mu_2 t + \frac{e^{2at} - 1}{4a} \operatorname{tr} \left(\Sigma_0 \begin{pmatrix} y_1^2 - y_1 & y_1 y_2 \\ y_1 y_2 & y_2^2 - y_2 \end{pmatrix} \right) \right. \\ &\quad + \frac{1}{4a} (\gamma_1 (y_1^2 - y_1) + \gamma_2 (y_2^2 - y_2)) \left(\frac{1}{2a} (e^{2at} - 1) - t \right) \\ &\quad + \frac{\lambda}{2ab_0} \left[\frac{b_1}{\Delta} \left(\arctan \left(\frac{2b_2 + b_1}{\Delta} \right) - \arctan \left(\frac{2b_2 e^{2at} + b_1}{\Delta} \right) \right) \right. \\ &\quad \left. \left. + \frac{1}{2} \ln \left(\frac{b_0 + b_1 + b_2}{b_2 e^{4at} + b_1 e^{2at} + b_0} \right) \right] + \frac{\lambda}{b_0} t - \lambda t \right\} \end{aligned}$$

with coefficients

$$\begin{aligned} b_0 &:= 1 + 4 \det(B - C) + 2 \operatorname{tr}(B - C), \\ b_1 &:= -8 \det(B) + 4 \operatorname{tr}(B) \operatorname{tr}(C) - 4 \operatorname{tr}(BC) - 2 \operatorname{tr}(B), \\ b_2 &:= 4 \det(B), \\ \Delta &:= \sqrt{4b_0b_2 - b_1^2} \end{aligned}$$

and matrices

$$B := \frac{1}{4a} \begin{pmatrix} y_1^2 - y_1 & y_1 y_2 \\ y_1 y_2 & y_2^2 - y_2 \end{pmatrix} \Theta, \quad C := \begin{pmatrix} \rho_{1y_1} & \rho_{12y_1} \\ \rho_{21y_2} & \rho_{2y_2} \end{pmatrix} \Theta.$$

Note that \arctan has to be understood as a function of complex argument to cover the case where the term in the square root of Δ is negative. If $\Delta = 0$, we obtain

$$\begin{aligned} E[e^{y_1 Y_t^1 + y_2 Y_t^2}] &= \exp \left\{ y_1 \mu_1 t + y_2 \mu_2 t + \frac{e^{2at} - 1}{4a} \operatorname{tr} \left(\Sigma_0 \begin{pmatrix} y_1^2 - y_1 & y_1 y_2 \\ y_1 y_2 & y_2^2 - y_2 \end{pmatrix} \right) \right. \\ &\quad + \frac{1}{4a} (\gamma_1 (y_1^2 - y_1) + \gamma_2 (y_2^2 - y_2)) \left(\frac{1}{2a} (e^{2at} - 1) - t \right) \\ &\quad \left. + \frac{\lambda}{2ab_0} \left[\frac{b_1}{2b_2 e^{2at} + b_1} - \frac{b_1}{2b_2 + b_1} + \frac{1}{2} \ln \left(\frac{b_0 + b_1 + b_2}{b_2 e^{4at} + b_1 e^{2at} + b_0} \right) \right] + \frac{\lambda}{b_0} t - \lambda t \right\}. \end{aligned}$$

Using $\det(A + B) = \det(A) + \det(B) + \operatorname{tr}(A) \operatorname{tr}(B) - \operatorname{tr}(AB)$ for $A, B \in M_2(\mathbb{R})$, the above formulae follow from

$$\det(I_2 - 2(H_y(s) + \rho^*(y))\Theta) = \det(I_2 - 2(e^{2as} - 1)B - 2C) = b_0 + b_1 e^{2as} + b_2 e^{4as}$$

and straightforward integration. Likewise, one can also derive a closed-form expression for $n = 4, 6, \dots$ using [22, 2.18(4)].

Consequently, one faces a trade-off at this point. One possibility is to retain the flexibility of different mean reversion speeds a_i by evaluating the remaining integral using numerical integration. Alternatively, one can restrict attention to identical mean reversion speeds in order to have a closed-form expression of the moment generating function at hand. The impact of this decision on the calibration performance is discussed in section 4.2.

Single-asset option pricing. For pricing single-asset options, one needs only the transforms of the marginal models, such that the above expressions simplify considerably. For example, the moment generating function of Y^1 is given by

$$E[e^{y_1 Y_t^1}] = \exp \left\{ y_1 \mu^1 t + \frac{e^{2a_1 t} - 1}{4a_1} (y_1^2 - y_1) \Sigma_0^{11} + \frac{1}{4a_1} \left(\frac{1}{2a_1} (e^{2a_1 t} - 1) - t \right) (y_1^2 - y_1) \gamma_L^{11} + \frac{\lambda}{2a_1 b_0} \ln \left(\frac{b_0 + b_1}{b_0 + e^{2a_1 t} b_1} \right) + \frac{\lambda t}{b_0} - \lambda t \right\},$$

where b_0 and b_1 simplify to

$$b_0 = 1 + \left(\frac{1}{2a_1} (y_1^2 - y_1) - 2\rho_1 y_1 \right) \Theta_{11} - 2\rho_{12} y_1 \Theta_{12},$$

$$b_1 = -\frac{1}{2a_1} (y_1^2 - y_1) \Theta_{11}.$$

Note that one can use the recursion formula stated in [22, 2.155] to obtain a closed-form expression for $\mathcal{W}_2(n, \Theta)$ -jumps with $n \in 2\mathbb{N}$, too. In the special case where the operator ρ is diagonal, i.e., if $\rho_{12} = \rho_{21} = 0$, the margins are (in distribution) Γ -OU BNS models, whose moment generating function has been derived in [37, Table 2.1].

Remark 4.2 (high dimensionality). The above model can also be defined for $d > 2$, but of course, the Fourier formula (3.1) becomes numerically infeasible in high dimensions. Nevertheless, if ρ is diagonal, the calibration of a high-dimensional OU–Wishart model is still possible by evaluating only options on just *two* underlyings. Using zero-strike spread options and provided the characteristic function is known explicitly, this means that one need only evaluate single integrals numerically, as in the univariate case. Indeed, combining [4, Proposition 4.5] and the fact that every symmetric submatrix of a Wishart distributed matrix is again Wishart distributed (cf. [23, Theorem 3.3.10]), it follows that the joint dynamics of each pair of assets follow a two-dimensional OU–Wishart model as above. Hence, we can calibrate the model using only two-asset options (e.g., spread options). The price to pay is that the resulting model incorporates only pairwise dependencies, since the respective covariances completely determine the underlying Wishart distribution.

Remark 4.3. If ρ is diagonal, we have equivalence in distribution of the margins of our model to a Γ -BNS model. This implies immediately that we need to use prices on multiasset options in order to infer all parameters from observed option prices. If ρ is nondiagonal, we have a Γ -BNS model with an additional (correlated) jump term. Due to this additional term, it might be possible to infer θ_{12} from single-asset options. However, one cannot obtain Σ_0^{12} in this way because it does not appear in the marginal moment generating function.

In many multifactor univariate models one can in general similarly not be sure whether one can uniquely determine all parameters from observed option prices. In many papers the parameters are calibrated and the procedure seems to work, but we are not aware of any reasonably complex multifactor model where the identifiability of the parameters based on option prices has been established. The reason is clearly the highly nontrivial relation between the parameters and the option prices.

4.2. Empirical illustration. The aim of this subsection is to show that a calibration of the OU–Wishart model to market prices is feasible. Since multiasset options are mostly traded over the counter, it is difficult to obtain real price quotes. To circumvent this problem, we proceed as in [44] and consider *foreign exchange rates* instead, where a call option on some exchange rate can be seen as a spread option between two others. Let us emphasize that our calibration routine should not be seen as a finished product, but much rather as a first test and proof of principle. A more detailed investigation as well as an extension to numerically more involved models with nondiagonal A is left to future research.

We consider a two-dimensional OU–Wishart model as above. Our first asset is the EUR/USD exchange rate $S^{\$/\epsilon} = S_0^{\$/\epsilon} e^{Y^1}$, that is, the price of 1 € in \$, and our second asset is the GBP/USD exchange rate $S^{\$/\pounds} = S_0^{\$/\pounds} e^{Y^2}$, i.e., the price of 1 £ in \$. We model directly under a martingale measure. Therefore we have, by Theorem 2.10, that

$$\mu_1 = r_{\$} - r_{\epsilon} - \int_{\mathbb{S}_d^+} (e^{\rho_1 X^{11} + \rho_{12} X^{12}} - 1) \kappa_L(dX).$$

Since κ_L is the intensity λ times a Wishart distribution with parameters $n = 2$ and θ , this simplifies to

$$\begin{aligned} \mu_1 &= r_{\$} - r_{\epsilon} - \lambda \left(\det(I_2 - 2 \begin{pmatrix} \rho_1 & \rho_{12} \\ 0 & 0 \end{pmatrix} \Theta)^{-1} - 1 \right) \\ &= r_{\$} - r_{\epsilon} - \lambda \frac{2\rho_1 \Theta_{11} + 2\rho_{12} \Theta_{12}}{1 - 2\rho_1 \Theta_{11} - 2\rho_{12} \Theta_{12}}. \end{aligned}$$

Likewise we have

$$\mu_2 = r_{\$} - r_{\pounds} - \lambda \frac{2\rho_2 \Theta_{22} + 2\rho_{21} \Theta_{12}}{1 - 2\rho_2 \Theta_{22} - 2\rho_{21} \Theta_{12}}.$$

Thus, for $\rho_{12} = 0$ or $\rho_{21} = 0$, we recover the martingale conditions of the Γ -OU BNS model. By [28, section 13.4], it follows that the price in \$ of a plain vanilla call option on $S^{\$/\epsilon}$ or $S^{\$/\pounds}$ is given by $e^{-r_{\$}T} E((S_T^{\$/\epsilon} - K)^+)$ or $e^{-r_{\$}T} E((S_T^{\$/\pounds} - K)^+)$, respectively. Now observe that the \$-payoff of a call option on the EUR/GBP exchange rate $S^{\pounds/\epsilon}$ is given by $S_T^{\$/\pounds} (S_T^{\pounds/\epsilon} - K)^+ = (S_T^{\$/\epsilon} - K S_T^{\$/\pounds})^+$; hence it can be regarded as a spread option on $S^{\$/\epsilon} - S^{\$/\pounds}$ where the initial value of the second asset is replaced by $K S_0^{\$/\pounds}$. Since it is a zero-strike spread option, we can use Proposition 3.2 to value it.

We obtained the option price data from EUWAX on April 29, 2010, at the end of the business day. The EUR/USD exchange rate at that time was $S_0^{\$/\epsilon} = 1.3249\pounds$, the GBP/USD exchange rate was $S_0^{\$/\pounds} = 1.5333\pounds$, and the EUR/GBP exchange rate was $0.8641\pounds$. As a proxy for the instantaneous riskless interest rate we took the 3-month LIBOR for each currency, viz. $r_{\epsilon} = 0.604\%$, $r_{\pounds} = 0.344\%$, and $r_{\$} = 0.676\%$. All call options here are plain vanilla call options of European style. We used 148 call options on the EUR/USD exchange rate, 67 call options on the GBP/USD exchange rate, and 105 call options on the EUR/GBP exchange rate, all of them for different strikes and different maturities, for a total of 320 option prices. We always used the midvalue between bid and ask price. A spread sheet containing all data used for the calibration can be found on the second author’s website.

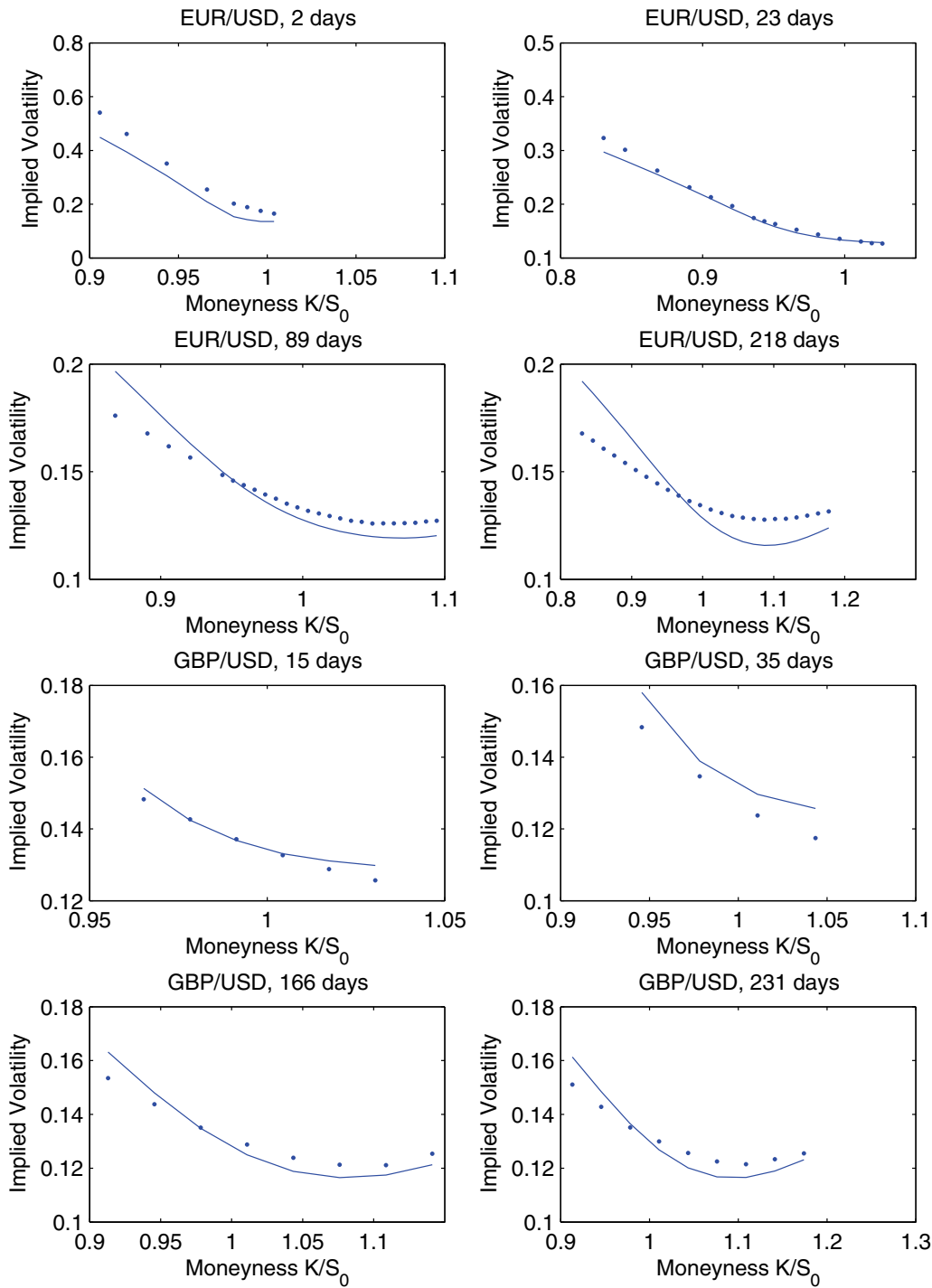


Figure 1. Comparison of the Black–Scholes implied volatility of market prices (dots) and model prices (solid line). The plots show only the results for the 12-parameter OU–Wishart model (Step A), since they do not change visually for the more complex models from Steps B to D.

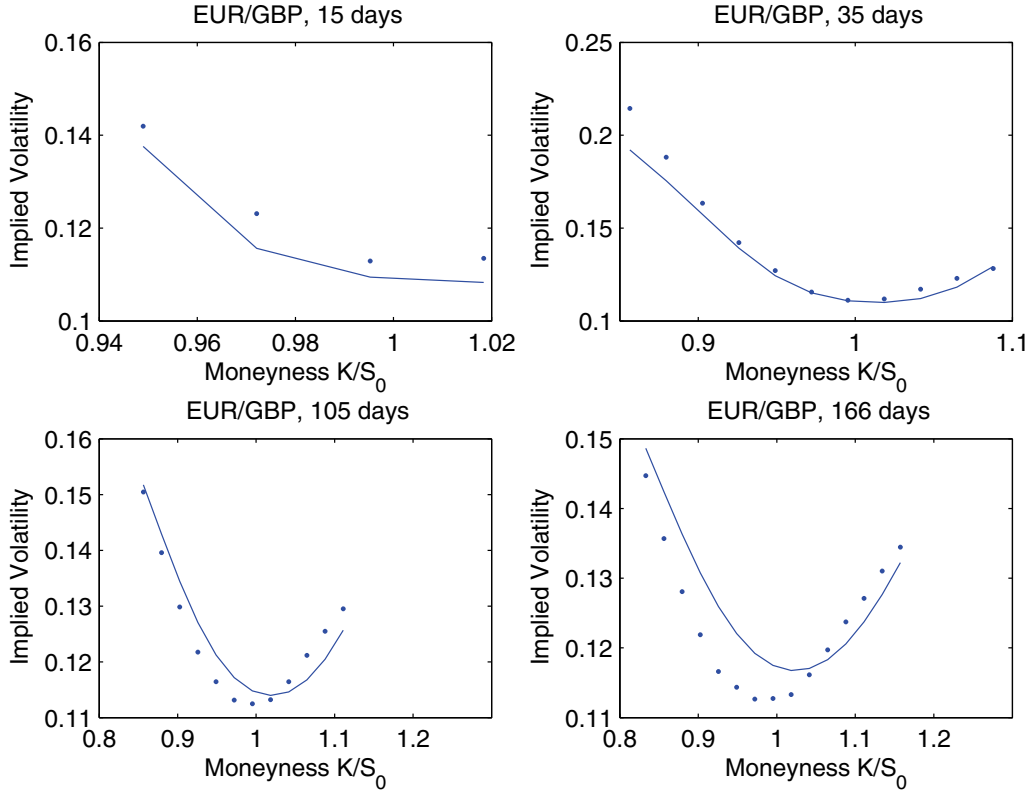


Figure 2. Comparison of the Black–Scholes implied volatility of market prices (dots) and model prices (solid line). The plots show only the results for the 12-parameter OU–Wishart model (Step A), since they do not change visually for the more complex models from Steps B to D.

The calibration was performed by choosing the model parameters so as to minimize the root mean squared error (RMSE) between the Black–Scholes volatilities implied by market and model prices. Note that the RMSE is the square root of the sum of the squared distances divided by the number of options. All computations were carried out in MATLAB and performed on a standard desktop PC with a 2.4GHz processor.

In *Step A*, we impose $a := a_1 = a_2$ and $\rho_{12} = 0 = \rho_{21}$; i.e., we make the assumption that the mean reversion parameters of both assets are equal and that ρ is diagonal. This is the most tractable case, since there is a closed-form expression for the moment generating function of (Y^1, Y^2) and the number of model parameters is reduced to 12. The starting and calibrated parameters can be found in Table 1. The overall RMSE is 0.0082, and the run time was 48 minutes; i.e., calibration of the model is feasible even on a standard PC. If one considers only the marginal models for EUR/USD and GBP/USD, one has an RMSE of 0.0106 and 0.0048, respectively. For visualization, we provide Figures 1 and 2, where market and model prices are compared in terms of Black–Scholes implied volatility for a few selected maturities. These results illustrate that even this simple model is able to fit the observed smiles rather well. For comparison, we calibrated two independent univariate Γ -OU BNS

models to the margins separately (see Table 1) and obtained a lower RMSE of 0.0071 and 0.0020, respectively. This stems from the fact that the additional dependence parameters do not enter the pricing formulas for single-asset options, whereas the intensity of the compound Poisson process is the same for all assets in our multivariate framework, unlike when using two univariate models. This means that we are *not overfitting* the marginal distributions with an excessive amount of additional parameters, but much rather using a simplified version of a standard model. Nevertheless, the calibration still performs quite well even when using this simplification.

Table 1

Calibrated parameters for different models. In decreasing order: models from Steps A to D; univariate BNS models for EUR/USD and GBP/USD; initial parameters.

Step	λ	a_1	ρ_1	ρ_{12}	Θ^{11}	Σ_0^{11}	γ_1
A	0.774	-2.392	-3.741	/	0.011	0.019	0.027
B	0.901	-3.008	-5.364	0.679	0.011	0.019	0.034
C	0.774	-2.392	-3.741	/	0.011	0.019	0.027
D	1.231	-7.562	-6.806	0.948	0.010	0.024	0.097
univ. 1	0.781	-32.177	-5.995	/	0.007	0.034	/
univ. 2	0.864	/	/	/	/	/	/
initial	0.800	-2.500	-3.000	/	0.010	0.020	0.020

Step	a_2	ρ_2	ρ_{21}	Θ^{22}	Σ_0^{22}	γ_2	Θ^{12}	Σ_0^{12}
A	/	-0.494	/	0.063	0.017	0.000	0.022	0.013
B	/	-0.661	0.896	0.067	0.018	0.000	0.023	0.013
C	-2.392	-0.494	/	0.063	0.017	0.000	0.022	0.013
D	-6.553	-0.535	1.188	0.102	0.021	0.000	0.030	0.016
univ. 1	/	/	/	/	/	/	/	/
univ. 2	-2.482	-0.471	/	0.050	0.017	0.012	/	/
initial	/	-0.500	/	0.030	0.015	0.011	0.010	0.010

As a further cross-check, Figure 3 depicts sample paths of the EUR/USD and the GBP/USD spot rates and their volatilities, simulated with our calibrated parameters, which show reasonable path properties.

In *Step B*, we allow for a nondiagonal leverage operator ρ . Although this introduces two additional parameters, ρ_{12} and ρ_{21} , a closed-form expression for the moment generating function is still available. As initial values, we take the parameters obtained in Step A and set ρ_{12} and ρ_{21} to zero. After 80 minutes, the optimizer finds a minimum with an RMSE of 0.0079. At the margins, we have RMSEs of 0.0104 and 0.0037, respectively. Hence, calibration is still feasible without resorting to higher-powered computers, but the gains in fitting accuracy appear to be only moderate for the option price surface at hand.

Next, we drop the assumption of an equal mean reversion parameter and allow for $a_1 \neq a_2$. Since the moment generating function of (Y^1, Y^2) is then no longer known in closed form, good starting values are particularly important in order to reduce computational time to an acceptable value. We distinguish the two cases where ρ is diagonal (Step C) and ρ is nondiagonal (Step D), and take as starting values the parameters obtained from Step A or Step B, respectively. Interestingly, in *Step C* the optimizer finds the minimum at the same parameters as in Step A; thus the additional freedom of different mean reversion parameters

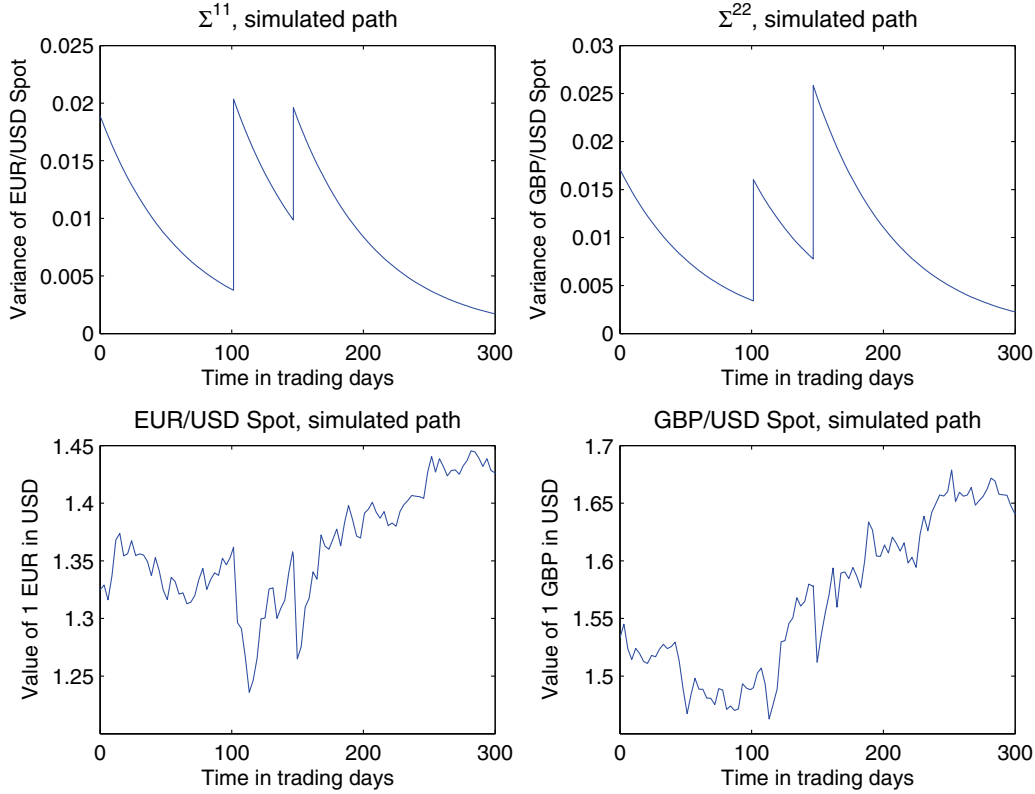


Figure 3. Simulated sample path of the EUR/USD and the GBP/USD spot rates and their volatilities.

does not yield a better fit in this case.

Finally, in *Step D*, we calibrate the full model with nondiagonal ρ and different mean reversion speeds a_1, a_2 . Due to the lack of a closed-form expression for the moment generating function and the high number of parameters (15), the run time increases to an unsatisfactory 10 hours on our standard PC, suggesting that higher-powered computing facilities and an optimized numerical implementation in a compiled instead of an interpreted language should be employed here. In contrast to *Step C*, we find an improvement by allowing for different mean reversion speeds: The overall RMSE is 0.0076. Then again, for the data set at hand, the improvement is again only slight compared to the simplest model considered in *Step A*.

Comparison with other bivariate models. We now compare our bivariate Wishart–OU model to some benchmarks from the literature. The canonical candidate would be the bivariate Wishart model, which also exhibits stochastic correlations between the assets and has very recently been calibrated to market prices by [11]. However, the involved parameter restrictions necessary for the existence of the Wishart process are not satisfied in the results of the calibration. This suggests that some kind of constrained optimization must be incorporated, which is beyond our scope here. However, we emphasize that the Wishart model should yield a comparable performance once these implementation issues have been resolved in a satisfactory

manner.

Instead, we use the multivariate variance gamma (henceforth VG) model of [34] and a generalization with stochastic volatility suggested therein for our comparison. In the multivariate VG model with parameters $(\theta_i, \sigma_i, \nu)$, $i = 1, 2$, the log-price processes Y^1, Y^2 are given by two independent Brownian motions with drift which are subordinated by a common gamma process. The joint moment generating function of the log-price processes under a risk neutral measure is shown to be given by

$$E[\exp(y_1 Y_t^1 + y_2 Y_t^2)] = e^{(y_1(r_s - r_\epsilon + w_1) + y_2(r_s - r_\epsilon + w_2))t} \left(1 - \nu \sum_{i=1}^2 \left(y_i \theta_i + \frac{1}{2} y_i^2 \sigma_i^2 \right) \right)^{-t/\nu}$$

with $w_i = \nu^{-1} \log(1 - \theta_i \nu - \frac{1}{2} \sigma_i^2 \nu)$. The parameters obtained from a calibration of this model to our option data set can be found in Table 2. The corresponding overall RMSE is 0.0134, which is roughly 63% higher than the RMSE obtained from the calibration of our 12-parameter OU–Wishart model from Step A. At the EUR/USD and GBP/USD margins the multivariate VG model has an RMSE of 0.0161 and 0.0107. Consequently, the performance of this model is much worse than for the OU–Wishart model, which is not surprising since it involves only 5 parameters.

To alleviate this issue, our second benchmark allows for stochastic activity driven by an OU-type process. More specifically, the log-price processes of the EUR/USD and GBP/USD spot rates are given by $Y_t^1 = X_{Z_t}^1$ and $Y_t^2 = X_{Z_t}^2$, where X^1 and X^2 are two independent VG processes with parameters $(\theta_i, \sigma_i, \nu_i)$, $i = 1, 2$, and $Z_t = \int_0^t z_s ds$ is an integrated OU process. The OU process $(z_s)_{s \in \mathbb{R}^+}$ is given by $dz_s = 2\alpha z_s ds + dN_{-2\alpha t}$, $z_0 = 1$, $\alpha < 0$, where N is a compound Poisson process with intensity ϑ and $\text{Exp}(\xi)$ distributed jumps. It can be shown that the moment generating function of Z_t (see, e.g., [42, section 7.2.2]), is given by

$$\Phi_{Z_t}(y) = \exp\left(\frac{y}{2\alpha}(\exp(2\alpha t) - 1) + \frac{2\alpha\vartheta(ty - \xi \log[-2\alpha\xi] + \xi \log[(\exp(2\alpha t) - 1)y - 2\alpha\xi])}{y + 2\alpha\xi}\right).$$

For the moment generating function of $Y_t = (Y_t^1, Y_t^2)$, conditioning on the stochastic activity process Z yields

$$\Phi_{Y_t}(y_1, y_2) = \Phi_{Z_t} \left(\log \Phi_{X_1^1}(y_1) + \log \Phi_{X_1^2}(y_2) \right)$$

with $\Phi_{X_1^i}(y_i) = (1 - y_i \theta_i \nu_i - \frac{1}{2} \sigma_i^2 y_i^2 \nu_i)^{-1/\nu_i}$, $i = 1, 2$. Thus, the joint moment generating function of the log-price processes Y_t^1, Y_t^2 under a risk neutral measure is given by

$$\Phi_{Y_t}(1, 0)^{-y_1} \Phi_{Y_t}(0, 1)^{-y_2} \Phi_{Y_t}(y_1, y_2).$$

A calibration of this model to our dataset leads to the parameters provided in Table 2; a plot depicting some of the respective implied volatilities can be found in Figure 4. The corresponding RMSE is 0.0129. Somewhat surprisingly, this is only 4% lower than for the model of [34], despite increasing the parameters from 5 to 9. At the margins, we have 0.0143 and 0.0095, which corresponds to improvements of 11%. Hence, there is quite some improvement

in fitting the margins, but the multivariate options are not fit much better. This suggests that stochastic correlations indeed seem necessary to recapture the features of our empirical dataset. However, let us emphasize again that this applies only to one specific dataset in the foreign exchange market. A more detailed empirical study is a challenging topic for future research.

Table 2

The first row shows the calibrated parameters for the multivariate VG model of [34]. The second row contains the calibrated parameters for two independent VG processes with a common integrated Γ -OU time change.

θ_1	θ_2	σ_1	σ_2	ν_1	ν_2	ϑ	α	ξ
-0.360	-0.327	0.090	0.093	0.106	0.106	/	/	/
-1.470	-2.190	0.001	0.050	0.022	0.001	0.468	-42.140	1.747

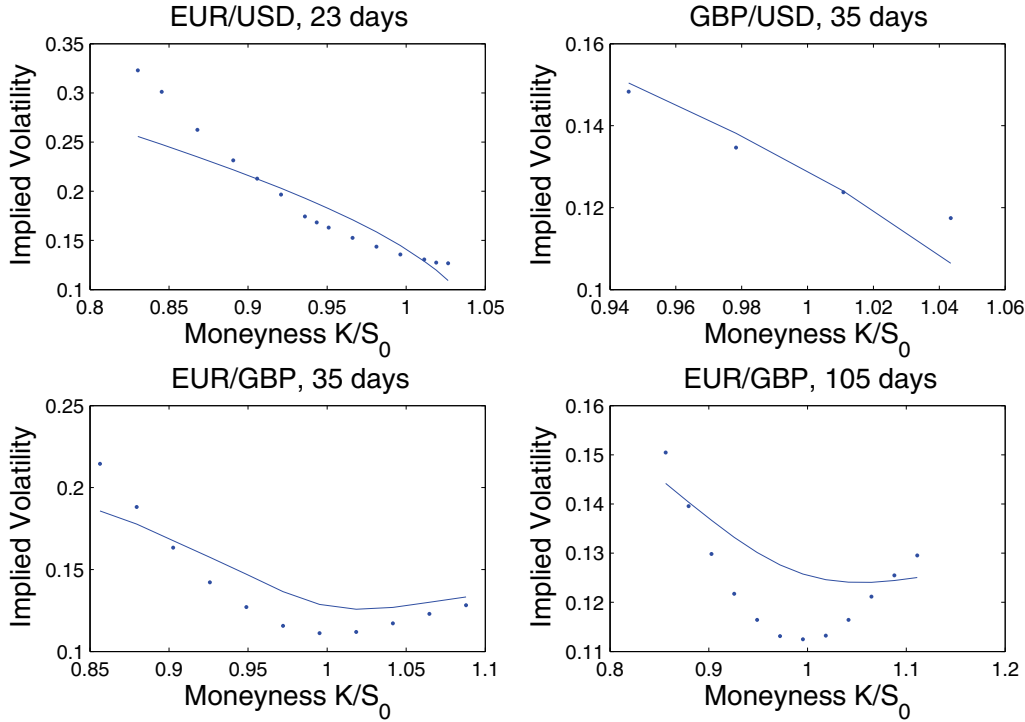


Figure 4. Comparison of the Black-Scholes implied volatility of market prices (dots) and model prices (solid line). The headers state the underlying and the days to maturity. The plots are for the benchmark model where the log-price processes are modeled by two independent VG processes with a common time change which is given by an integrated Γ -OU process. The plots for the multivariate VG model from [34] look very similar.

5. Covariance swaps. In this final section, we show that it is possible to price swaps on the covariance between different assets in closed form. This serves two purposes. On the one hand, options written on the realized covariance represent a family of payoffs that make sense

only in models where covariances are modeled as stochastic processes rather than constants. On the other hand, the ensuing calculations exemplify once more the analytical tractability of the present framework.

We consider again our multivariate stochastic volatility model of OU type under an EMM Q . In addition, we suppose that the matrix subordinator L is square integrable, i.e., $\int_{\{|X|>1\}} \|X\|^2 \kappa_L(dX) < \infty$. The pricing of options written on the realized variance or the quadratic variation as its continuous-time limit has been studied extensively in the literature; cf., e.g., [6] and the references therein. Since we have a nontrivial correlation structure in our model, one can also consider *covariance swaps* on two assets $i, j \in \{1, \dots, d\}$, i.e., contracts with payoff $[Y^i, Y^j]_T - K$ with *covariance swap rate* $K = E([Y^i, Y^j]_T)$ (see, e.g., [7, 12] or [43] for more background on these products). Now, we show how to compute the covariance swap rate. We have

$$[Y^i, Y^j]_T = [Y^i, Y^j]_T^c + \sum_{s \leq T} \Delta Y_s^i \Delta Y_s^j = (\Sigma_T^+)^{ij} + \rho^i(X) \rho^j(X) * \mu_T^L(dX).$$

Since $\kappa_L(dX)dt$ is the compensator of μ^L , this yields

$$(5.1) \quad E([Y^i, Y^j]_T) = (E(\Sigma_T^+))^{ij} + T \int_{\mathbb{S}_d^+} \rho^i(X) \rho^j(X) \kappa_L(dX),$$

where Σ_T^+ was defined in (2.4). Note that by [38, Proposition 2.4] and since $|\rho^i(X) \rho^j(X)| \leq \|\rho\|^2 \|X\|^2$, our integrability assumption on L implies that the expectation is finite. The first summand can be calculated as follows. By setting $y = 0$ in Theorem 2.5, we obtain the characteristic function of Σ_t . Differentiation yields

$$E(\Sigma_T) = e^{AT} \Sigma_0 e^{A^T T} + e^{AT} \mathbf{A}^{-1} (E(L_1)) e^{A^T T} - \mathbf{A}^{-1} (E(L_1)),$$

where $E(L_1) = \gamma_L + \int_{\mathbb{S}_d^+} X \kappa_L(dX)$. Using (2.4), we obtain

$$E(\Sigma_T^+) = \mathbf{A}^{-1} (E(\Sigma_T) - T E(L_1) - \Sigma_0),$$

so we need only know $E(L_1)$. The second summand in (5.1) can analogously be computed by differentiating the characteristic function of the matrix subordinator L .

In our OU–Wishart model, where L is a compound Poisson matrix subordinator plus drift with $\mathcal{W}_d(n, \Theta)$ -distributed jumps, we have by [23, Theorem 3.3.15] that

$$E(L_1) = \gamma_L + \lambda n \Theta.$$

If ρ is diagonal, the second term in (5.1) simplifies to

$$T \rho_i \rho_j \int_{\mathbb{S}_d^+} X_{ii} X_{jj} \nu(dX) = T \rho_i \rho_j \lambda n (2\Theta_{ij}^2 + n \Theta_{ii} \Theta_{jj}),$$

again by [23, Theorem 3.3.15]. Thus we have a closed-form expression for the covariance swap rate:

$$K = \left(\mathbf{A}^{-1} \left[e^{AT} (\Sigma_0 + \mathbf{A}^{-1} (\gamma_L + \lambda n \Theta)) e^{A^T T} - \mathbf{A}^{-1} (\gamma_L + \lambda n \Theta) - T (\gamma_L + \lambda n \Theta) - \Sigma_0 \right] \right)^{ij} + T \rho_i \rho_j \lambda n (2\Theta_{ij}^2 + n \Theta_{ii} \Theta_{jj}).$$

For example, in the two-dimensional OU–Wishart model from section 4.1 we have, for $i = 1$ and $j = 2$,

$$K = \frac{1}{a_1 + a_2} \left[\left(e^{(a_1 + a_2)T} - 1 \right) \left(\Sigma_0^{12} + \frac{\lambda n \Theta_{12}}{a_1 + a_2} \right) - T \lambda n \Theta_{12} \right] + T \rho_1 \rho_2 \lambda n (2\Theta_{12}^2 + n \Theta_{11} \Theta_{22}).$$

As an illustration we provide, in Figure 5, a plot of the normalized covariance swap rate measured in volatility points, i.e., $T \mapsto \sqrt{\frac{1}{T} E([Y^1, Y^2]_T)}$, for our calibrated 12-parameter OU–Wishart model from section 4.2 (Step A).

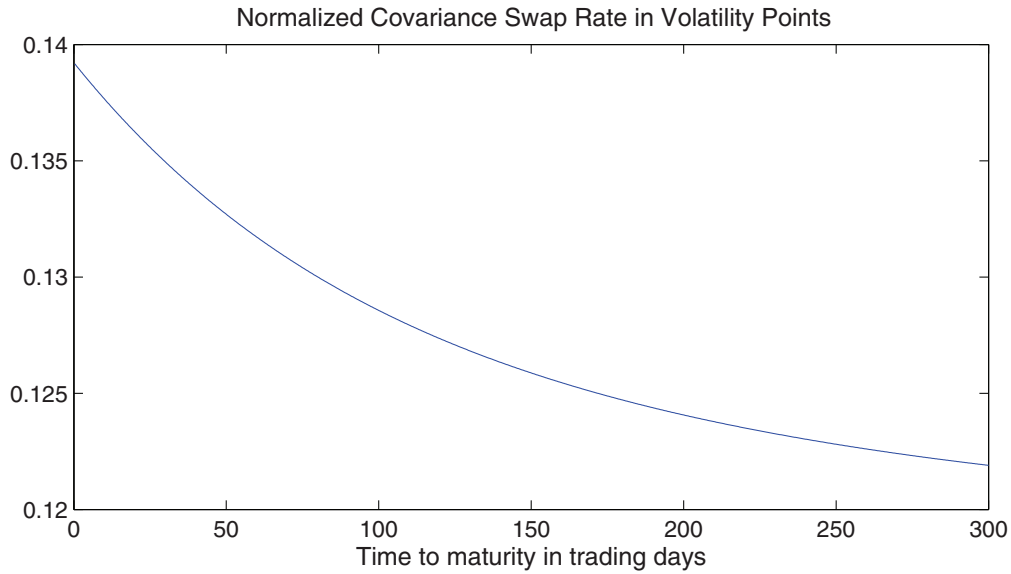


Figure 5. Normalized covariance swap rate for the calibrated 12-parameter OU–Wishart model.

Finally, we remark that similarly as in [6], pricing of options on the covariance can be dealt with using the Fourier methods from section 3, since the joint characteristic function of $(\Sigma^+, \rho^i(X) \rho^j(X) * \mu^L(dX))$ can be calculated similarly as in the proof of Theorem 2.5.

Appendix A. The following result on multidimensional analytic functions is needed in the proof of Lemma 2.7.

Lemma A.1. Let $D_\epsilon = \{z \in \mathbb{C}^n : \|\operatorname{Re}(z)\| < \epsilon\}$ for some $\epsilon > 0$. Suppose $f : D_\epsilon \rightarrow \mathbb{C}$ is an analytic function of the form $f = e^F$, where $F : D_\epsilon \rightarrow \mathbb{C}$ is continuous. Then F is analytic in D_ϵ .

Proof. Let $z = (z_1, z_2, \dots, z_n) \in D_\epsilon$, and define $z_{-1} = (z_2, \dots, z_n)$. Then $f_{z_{-1}} : w \mapsto f(w, z_{-1})$ defines an analytic function without zeros on the open convex set $D_{\epsilon, z_{-1}} := \{w \in \mathbb{C} : (w, z_{-1}) \in D_\epsilon\}$. By, e.g., [19, Satz V.1.4], there exists an analytic function $g_{z_{-1}}^1 : D_{\epsilon, z_{-1}} \rightarrow \mathbb{C}$ such that $\exp(g_{z_{-1}}^1) = f_{z_{-1}}$. Hence $F(w, z_{-1}) - g_{z_{-1}}^1(w) \in 2\pi i\mathbb{Z}$ on $D_{\epsilon, z_{-1}}$. Since both F and g are continuous, their difference is constant and it follows that $w \mapsto F(w, z_{-1})$ is analytic on $D_{\epsilon, z_{-1}}$. Analogously, one shows analyticity of F in all other components. The assertion then follows from Hartog’s theorem (cf., e.g., [25, Theorem 2.2.8]). ■

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