
1, Diestel 3.5: Deduce the $k = 2$ case of Menger's theorem (3.3.1) from Proposition 3.1.1.

Let G be 2-connected, and let A and B be 2-sets.

We handle some special cases (thus later in the induction if these occur then we are done, so we may assume these do not occur): if $A = B$ then A is a pair of disjoint A–B paths. If $|A \cap B| = 1$ then let $A \cap B = \{v\}$. v is an A–B path, and $G - v$ is connected (because G was 2-connected), so there remains an A–B path in $G - v$, and this path together with v are two disjoint A–B paths.

Otherwise, there are four vertices in $A \cup B$: a_1, a_2, b_1, b_2. G is 2-connected, so by proposition 3.1.1, G is a cycle or G is formed from a 2-connected graph H plus an H-path. We go by induction on this structure of H:

Base Case: When $G = C = v_0v_1v_2 \cdots v_kv_0$ is a cycle, then let $a_1 = v_i$, $a_2 = v_j$, $b_1 = v_x$, $b_2 = b_y$. WLOG $i < j < x < y$ or $i < x < j < y$, because i, j and x, y can be flipped, as well as A and B, and we care only about order of appearance on a cycle. In the first case, $v_iCv_0 \overset{\circ}{v}_k Cv_y$ and v_jCv_x are two disjoint A–B paths, and in the second v_iCv_x and v_jCv_y are.

Induction Step: Let the H-path be P. We go by cases

- If none of the vertices lie on the interior of P, then we delete P and find the paths by the induction hypothesis.
- If one of A and one of B are in the interior of P (WLOG a_1 and b_1) then we take a_1Pb_1 as one path, and find the other in H (because the remaining graph is connected).
- If two of A and none of B are on the interior of P, then we set A' to be the endpoints of P, find two disjoint A'–B paths on H, and extend them along P to a_1 and a_2 in the obvious way.
- Two of B and none of A is proved by the previous case WLOG.
- If one of A and none of B on the interior of P, then one endpoint of P is not in A, and we so we take that and the other vertex of A to form A'. The induction hypothesis gives the A'–B paths in H, and we extend on e path the the vertex on P.
- If two of A and one of B are on the interior of P, then WLOG b_1 is in the interior of P, and P contains vertices in the order b_1, a_1, a_2 or a_1, b_1, a_2. In either case we take b_1Pa_1 as one path, find a path in H from b_2 to the endpoint of P closer to a_2 on P than a_1, and extend that path to a_2.
- Two of B and one of A is proved by the previous case WLOG.
- If all four vertices lie on the interior of P, then the endpoints of P are connected in H, so we can add a path between them to P to make a cycle. Then this case was proved by the base case.

2, Diestel 3.17 (i): Find the error in the following 'simple proof' of Menger's theorem (3.3.1). Let X be an A–B separator of minimum size. Denote by G_A the subgraph of G induced by X and all the components of $G - X$ that meet A, and define G_B correspondingly. By the minimality of X, there can be no A–X separator in G_A with fewer than $|X|$ vertices, so G_A contains k disjoint A–X paths by induction. Similarly, G_B contains k disjoint X–B paths. Together, all these paths form the desired A–B paths in G.

The problem is that G_A might be equal to G. In this case, no matter what the induction is on (vertices, edges, etc.), we cannot argue about G_A using the induction hypothesis, because we did not modify it from G. Formally: if the induction is on an arbitrary graph invariant, then this invariant does not change because $G_A \cong G$, so assuming the IH for graphs with smaller values of the invariant does not say anything about G_A.

To prove that this can occur, consider when $G = K_n$, B is a set of one vertex and A is the other $n-1$. B is the unique minimum-size A–B separator, but with $X = B$, $G_A = K_n$ (because all vertices are in A or B, so they either meet A or are in X. So, when Diestel says "by induction" he leaves unhandled the cases where the only minimum A–B separators are ones whose deletions leave all components meeting A. In fact, the same problem can occur with $G_B \cong G$.

3, Diestel 3.18: Prove Menger's theorem by induction on $||G||$, as follows. Given an edge $e = xy$, consider a smallest A–B separator S in $G - e$. Show that the induction hypothesis implies a solution for G unless $S \cup \{x\}$ and $S \cup \{y\}$ are smallest A–B separators in G. Then show that if choosing neither of these separators as X in the previous exercise gives a valid proof, there is only one easy case left to do.

Base Case: ($||G|| = 0$) is the same as in the original proof of Menger's theorem. We observe the minimum separator is $X = A \cap B$, and $A \cap B$ is in fact the maximum size set of disjoint A–B paths (all of which happen to be trivial).

Induction Step: Let S be a smallest separator of $G - e$ and let $|S| = k$. If one of $S \cup \{x\}$ and $S \cup \{y\}$ is not a smallest A–B separator in G, WLOG $S \cup \{x\}$ is not, then we know $S \cup \{x\}$ is a separator of G (because the only A–B paths in G not in $G - e$ are those that include e, and these paths go through x; all other paths go through S by its definition). So the minimum size separator of G has size less than $|S \cup \{x\}| = k + 1$, meaning size at most k. It is also size at least $|S|$ because any separator of G is a separator of $G - e$, so we need to find k disjoint A–B paths in G. Such paths are given in $G - e$ by the IH, and those are paths in G as well.

In the case that both $S \cup \{x\}$ and $S \cup \{y\}$ are smallest separators, we observe that either none or both of x and y are in S (as otherwise they are both smallest but with different sizes). If neither is in S, then we show that one of $S \cup \{x\}$ and $S \cup \{y\}$, WLOG $S \cup \{x\}$ gives a $||G_A|| < ||G||$ and $S \cup \{y\}$ gives a $||G_B|| < ||G||$ (as defined in 3.15), which allows us to use the IH (because the new graphs in which we find paths are strict subgraphs) to get $k + 1$ disjoint A–$S \cup \{x\}$ paths is G_A and $k + 1$ disjoint B–$S \cup \{y\}$ paths in G. We can concatenate paths that have endpoints in S, and then for the A–x path and the B–y path put e in between to form a new system of $k + 1$ disjoint A–B paths.

To argue that $||G_A||, ||G_B|| < ||G||$, we consider $G - S$, which has an A–B path, $P = a \ldots b$ (with $a \in A$) because $S \cup \{x\}$ is larger than S and is a minimum separator. Because S is a separator of $G - e$, this A–B path must include e, and WLOG the path has x closer to A than y. Then since $S \cup \{x\}$ is an A–B separator, and yPb does not intersect $S \cup \{x\}$, all A–y paths must intersect $S \cup \{x\}$. Then, since $y \notin S$, and there are no A–y paths in $G - S \cup \{x\}$, we conclude $y \notin G_A$, and $e \notin G_A$. The symmetric argument gives $x \notin G_B$ and $e \notin G_B$. Thus, the use of the IH was appropriate because $e \notin G_A, G_B \Rightarrow ||G_A||, ||G_B|| < ||G||$.

Otherwise, x and y are both in S, so S is a smallest separator of G, and then we observe that the k disjoint A–B paths in $G - e$, which are given by the IH, suffice.

In all cases, the other direction (that there are no more paths than the size of the minimum separator of G) remains clear and need not use the IH.

4, Diestel 3.21: Let $k \geq 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

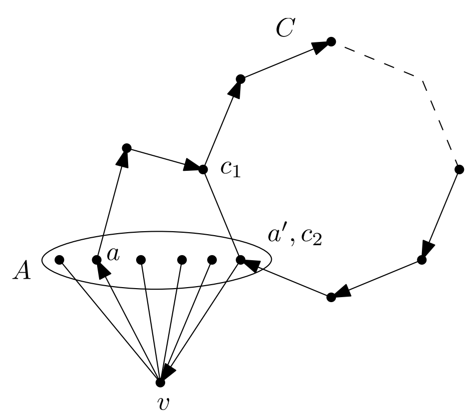
Let $k \geq 2$ and let G be a k-connected graph with $|G| \geq 2k$. As G is k-connected, it is connected, and as $\delta(G) \geq \kappa(G) \geq k \geq 2$, it has no leaves, so it is not a tree, so it has a cycle.

Let C be a largest cycle in G. First, as $\delta(G) \geq \kappa(G) \geq k$, and G has a cycle, $|C| \geq k+1$ by Diestel Proposition 1.3.1. Assume for the sake of contradiction that $|C| < 2k$. Then there is $v \in G \setminus C$. Let $A = N(v)$ and $B = V(C)$. As $\delta(G) \geq \kappa(G) \geq k$, $|A| \geq k$. Furthermore, any set X of size less than k cannot separate A and B as that would disconnect v and some $c \in C$, contradicting that G is k-connected. Thus the size of a minimum separator is at least k, and by Menger's theorem, there are at least k disjoint A–B paths.

By the pigeon-hole principle (with vertices in A as pigeons and edges in C as holes), there are $a, a' \in A$ and $c_1, c_2 \in C$ such that $c_1c_2 \in E(G)$ there are distinct a–c_1 and a'–c_2 paths P_a and $P_{a'}$. (Note that these paths may be of length zero if a vertex of C is adjacent to v.) Let P be the c_1–c_2 path in C of size at least two. Then

$$C' = vP_a\mathring{P}P_{a'}v$$

has size at least one larger than C, contradicting the maximality of C.



We conclude $|C| \geq 2k$.

5, Diestel 3.22: Let $k \geq 2$. Show that in a k-connected graph any k vertices lie on a common cycle.

Let $k \geq 2$ and let G be k-connected. Let $v_1, \ldots, v_k$ be k arbitrary vertices in G. As $\delta(G) \geq \kappa(G) \geq k$, G does not contain any leaves, but it is connected, so it must contain a cycle. Let C be the cycle containing the largest number of the k designated vertices and assume for the sake of contradiction that it contains $j < k$ of them. Without loss of generality, label those vertices $v_1, \ldots, v_j$. Let $A = N(v_k)$ and $B = V(C)$. Consider two cases.

First, if $|C| < 2k$, then as in the solution to Diestel 3.18 above, we find C' containing $v_1, \ldots, v_j, v_k$, contradicting the maxmimality of C.

Thus assume $|C| \geq 2k$. Then partition C into j independent paths, each beginning with v_i and ending with the vertex before v_{i+1} (or v_1 in the case we began at v_j). As $|A| \geq k$, and $|B| \geq k$, any A–B separator, S, has size at least k, because it either contains all of A or B or it leaves a vertex $a \in A$ disconnected from a vertex $b \in B$ in $G - S$. In the latter case, $|S| \geq k$ because otherwise S contradicts the k-connectedness of G.

By Menger's theorem (3.3.1) there are at least k A–B paths, $P_1, \ldots, P_k$ so by the pigeon-hole principle there is a segment in which two paths terminate. Call those paths P_i and P_j. We can follow C until the first of those paths (WLOG P_i), take P_i to v_k (either adding the edge to v_k at the end, or terminating early if P_i intersects v_k. Then we take the other path back to C (again either with the edge to a neighbor and then P_j or starting on P_j at v_k). Following the rest of C after this yields C' containing $v_1, \ldots v_j$ and v_k, contradicting the maxmimality of C.

Thus we conclude C must contain all k of the designated vertices.