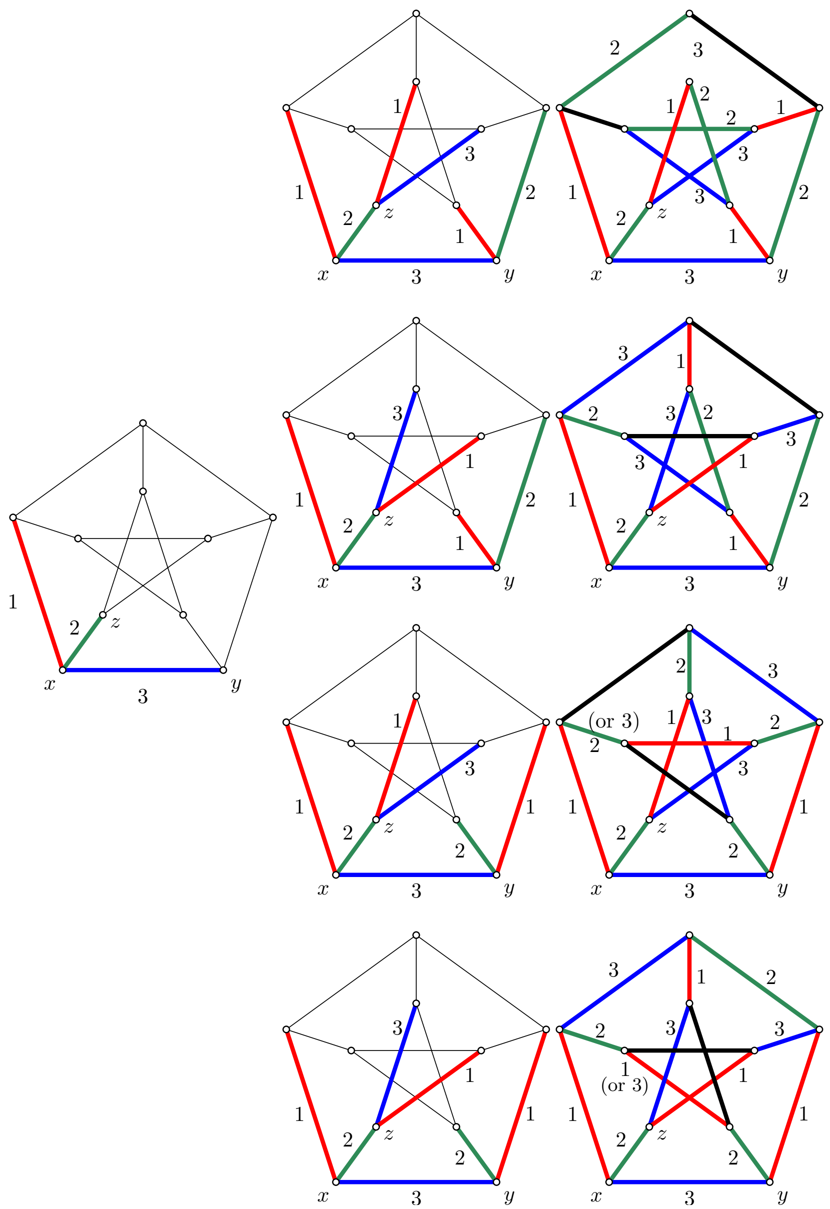

1: Determine, with proof, the edge-chromatic number of the Petersen graph.

The Petersen graph has maximum degree 3, so by Vizing's theorem its edge-chromatic number is 3 or 4. We will prove that in fact the edge-chromatic number is not 3, by considering possible 3-edge-colorings and finding contradictions.

First, we pick a vertex x and say that its edges are colored 1, 2, and 3 WLOG. Let y be the neighbor along the 3 edge, and let z be the neighbor on the 2 edge. y has two other edges, and we consider two cases where they are numbered 1, 2 and 2, 1. z has two other edges, and we consider two other cases where those are numbered 1, 3 and 3, 1. Combining these cases gives 4 cases. In all 4 cases, we can fill in forced edge colors until there is a contradiction [the order in which you force edges will affect which edges are colored and which are uncolorable in the end, but in any case you find a partial coloring forced by the assumptions that cannot be extended to a proper coloring]. We display the the initial assumption, then the four cases, and then the forced colors. This proves $\chi'(G) = 4$.



2: Show that no regular, self-complementary graph has edge-chromatic number equal to its maximum degree.

An r-regular self-complementary graph has $r = \frac{n-1}{2}$ (since $n - 1 - r = r$). Because r is an integer, n is odd. Odd order regular graphs are all class 2 because each vertex must be missing a color in any edge-coloring, so a $\Delta(G)$-edge-coloring is not possible.

3: Show that no regular graph with a cut vertex has edge-chromatic number equal to its maximum degree.

We show the contrapositive, that a regular class 1 graph has no cutvertex.

Let G be a regular graph with vertex v and let φ be a Δ-edge-coloring of G. Let x and y be neighbors of v. Let $\alpha = \varphi(vx)$ and let $\beta = \varphi(vy)$. In $G-v$, x is missing α (and y is missing β). Let P be the α/β path starting at x. We prove that P ends at y: No vertex besides x and y is missing α or β in $G-v$, because the edges of v besides vx and vy cannot be colored α or β, and in G no vertex is missing any color. x is missing only α, because its other edges remain in $G-v$. Therefore, the last vertex of P cannot be x (it has a β edge), and it cannot be any vertex other than x and y (they have an α edge and a β edge), so the path ends at y.

Because any two neighbors of v have a path between them in $G-v$, v is not a cut-vertex! Suppose a' and b' are connected in G and disconnected in $G-v$; let P be an a'–b' path; v lies on P (otherwise P is an a'–b' path in $G-v$), so call a and b the neighbors of v on P (with a closer to a' and b closer to b'). There is an a–b path in $G-v$, Q, and so $a'PaQbPb'$ is an a'–b' path in $G-v$.

4, Diestel 7.4: Determine the value of $\text{ex}(n, K_{1,r})$ for all $r, n \in \mathbb{N}$.

First, observe that if $n \leq r$, no graph with n vertices contains a star graph on $r+1$ vertices, so in this case, the extremal number is $\binom{n}{2}$, the maximum number of edges a graph can have.

In the non-trivial case, we will show that the extremal number is $\lfloor \frac{n(r-1)}{2} \rfloor$. The idea is that this is the number of edges we get from a $(r-1)$ regular graph. Such a graph is clearly edge-maximal without a r-star, and moreover, it is extremal, as every vertex is contributing as much as it possibly can without creating an r-star. (You can also see this as any graph with more edges would have to have a vertex of degree r by averaging, so at least, our bound is an upper bound on the extremal number. The rest of the proof is showing that the upper bound can be realized.)

First, let's assume $n(r-1)$ is even. This is when we won't need a floor. Let's start with an n-cycle, which is 2-regular. (If $r = 1$, the graph can not have edges so the bound is trivial.) For every i in $2 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$, we will add an edge between every vertex at distance i on the original cycle. If $(r-1)$ is even, we have created an $r-1$ regular graph, as desired. Otherwise, we have an $(r-2)$ regular graph, and n has to be even. In this case, we will add edges between all vertices that are at a distance $\frac{n}{2}$ from each other. Observe that these additional edges give us a perfect matching. So every degree increases by 1 to give us an $(r-1)$ regular graph, and we are again done.

So now we just have to worry about the case when both n and $(r-1)$ are odd. The previous construction can be used to create an $r-2$ regular-graph, and we add a perfect matching of edges between pairs of $n-1$ of the vertices, leaving exactly one vertex of degree $r-2$. Thankfully, the resulting graph also matches our bound, and is extremal since there is no graph with total degree being odd.