
1: Given k and a k-coloring of a k-chromatic graph, prove that for any color c there is a vertex of color c which is adjacent to vertices of every other color.

Let a k-chromatic graph have a k-coloring given. For readability, assume WLOG c = Red

Suppose all red vertices v_i have some color other than red, c_i, such that v_i is not adjacent to a vertex of color c_i. We modify the k-coloring by coloring each v_i with the color c_i instead. The is still a proper coloring, because any pair of adjacent vertices were originally both not colored red, and are unchanged, or had one colored red, but in that case the red vertex was recolored to a color other than its adjacent vertices' colors. (No pair of adjacent vertices were both recolored, because no pair of adjacent vertices were both colored red originally). This procedure produces a proper $k-1$ coloring, which contradicts the assumption that the graph is k-chromatic.

2, Diestel 5.18: Given a graph G and $k \in \mathbb{N}$ let $P_G(k)$ denote the number of vertex colourings $V(G) \to \{1, \ldots, k\}$. Show that P_G is a polynomial in k of degree $n := |G|$, in which the coefficient of k^n is 1 and the coefficient of k^{n-1} is $-||G||$. (P_G is called the *chromatic polynomial* of G.)
(Hint. Apply induction on $||G||$.)

A counting argument shows that when G has 0 edges and n vertices, there are k^n ways to color the graph with k colors: for each of n vertices choose any of k colors. So P_G is a polynomial in k of degree n with 1 as the coefficient of k^n and $0 = ||m||$ as the coefficient of k^{n-1}. This completes the base case of induction on $||G||$.

Let $e \in G$ be arbitrary. By the induction hypothesis, P_{G-e} is a polynomial of degree n with 1 as the coefficient of k^n and with $-||G-e|| = -(||G||-1)$ as the coefficient of k^{n-1}. Similarly, $P_{G/e}$ is a polynomial of degree $n-1$ with 1 as the coefficient of k^{n-1} (we do not worry about the coefficient of k^{n-2} here, since contraction can remove multiple edges).

We find $P_G = P_{G-e} - P_{G/e}$ by inclusion-exclusion: the colorings of $G-e$ are all proper in G except exactly those which correspond to proper colorings of $P_{G/e}$; this is because a proper coloring of $P_{G/e}$ is proper with respect to all edges except e, and assigns the same color to both vertices of e.

More formally [this is the above argument expanded as a refresher, but not entirely necessary if the above is clear]: let B_f be the event that a coloring colors both vertices of f the same color. There are k^n (possibly improper) colorings, so the number of proper colorings is

$$k^n - \left| \bigcup_{f \in E(G)} B_f \right| = k^n - \left| \bigcup_{f \in E(G-f)} B_f \right| - |B_e| + \left| B_e \cap \left(\bigcup_{f \in E(G-f)} B_f \right) \right|.$$

Clearly $\left|\bigcup_{f \in E(G-e)} B_f\right| = k^n - P_{G-e}$ and $\left|\bigcup_{f \in E(G)} B_f\right| = k^n - P_G$ by the definition of P_G and of proper colorings. The other two terms subtract the colorings in B_e that are proper for every other edge. By extending a coloring of G/e to be a coloring of G (where both vertices of e are given the color their contracted vertex was given), we form a bijection between proper colorings of G/e and the set whose cardinality is subtracted (colorings that are proper everywhere except e and improper at e). This gives $P_G = P_{G-e} - P_{G/e}$.

From the facts about P_{G-e} and $P_{G/e}$ we got from the IH, this means that P_G is a polynomial in k of degree at most n. k^n has coefficient $1 - 0 = 1$, and this also establishes that the degree is n. k^{n-1} has degree $-(||G||-1) - 1 = -||G||$, so the induction step is complete.

3, Diestel 5.19: Determine the class of all graphs G for which $P_G(k) = k(k-1)^{n-1}$. (As in the previous exercise, let $n := |G|$, and let P_G denote the chromatic polynomial of G.)
Hint: A graph with n vertices is a tree if and only if it is connected and has $n-1$ edges.

As the hint betrays, the class is trees.

We establish that trees have this polynomial by induction on the order, where the base case is a tree with one vertex whose chromatic polynomial is clearly k. For the induction, given G a tree with $n+1$ vertices, we let v be a leaf of G with neighbor u. We have that $P_{G-v} = k(k-1)^{n-1}$ by the induction hypothesis. For each proper coloring of $G-v$, there is some color assigned to u, and so there are $k-1$ colors not used at u that can be used at v. So, we can form a correspondence between the proper colorings of G and the proper colorings of $G-v$, where $k-1$ colorings in the first set are mapped to each coloring in the second, to conclude $P_G = (k-1)P_{G-v} = k(k-1)^n$.

In the reverse direction, we prove that if $P_G(k) = k(k-1)^{n-1}$ then G is connected, and has $n-1$ edges (and therefore a tree by the hint). First, by algebra, the coefficient of k^{n-1} is $-(n-1)$, so G has $n-1$ edges. Second, we prove that a graph with c components has k as a root with multiplicity at least c ($k(k-1)^{n-1}$ therefore is connected, since it is factored and this shows that the multiplicity of k is 1).

All chromatic polynomials have k as a root of multiplicity at least 1 because we can choose a vertex v and partition the colorings by the color of v. The classes of this partition have the same cardinality, because there is a bijection between them by swapping the colors of two classes (color all blue vertices red and all red vertices blue) so that v changes to the desired color. Thus $k \mid P_G$.

For $c > 1$ let $C_1, \ldots, C_c$ be the components of G. Clearly $P_G = \prod_{i=1}^{c} P_{C_i}$ by counting. Since $k \mid C_i$ for each C_i, $k^c \mid P_G$, and so k has multiplicity at least c.