1, Diestel 7.17: Prove the Erdős-Sós conjecture for the case when the tree considered is a path. (Hint. Use Exercise 9 of Chapter 1.)

We seek to prove that the maximal number of edges in a graph with n vertices that does not have a path of length k as a subgraph is $\frac{1}{2}(k-1)n$ for each $k \geq 2$. To do this, we will prove the contrapositive: if G is a graph with $||G|| > \frac{1}{2}(k-1)|G|$, then G contains a path of length k as a subgraph. We prove this claim for fixed k by induction on $|G|$.

When $|G| \leq k$, G has at most $\binom{k}{2} = \frac{1}{2}(k-1)k \geq \frac{1}{2}(k-1)n$ edges, so the implication is vacuously true. Thus for the base case, let $|G| = k+1$. If $||G|| \leq \frac{1}{2}(k-1)|G|$, the claim holds. Thus assume

$$||G|| > \frac{1}{2}(k-1)|G| = \frac{1}{2}(|G|-2)|G|.$$

We wish to bound $\delta(G)$. Note that at least one vertex has degree $\delta(G)$; other than this vertex, the other $|G|-1$ vertices may form a complete graph. Thus we have

$$||G|| \leq \delta(G) + \binom{|G|-1}{2} = \delta(G) + \frac{1}{2}(|G|-2)(|G|-1).$$

Combining these inequalities, we get

$$\delta(G) + \frac{1}{2}(|G|-2)(|G|-1) > \frac{1}{2}(|G|-2)|G| \Rightarrow \delta(G) > \frac{1}{2}(|G|-2).$$

Thus $2\delta(G) > |G|-2$, so $2\delta(G) \geq |G|-1$ and $\delta(G) \geq \frac{|G|-1}{2} = \frac{k}{2}$. By exercise 8 in chaper 1 of Diestel, G contains a path of length at least $\min\{2\delta(G), |G|-1\}$. We have $2\delta(G) \geq 2\frac{k}{2} = k$ and $|G|-1 = k+1-1 = k$, so in either case we are guaranteed a path of length at least k, which of course contains as a subgraph a path of length k.

Assume, for some fixed natural $n > k+1$, that $\text{ex}(j, P_k) \leq \frac{1}{2}(k-1)j$ for all $j < n$. Let G be a graph on n vertices and consider two cases.

Case 1) $\delta(G) \geq \frac{k}{2}$. Then we have $2\delta(G) \geq k$ and $|G|-1 = n-1 > k+1-1 = k$, so once again by exercise 8 of chapter 1 in Diestel, we have a path of length k.

Case 2) $\delta(G) < \frac{k}{2}$. If $||G|| \leq \frac{1}{2}(k-1)|G|$, then the implication holds vacuously, so assume otherwise. Let v be a vertex of degree $\delta(G)$. Then $2\deg(v) < k$ so $2\deg(v) \leq k-1$ and $\deg(v) \leq \frac{k-1}{2}$. Let H be $G-v$. We have $|H| = |G|-1$, and $||H|| = ||G|| - \deg(v) \geq ||G|| - \frac{k-1}{2}$. Then

$$\begin{aligned}
||G|| &> \frac{1}{2}(k-1)|G| && \text{By assumption}\\
||G|| - \frac{k-1}{2} &> \frac{1}{2}(k-1)|G| - \frac{k-1}{2}\\
||G|| - \frac{k-1}{2} &> \frac{1}{2}((k-1)|G| - (k-1))\\
||G|| - \frac{k-1}{2} &> \frac{1}{2}(k-1)(|G|-1)
\end{aligned}$$

and

$$||H|| \geq ||G|| - \frac{k-1}{2} > \frac{1}{2}(k-1)(|G|-1) = \frac{1}{2}(k-1)|H|$$

so as $|H| < n$, by our induction hypothesis H contains a path of length k and thus so does G.

2, Diestel 7.18: Can large average degree force the chromatic number up if we exclude some tree as an induced subgraph? More precisely: For which trees T is there a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$, every graph of average degree at least $f(k)$ either has chromatic number at least k or contains an induced copy of T?

We claim the set of such trees is exactly the set of stars. (Note: an r-star is a tree with only leaves and one vertex of degree r.)

First, let T be an r-star. Let $f(k) = r(k-2)$. Let G be a graph such that $d(G) \geq r(k-2)$. If $\chi(G) \geq k$, we're done, so assume $\chi(G) < k$ and let φ be a $k-1$-coloring of G. As $d(G) \geq r(k-2)$, there must be a vertex of degree at least $r(k-2)$, say v. Consider the partition of $N(v)$ into φ-color classes. As, without loss of generality, v is colored 1, there are at most $k-2$ classes. By the pigeon-hole principle, at least one class, say color j, must be of size r (as v has $r(k-2)$ neighbors). The vertices of color j are not adjacent, so taking the incuded subgraph of r of those vertices and v gives an r-star. Thus T is an induced subgraph of G.

Now let T be a graph such that there is a function f such that for any graph G and any k, $d(G) \geq k \Rightarrow \chi(G) \geq k$ or T is an induced subgraph of G. Fix $k > 2$ (the $k \leq 2$ cases are trivial) and let $G = K_{f(k),f(k)}$. Note that G is bipartite, and thus $\chi(G) = 2 < k$, and $d(G) = \delta(G) = f(k) \geq f(k)$, so G contains T as an induced subgraph. Let A and B be the parts of G. Let $A' = A \cap T$ and $B' = B \cap T$. Assume, without loss of generality, $|A'| \leq |B'|$. First we claim $|A'| \leq 1$. Assume otherwise. Then $a_1, a_2 \in A'$ and $b_1, b_2 \in B'$ and $a_1 b_1 a_2 b_2 a_1$ is a cycle in T, contradicting that it is a tree. If $|A'| = 0$, then $|B'| \leq 1$, and T is a trivial star. Otherwise, $|A'| = 1$ and T is a $|B'|$-star. In either case, T is a star.

Thus we conclude the set of trees with the given property are exactly the star graphs.

3, Diestel 7.21: Given a graph G with $\varepsilon(G) \geqslant k \in \mathbb{N}$, find a minor $H \preccurlyeq G$ such that $\delta(H) \geqslant k \geqslant |H|/2$.

Let G be a graph with $\varepsilon(G) \geq k$. Let G' be a minor of G minimal by subgraph inclusion such that $\varepsilon(G') \geq k$. (We know such a minor exists as G is one.) Note $\varepsilon(G') \geq k \Rightarrow d(G') \geq 2k$. We claim G' contains a vertex of degree at most $2k$. Otherwise, every vertex has degree at least $2k+1$ and removing any edge gives a more minimal minor while still having $\varepsilon(G) = \frac{1}{2}d(G) \geq \frac{1}{2}\delta(G) = k$. Thus there is $x \in G'$ such that $d(x) \leq 2k$. Consider $y \in N(x)$. Note if $|N(x) \cap N(y)| < k$, then G'/xy is a more minimal minor as it has one fewer vertex but at most $k-1+1 = k$ fewer edges. Let H be the graph induced by $N(x)$. We have $|H| \leq 2k \Rightarrow k \geq \frac{|H|}{2}$ and $\delta(H) \geq k$ as each vertex had at least k neighbors in common with x. Thus H is the minor we seek.

4, Diestel 7.24: Show that any function h as in Lemma 3.5.1 satisfies the inequality $h(r) > \frac{1}{8}r^2$ for all even r, and hence that Theorem 7.2.3 is best possible up to the value of the constant c.

For positive, even r, the "easiest" way to find a TK^r in a $K_{s,s}$ is to select $\frac{r}{2}$ vertices in each partition to be non-subdivided vertices. Then, each partition will need to have an additional $\binom{r/2}{2}$ vertices in the opposite partition in order to add the missing edges. This is because there are $\binom{r/2}{2}$ pairs of selected vertices in one partition, and each such pair is missing an edge (by the definition of bipartite), and so the subdivision will require the use of at least one other vertex (and possibly more) in the opposite partition. Thus, the subdivision will require a total of at least

$$\frac{r}{2} + \binom{r/2}{2} = \frac{r}{2} + \frac{\frac{r}{2}(\frac{r}{2}-1)}{2} = \frac{1}{8}r^2 + \frac{1}{4}r$$

vertices in each partition. With this preliminary calculation in mind, we tackle the general case.

Observe that if more than $r/2$ vertices are selected in one partition, it is in fact the *other* partition that will require more vertices (because of the quadratic number of missing edges). To formalize this, say that $r/2 + x$ with $x \geq 0$ vertices are selected in partition U. Then, assuming $r \geq 4$, we have

$$s \geq |V| \geq \frac{r}{2} - x + \binom{\frac{r}{2}+x}{2} = \frac{r}{2} - x + \frac{(\frac{r}{2}+x)(\frac{r}{2}+x-1)}{2} = \frac{r^2+4x^2+(4r-12)x+2r}{8} \geq \frac{r^2+2r}{8}.$$

In the event that r is divisible by 4, this shows that $K_{r^2/8,r^2/8}$ has no subdivision of K^r. In the event that r is congruent to 2 mod 4, this shows that the complete, balanced bipartite graph with $\frac{r^2+2r}{8} - 1 > \frac{r^2}{8}$ vertices in each partition has no subdivision of K^r. In both cases, we have exhibited graphs with average degree at least $\frac{r^2}{8}$ containing no subdivision of K^r. We consider separately the $r = 2$ case, because h is a function to natural numbers, and it is clear we must require the average degree to be at least 1, because all graphs with average degree 0 contain no edges, and therefore no TK^2.

5, Diestel 7.33: Prove Hadwiger's conjecture for $r = 4$ from first principles. Hint: Use induction on $|G|$. Color a shortest cycle in G (if G has a cycle) and extend the coloring to the rest of the graph.

We prove by induction on the order of G that if G has no K^4 minor then G is 3-colorable. When G has fewer than 4 vertices, we have at least one color per vertex. Also, when G has no cycles, G is a forest. So each component of G is a tree, and we can 2-color each tree by beginning at an arbitrary root and coloring according to the parity of the distance from the root.

Otherwise, G has a cycle, and we let C be a minimum cycle in G. C has no chords, as a chord would create a smaller cycle. So C is an induced cycle. It is possible therefore to 3-color $G[V(C)]$ by traveling around the cycle alternating between two colors and coloring the last vertex with the third color. Fix such a coloring of C. If $G = C$ we are done, otherwise $G - C$ is non-empty.

Let $S_1, \ldots, S_k$ be the components of $G - C$. Each has order strictly smaller than G, and is K^4-minor free (by transitivity of the minor relationship). Let $1 \leq i \leq k$ be arbitrary. S_i has at most 2 neighbors in C, because otherwise contracting S_i to one vertex and contracting along C to remove all vertices of C except 3 neighbors forms a K^4.

If S_i has no neighbors in C we get a coloring of S_i from the induction hypothesis and need not modify it.

If S_i has only one neighbor in C, or two neighbors that both received the same color in the coloring of C, take $G[V(C) \cup V(S_i)]$ and contract C to a single vertex, v_C, to get a graph that is K^4-minor free (transitivity) and of smaller order, so 3-colorable by the induction hypothesis. Take a 3-coloring of this graph, and exchange colors so that v_C receives the same color as the neighbor(s) of S_i did in the coloring of C. The result is a 3-coloring of S_i that is proper on S_i but also proper with respect to the coloring of C that was fixed.

Otherwise, S_i has neighbors x and y in C that received different colors. In this case take $G[V(C) \cup V(S_i)]$ and contract C until only x and y remain from C. The result is again K^4 minor free and of smaller order, so 3-colorable. Take a 3-coloring of this graph, and permute colors so that x gets the same color as it had in the fixed coloring of C and y gets the same color as it had in C. This is possible because x and y were adjacent in the original coloring, so they received different colors, just as they do in the coloring of C. The result is again a 3-coloring of S_i that is proper with respect to the coloring of C.

By performing the above for each S_i, we obtain a 3-coloring on the entire graph G that is proper, completing the proof.

6, Diestel 7.35: Prove Corollary 7.3.5: A graph with $n > 2$ vertices and no K^5 minor has at most $3n - 6$ edges.

We assume that $n \geq 3$ from the context of the problem (and because otherwise the statement is false.)

We prove by induction on the structure given by Theorem 7.3.4 for the graph. For our base case, we observe that a planar graph has at most 3n - 6 edges (4.2.10). We also count that W has 8 vertices and 12 edges, so it satisfies the bound.

Finally, we prove that pasting two graphs that satisfy the bound along an edge or along a triangle preserves the bound. Let G_1 have

$$||G_1|| \leq 3|G_1| - 6$$

and

$$||G_2|| \leq 3|G_2| - 6.$$

Say the graph G is formed by pasting G_1 and G_2 along an edge. Then

$$||G|| = ||G_1|| + ||G_2|| - 1 \leq 3|G_1| + 3|G_2| - 13.$$

Also

$$|G| = |G_1| + |G_2| - 2$$

so

$$||G|| \leq (3|G_1| + 3|G_2| - 6) - 7$$

and

$$||G|| \leq 3|G| - 7 < 3|G| - 6.$$

Now say that G is formed by pasting G_1 and G_2 along a triangle. Then

$$||G|| = ||G_1|| + ||G_2|| - 3 \leq 3|G_1| + 3|G_2| - 15.$$

Also

$$|G| = |G_1| + |G_2| - 3$$

so

$$||G|| \leq (3|G_1| + 3|G_2| - 9) - 6$$

and

$$||G|| \leq 3|G| - 6.$$