1: Suppose that 13 people are each dealt 4 cards from a standard 52-card deck. Show that it is possible for each of them to select one of their cards so that no two people have selected a card of the same rank.

We construct a simple bipartite graph G with the 13 people as the vertices of partition P, the 13 ranks as the vertices of the other partition R, and an edge pr connecting the person p with the rank r if p has at least one card of rank r.

Let $S \subseteq P$ be an arbitrary set of people. These people hold a total of $4|S|$ cards. Each rank appears on exactly 4 cards (one for each suit) in the deck. So each rank that appears within those $4|S|$ cards appears on at most 4 of those cards. Therefore by the pigeonhole principal, there must be at least $|S|$ distinct ranks that appear on those cards. In terms of G, this means

$$|N(S)| \geq |S|,$$

which is exactly Hall's Condition. Hall's Theorem then implies the existence of a matching of P. This matching indicates a rank in each player's hand that the player can select such that no two players select the same rank.

2, Diestel 2.4: Moving alternatively, two players jointly construct a path in some fixed graph G. If $v_1, \ldots, v_n$ is the path constructed so far, the player to move next has to find a vertex v_{n+1} such that $v_1, \ldots, v_{n+1}$ is again a path. Whichever player cannot move loses. For which graphs G does the first player have a winning strategy, for which the second?

The second player has a winning strategy if and only if the graph contains a perfect matching. A property of this type of game is that the second player has no winning strategy if and only if the first player has a winning strategy.

If $M \in E(G)$ is a perfect matching, then we prove that player 2 has a winning strategy by taking matched edges on every turn. We go by induction on $|M|$.

- **Inductive Hypothesis:** Player 2 has a winning strategy if G has a perfect matching of size n
- **Base Case:** When G has an empty perfect matching, player 1 loses immediately as there are no vertices in G by the definition of perfect matchings.
- **Induction Step:** When G has a perfect matching of size $n+1$, after player 1 chooses a vertex a, player 2 can use $m \in M$ to choose a matched vertex b. The remaining vertices form $G - m$, which has a perfect matching $M - m$ of size n. By the induction hypothesis, no matter which vertex player 1 chooses next (as his "first" move in $G - m$), player 2 has a winning strategy in the remaining game on $G - m$. Since player 1's choice of a was arbitrary, player 2 has a winning strategy in the entire game on G.

We now prove that if G has no perfect matching, player 1 has a winning strategy. Let M be a maximal matching in G. M is not a perfect matching, so there exists a vertex a that is unmatched in M. Player 1 chooses a as his first move. For each other move, Player 1 takes an edge m in M. Player 1 may do this until one of two things happens (and we prove neither can happen):

- **Case 1: Player 2 chooses an unmatched vertex.** In this case, the path in the game so far is an augmenting path (since every other edge is chosen as a matched edge by player 1, and player 2 cannot pick a matched edge since a is unmatched, and after every other turn by player 1, the matched edge incident with the current vertex was just traversed). This augmenting path can be used to extend M into a larger matching, which contradicts M being maximal.
- **Case 2: Player 2 chooses a matched vertex x, but x is matched to a vertex y that has already been traversed.** Neither of x and y is a, since x and y are matched. Player 1 has not moved to y because x had not been traversed and player 1 only takes edges in the matching. If player 2 had moved to y earlier, then x would already have been traversed.

So long as M is maximal, we have shown that player 1 can employ this strategy indefinitely in response to player 2's moves (in reality, until player 2 loses). Therefore, player 1 has a winning strategy.

We have shown that player 2 has a winning strategy when there is a perfect matching, and player 1 has a winning strategy when there is no perfect matching. This partitions the games, and by the unproved theorem, there are no other games where player 1 or 2 has a winning strategy.

3, Diestel 2.10: Prove *Sperner's theorem:* in an n-set X there are never more than $\binom{n}{\lfloor n/2 \rfloor}$ subsets such that none of these contain another.

(Hint. Construct $\binom{n}{\lfloor n/2 \rfloor}$ chains covering the power set lattice of X.)

Let $m = \lfloor n/2 \rfloor$. Consider $C = \binom{X}{m}$, the family of subsets of X of cardinality m. As the notation in the definition of A suggests,

$$|C| = \binom{n}{m} = \binom{n}{\lfloor n/2 \rfloor}.$$

We will extend each subset in C into a chain in such a way that the chains formed cover the power set lattice of X. Then, any *antichain* (set of subsets of X such that no subset contains another) contains at most one element of each chain (since all elements in a chain are comparable pairwise, and all elements of an antichain are pairwise incomparable), and therefore an antichain has no more than $\binom{n}{\lfloor n/2 \rfloor}$ subsets.

We prove by induction on k that there exists a set C_m^k for all integers $k \in [m, n]$, where C_m^k is a set of $\binom{n}{m}$ chains that contain (in their union) all subsets of X of cardinality j for $m \leq j \leq k$, and each chain contains a subset of each such cardinality. (This is our induction hypothesis).

Base Case: Let $C_m^m = C = \left\{a_i : 1 \leq i \leq \binom{n}{m}\right\}$.

Induction Step: We extend a set of chains C_m^k (given by the Induction Hypothesis) using Hall's theorem. Each subset $s \in \binom{X}{k}$ is an element of a chain of C_m^k. For any s there are $n - k$ subsets of size $k + 1$ that contain s (one for each element of $X \setminus s$). For any subset s' of X of size $k + 1$, there are $k + 1$ subsets of size k contained in s' (one for each element of s'). Consider G, the bipartite graph with one partition $B = \binom{X}{k}$, the other partition $A = \binom{X}{k+1}$, and edges ss' when $s \subseteq s'$. We will show that for all $S \subseteq A$

$$|N(S)| \geq |S|.$$

By Theorem 2.1.2, this implies the existence of a matching M of A. By adding to each chain that has $s \in B$ such that $ss' \in M$ the matched set $s' \in A$, we extend some of the chains of C_m^k. We extend any chains which have $s \in B$ that is not matched in M arbitrarily (say, by adding the least element of $X \setminus s$ to create C_m^{k+1}. By the definition of G and of matchings, all subsets of size $k+1$ are covered by C_m^{k+1}, and each chain contains a subset of cardinality $k + 1$ (because chains not extended according to the matching were extended arbitrarily). Adding the assumptions about C_m^k in the induction hypothesis we conclude that each chain has each cardinality it should, and all subsets of the asserted cardinalities are indeed covered. However, we still must verify Hall's condition.

Let $S \subset A$ be arbitrary. Each vertex in A has $k + 1$ incident edges by the earlier observation, so there are a total of $|S|(k + 1)$ edges from S to $N(S)$. Meanwhile, each vertex in $N(S)$ is a vertex of B, and has degree $n - k$ by the earlier observation. Therefore,

$$|N(S)|(n - k) \geq |S|(k + 1).$$

Finally, we recall that $k \geq m = \lfloor n/2 \rfloor$, so we conclude $k + 1 \geq (n - k)$, and since both are positive (as are S and $N(S)$), we may divide to find

$$|N(S)| \geq |N(S)|\frac{n-k}{k+1} \geq |S|.$$

We now prove that there exists a set C_k^n for all integers $k \in [0, m]$, where C_k^n is a set of $\binom{n}{m}$ chains that contain all subsets of X of cardinality j for $k \leq j \leq n$, where each chain contains a subset of each such cardinality. This proof is also by induction on k, now starting at $k = m$ and subtracting in the induction step. So, the base case in C_m^k which is given from the previous induction. The induction step is symmetrical: we have a bipartite graph with $A = \binom{X}{k-1}$ and $B = \binom{X}{k}$ (and the edge set defined as before), and we are extending chains from their smallest cardinality element by adding an element of smaller cardinality. Here, elements $s \in A$ have $n - (k-1)$ neighbors (corresponding to the elements of $X \setminus s$) and elements $s' \in B$ have k neighbors (corresponding to the k elements of s'. We check Hall's condition again:

$$|N(S)|k \geq |S|(n - k + 1).$$

Because $k \leq m$ we have $(n - k + 1) \geq k$ so we divide (everything's still positive) to find

$$|N(S)| \geq |N(S)|\frac{k}{(n-k+1)} \geq |S|.$$

So, Hall's condition still holds, and the rest of the argument is symmetrical (except we are removing elements from the subsets of smallest cardinality according to the matching and arbitrarily instead of adding elements).

We now have the existence of C_0^n, a set of $\binom{n}{m}$ chains that cover the power set lattice of X. The comment from the beginning shows that the proof is complete, and no antichain can have more than $\binom{n}{m}$ elements.

4, Diestel 2.20: Show that a graph G contains k independent edges if and only if $q(G-S) \leq |S| + |G| - 2k$ for all sets $S \subseteq V(G)$.

Let G be a graph. First assume G contains k independent edges. Let $H = G * K^{|G|-2k}$; that is, add $|G| - 2k$ vertices to G and make them adjacent to each other and every vertex in G. Then H has a perfect matching as every vertex not matched by the k independent edges can be matched to the new vertices. By Tutte's Theorem, $\forall S' \subseteq V(H)$,

$$q(H - S') \leq |S'|$$

For every $S \subseteq V(G)$, let S' be S together with the $|G| - 2k$ vertices we added. Then $G - S = H - S'$ and

$$q(G - S) = q(H - S') \leq |S'| = |S| + |G| - 2k.$$

To prove the other implication, assume $\forall S \subseteq V(G), q(G-S) \leq |S| + |G| - 2k$. Once again, define $H = G * K^{|G|-2k}$.

Note that H has $|G| + |G| - 2k = 2(|G| - k)$ vertices. Thus $q(H - \emptyset) = 0$. In addition, to disconnect H one must remove all of the vertices we added. Thus for any S' not containing every vertex we added, $q(H - S') \leq 1$.

Thus let S' be a set containing every vertex we added. Let $S = S' \cap G$. Then

$$H - S' = (G \cup K^{|G|-2k}) - S' = G - S$$

so

$$q(H - S') = q(G - S) \leq |S| + |G| - 2k = |S'|$$

Thus H satisfies Tutte's condition and contains a perfect matching. However, of the vertices in G, at most $|G| - 2k$ of them are matched using edges we added. We conclude at least $|G| - (|G| - 2k) = 2k$ vertices are matched using edges originally in G and thus conclude G has k independent edges.

5, Diestel 2.27: Show that if G has two edge-disjoint spanning trees, it has a connected spanning subgraph all whose degrees are even.

Let G be a graph with edge-disjoint spanning trees T_1 and T_2. We demonstrate how to construct H, a spanning subgraph of G with only vertices of even degree.

First, add all of the edges of T_1 to H.

For a first proof, note that T_1 has an even number of vertices of odd degree, so pair them and choose paths $P_1, P_2, \ldots, P_k$ in T_2 between the pairs. Add an edge from T_2 to H iff it occurs in an odd number of the paths. If v is a vertex of even degree in T_1, then each of the paths enters and exits v, so the sum over edges incident to v (number of times the edge is used in the paths) is an even number. This implies (much like in the proof that every graph has an even number of vertices of odd degree), that the number of edges incident with v that are used an odd number of times is even. Thus, adding the edges which occur in an odd number of the paths, leaves v with even degree. In a similar manner, if v is a vertex of odd degree in T_1, there is exactly one path that doesn't both enter and exit v, so there are an odd number of edges incident with v that occur in an odd number of the paths. Again, adding the edges which occur in an odd number of the paths, leaves v with even degree. Thus, H spans, is connected, and has all vertices of even degree.

For a second proof, choose $v_0 \in G$ and root T_2 at v_0. Let $L_i = \{v \in G : \mathrm{dist}_{T_2}(v, v_0) = i\}$; that is, L_i is the set of vertices in the i^{th} level of the level graph of T_2 rooted at v_0. Let k be maximal such that L_k is nonempty.

For each L_i, $i > 0$, starting with L_k, enumerate the vertices of L_i. As paths in trees are unique, for each $v \in L_i$, $i > 0$, there is a unique edge from v to $v' \in L_{i-1}$. For each vertex v, if $\deg(v)$ is odd in H, add the unique edge from v to $v' \in L_{k-1}$ and otherwise do not add that edge. Continue this process until reaching L_0, at which point we claim H will be spanning and only contain vertices of even degree.

First, the algorithm clearly terminates as the number of iterations is bounded by k. Let H be the output of the algorithm. Clearly H is spanning as it includes every edge from T_1. It remains to prove that every vertex of H has even degree.

First note each edge in T_2 is considered exactly once as edges are considered only when they are the last edge in the unique path from v_0 to some v and each v is considered only once.

Let $v \in G$. First assume that $v \neq v_0$. Then $v \in L_i$ for some $i > 0$. When the algorithm considered v, every edge from v to L_{i+1} had already been considered and would not be added. Thus, if v had odd degree, the single edge to L_{i-1} would have been added to H, giving it even degree. On the other hand, if it already had even degree, the edge would not be added to H and v would still have even degree. At this point, the degree of v in H is set as every edge in T_2 adjacent to v has been considered. Thus $v \neq v_0$ has even degree. As graphs must have an even number of vertices of odd degree, we conclude v_0 cannot have odd degree, and thus it has even degree.

Thus the algorithm must terminate and produce a spanning subtree such that every vertex has even degree.

6: Show that a tree T has a perfect matching if and only if $q(T-v)=1$ for every $v \in V(T)$.

First assume T has a perfect matching. By Tutte's Theorem, we can conclude T has an even number of vertices (by letting $S=\emptyset$) and that removing any vertex of T leaves at most one odd component of T. As T has an even number of vertices, $T-v$ has an odd number of vertices for any $v \in V(T)$, so $T-v$ must have at least one odd component. Thus we conclude for any $v \in V(T)$, $T-v$ has exactly one odd component.

Now assume for every $v \in V(T)$, $q(T-v)=1$. In particular, if we delete a leaf from T, then we are left with a single, odd component. We conclude $|T|$ is even, say of size $2k$. We proceed by induction on k.

When $k=1$, $|T|=2$ so $|T|$ clearly has a perfect matching.

Assume the claim holds for all trees of size less than $2k$ for some fixed, natural k.

Let T be a tree of size $2k$ such that $q(T-v)=1$ for every $v \in V(T)$. Let ℓ be a leaf in T, and let v be the vertex adjacent to ℓ. Consider $T-v$. It must have exactly one odd component, and that component must be ℓ. Let the other components be $C_1, \ldots, C_k$. Note that $|C_i|$ is even for all i.

We claim C_i also satisfies $q(C_i - v')=1$ for all $v' \in C_i$. Note $\sum_{j \neq i} |C_j| + \ell + v$ is even. Thus components of $C_i - v'$ are either identical to their counterparts in $T-v'$ or differ in an even number of vertices and thus in both cases have the same parity.

Thus, by our induction hypothesis, each C_i has a perfect matching. To form a matching in T, match ℓ to v and take the union of the matchings of the C_i's.

Thus, by induction, if $\forall v \in V(T)$, $q(T-V)=1$, then T has a perfect matching.