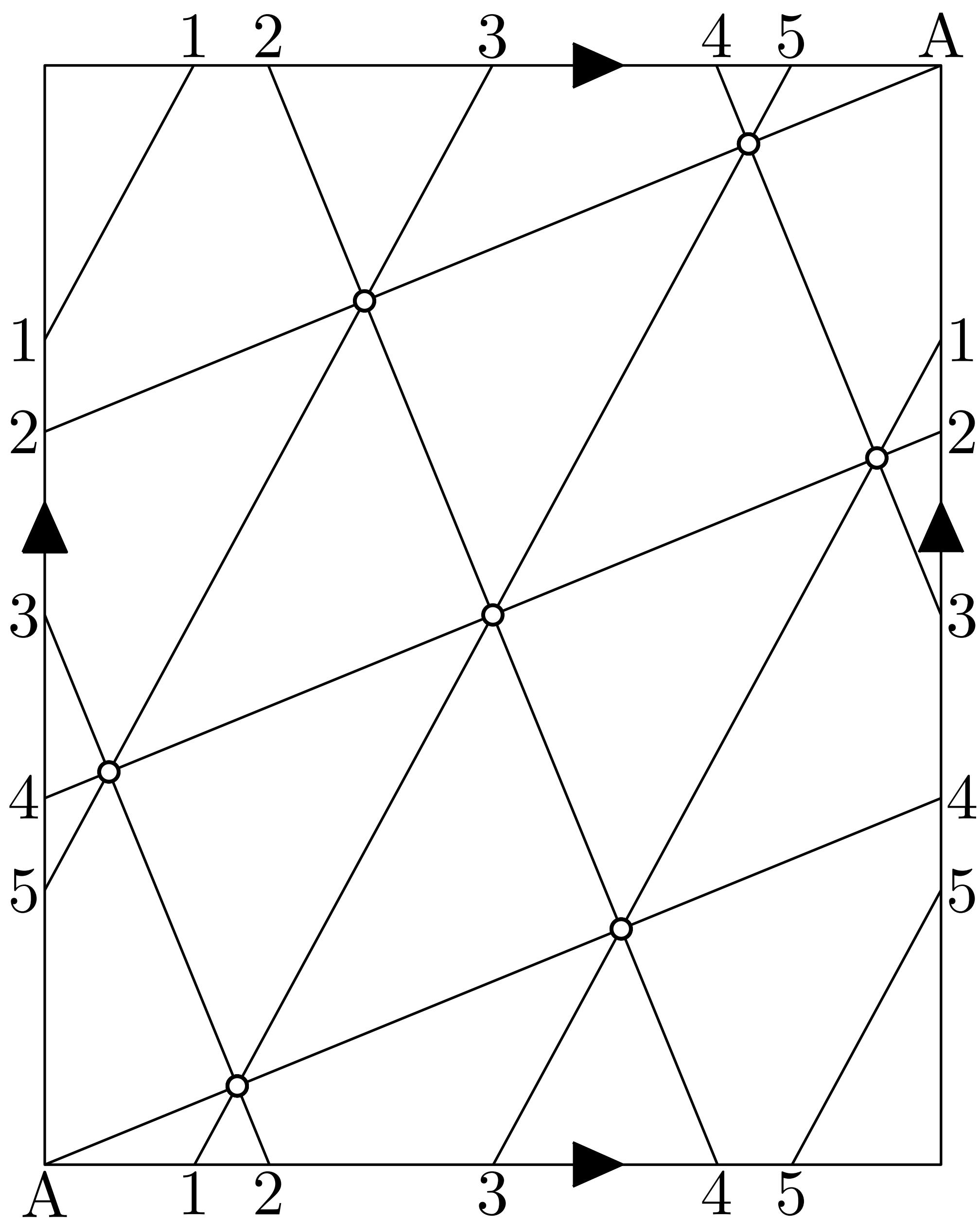
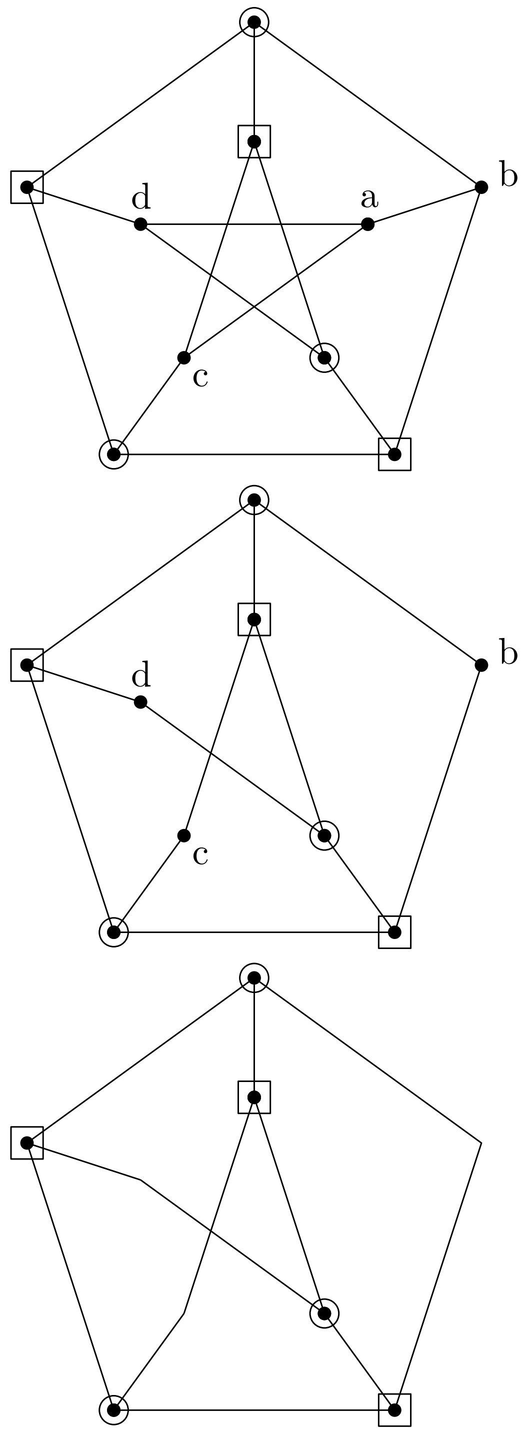
1: Draw K^7 on a torus with no edge crossings.

A quick calculation reveals that an embedding of K^7 on the torus is a 2-cell embedding. At that point, it is hard to go wrong if you start drawing C^3 faces, although making all edges straight is slightly harder. Note that the point marked A can be viewed as traveling through the horizontal borders and then the vertical borders to move to the opposite corner. It can be redrawn to only use one border at a time quite easily.



2: Find a TK^5 or a $TK_{3,3}$ in the Petersen Graph.

Delete a, then suppress b, c, and d to get the minor. Reversing the suppression is subdivision, and this yields $P - a$ (where P is the Petersen graph). Clearly $P - A \subseteq P$. [1]



[1] See animation on Wikipedia: `http://en.wikipedia.org/wiki/File:Kuratowski.gif`

3, Diestel 4.4: Show that every planar graph is a union of three forests.

If G is a planar graph, then every induced subgraph of G is planar (use embedding of G to get an embedding of the subgraph). Planar graphs satisfy

$$m \leq 3n - 6,$$

so the conditions of Theorem 2.4.3 are satisfied. This theorem implies that the edges of G can be partitioned into three trees, and we can group any isolated vertices together with one of the three trees to obtain three forests whose union makes the original graph.

4, Diesel 4.23: A graph is called *outerplanar* if it has a drawing in which every vertex lies on the boundary of the outer face. Show that a graph is outerplanar if and only if it contains neither K^4 nor $K_{2,3}$ as a minor.

Consider a graph G and let G' be formed from G by adding a new vertex v which is adjacent to all other vertices.

Lemma 1: Let J' be a vertex transitive graph, let $v \in J'$ and let $J = J' - u$. J is a topological minor of G if and only if J' is a topological minor of G'.
Proof: ($\Rightarrow$) Suppose $H \subseteq G$ is a subdivision of J. Let H' be formed from H by adding a new vertex v which is adjacent to the vertices of J (but not vertices added through subdivision). $H' \subseteq G'$, and H' is a subdivision of $J * K^1$ (by performing all of the subdivisions in H), so $J * K^1$ is a topological minor of G'. J' is a topological minor of $J * K^1$, so by transitivity of topological minors, J' is a topological minor of G'.

($\Leftarrow$) Suppose $H' \subseteq G'$ is a subdivision of J'. We consider 3 cases:

- If $v \notin H'$ then $H \subseteq G$, and so J is a topological minor of J' which is a topological minor of G, and we finish by transitivity.
- If $v \in H'$ was added during subdivision, then let the edge $xy \in J'$ be the edge that was subdivided to form v. Let H be a subdivision of $J - x$ formed by the same subdivisions as in H' (except in edges that no longer exist, and particularly xy). Since $x \notin J - x$, $v \notin H$, and so $H \subseteq G$. Therefore, $J - x$ is a topological minor of G.
- If $v \in H'$ was not added during subdivision, the let H be a subdivision of $J - v$ formed by the same subdivisions as in H' (except the edges that went to v). Since $v \notin H$, $H \subseteq G$, and since H is a subdivision of $J - v$, $J - v$ is a topological minor of G. $\square$

Lemma 2: G has a K^4 or $K_{3,2}$ minor if and only if G' has a K^5 or $K_{3,3}$ minor.
Proof: By Lemma 1, G has a K^4 topological minor if and only if G' has a K^5 topological minor, and G has a $K_{3,2}$ topological minor if and only if G' has a $K_{3,3}$ topological minor. Adding Proposition 1.7.4 and Lemma 4.4.2 we finish the lemma. $\square$

By Lemma 2, G has one of the forbidden minors if and only if G' has one of K^5 and $K_{3,3}$ as a minor. Adding Kuratowski's theorem, we have that G has neither of the minors if and only if G' is planar. If G' is planar, we take an embedding of G' and delete v, and then homeomorph the drawing on the sphere so that the face that v was in is the outer face. This gives an outerplanar drawing of G which witnesses the outerplanarity of G. In the other direction, G cannot be outerplanar if it has one of the forbidden minors, because an embedding of G on the plane with all vertices on the outer face can be transformed into a planar embedding of G' by adding v in the outer face, and drawing an arc from v to each vertex such that no arc intersects.

5: Prove or disprove: Every planar bipartite graph has a vertex of degree at most 3.

In class we proved that 2-edge-connected bipartite graphs satisfy

$$m \leq 2n - 4$$

by using Euler's formula, and counting edges in two ways (recall that bipartite graphs have at least 4 edges per cycle). We argue that this holds even for graphs that are not 2-edge-connected so long as they have order at least 3. First, we can apply the argument to each component of a disconnected graph and combine the results. Second, if we count the edges that lie on two faces once for each face, and count bridges (which are only on the frontier of a single face) twice for their face, we find that each face still counts at least 4 edges and each edge is counted twice, with the exception of faces that have no edges from a cycle on their frontier. This happens if and only if the graph is a forest, and we know that forests satisfy the result except when there are fewer than 3 vertices.

Because (non-empty) forests on fewer than 3 vertices must have a vertex of degree at most 3, need to only finish on the graphs where the inequality holds. We know that the average degree of a graph is $\frac{2m}{n}$, and so the average degree of planar graphs is at most $\frac{4n-8}{n}$ which is strictly less than 4. Since degrees are integers, there is a vertex of degree at most 3.

6: Show that the complement of any planar graph with 11 or more vertices is not planar. (It has been shown that the above statement is true when 11 is replaced by 9, and there are examples of planar graphs with 8 vertices having planar complement.)

Let our planar graph have n vertices and m edges. We have (when $n \geq 3$):

$$m \leq 3n - 6.$$

The complement has $\frac{n(n-1)}{2} - m$ edges, so for the complement to be planar we would have

$$\frac{n(n-1)}{2} - m \leq 3n - 6$$

or

$$m \geq \frac{1}{2}n^2 - \frac{7}{2}n + 6$$

Combining these two gives

$$n^2 - 7n + 12 \leq 6n - 12$$
$$n^2 - 13n + 24 \leq 0.$$

By the quadratic formula (okay, actually just Wolfram Alpha), this inequality only holds for $n \in [3, 10]$ (because n is an integer), finishing the proof. Note that $n < 3$ is in fact possible because our equation did not cover that case.