Diestel 1.2: Let $d \in \mathbb{N}$ and $V := \{0,1\}^d$; thus, V is the set of all 0–1 sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the *d-dimensional cube*. Determine the average degree, number of edges, diameter, girth and circumference of this graph.

(Hint for the circumference: induction on d.)

All vertices have degree d (one adjacent vertex for each position in the sequence), so the average degree is d.

There are 2^d vertices, each of degree d, so

$$|E| = \frac{1}{2}\sum_{v \in V} \deg v = \frac{1}{2}2^d d = 2^{d-1}d$$

The diameter is d: 0^d (d occurrences of the coordinate 0) and 1^d have distance at least d, because each edge changes one position, and d positions must be changed. For arbitrary $x, y \in V$ that differ in $c \leq d$ coordinates, let $x_0 = x$, and x_{i+1} be x_i with the next index that differs from y corrected. $x_c = y$, and $x_0 x_1 \cdots x_c$ is a path of length at most d.

For $d < 2$, there are no cycles (since there are fewer than 3 vertices), so the girth is ∞. For $d \geq 2$, there is a 4-cycle holding all indices beyond the first two constant at 0: $00\cdots$, $01\cdots$, $11\cdots$, and $10\cdots$. Note that there cannot be a 3-cycle as if v is adjacent to x_1 and x_2 then x_1 and x_2 disagree on two indicies.

For $d < 2$, the circumference is 0, as above. Otherwise, the circumference is 2^d, by induction. The base case is by inspection. For $d \geq 3$, a d-cube is the Cartesian product of a $(d-1)$-cube with K_2 (that is, two $(d-1)$-cubes with corresponding vertices adjacent). By the induction hypothesis, a $d-1$ cube has a 2^{d-1}-cycle, say $x_1 x_2 \cdots x_{2^{d-1}}$. The path $x_1 x_2 \cdots x_{2^{d-1}}$ (but not the edge $x_{2^{d-1}} x_1$ in one $d-1$ cube, followed by the edge to the other cube, followed by the reversed path $x_{2^{d-1}} \cdots x_2 x_1$, and then an edge back to the first cube is a 2^d cycle. The circumference cannot be greater because there are 2^d vertices.

Diestel 1.8: Show that graphs of girth at least 5 and order n have a minimum degree of $o(n)$. In other words, show that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)/n \to 0$ as $n \to \infty$ and $\delta(G) \leq f(n)$ for all such graphs G.

Let v be an arbitrary vertex of G. Since there are no 3-cycles in G, the neighborhoods of the neighbors of v do not intersect with the neighbors of v. Since there are no 4-cycles in G, the neighborhoods of the neighbors of v are pairwise disjoint, except for containing v. This yields $n = |G| \geq 1+\delta+\delta(\delta-1) > \delta^2$. So $\delta \leq \sqrt{n}$, and since $\sqrt{n}$ is $o(n)$, we are done.

Diestel 1.9: Show that every connected graph G contains a path of length at least $\min\{2\delta(G), |G|-1\}$.

Let G be a connected graph, let $\delta = \delta(G)$, and let $P = x_0x_1\cdots x_k$ be a path of maximal length. If P is of length at least $|G|-1$, we are done. Otherwise, the set $O = V(G) \setminus V(P)$ is nonempty, and since the graph is connected, there must be a $V(P)$–O path, $P' = y_0y_1\cdots y_m$ (and this path is clearly non-trivial). If P is of length less than 2δ, we will prove that there is a cycle spanning $V(P)$. Deleting an edge of this cycle incident with y_0 allows us to extend the remaining path with P', forming a path longer than P, which is a contradiction.

It remains to prove that there is a cycle spanning $V(P)$. Recall we denote the set of neighbors of v as $N(v)$. First, observe that $N(x_0) \subset P$ and $N(x_m) \subset P$, because if either end of P is adjacent to a vertex outside of P, then the path can be extended and is not maximal. If x_0x_{i+1} and x_mx_i are both edges in P, then there is a cycle: $x_0\ldots x_ix_m\ldots x_{i+1}x_0$. So, it suffices to show by the Pigeonhole Principal that such a pair of edges must occur. Note first that a special case of this is where there is an edge x_0x_m, since the other edge is given in the path. The vertex x_0 has at least $\delta-1$ neighbors out of $\{x_2\cdots x_{m-1}\}$ (because it is adjacent to x_1 and not adjacent to x_m), and for each neighbor x_i there is a corresponding vertex x_{i-1} to which x_m is not adjacent. So, x_m must have at least $\delta-1$ neighbors out of $\{x_1\cdots x_{m-2}\}$, and of those $\delta-1$ are forbidden. Since $m < 2\delta$, there are fewer than $2\delta-2$ possible neighbors, so by the Pigeonhole Principal one of the neighbors is forbidden, so there is a cycle, and so the path is not maximal.

Diestel 1.13: Determine $\kappa(G)$ and $\lambda(G)$ for $G = P^m, C^n, K^n, K_{m,n}$ and the d-dimensional cube (Exercise 2); $d, m, n \geq 3$.

Label $P^m = x_0 x_1 \cdots x_m$. Arbitrary vertices x_i and x_j are connected in P^m because, without loss of generality, $i < j$ and $x_i x_{i+1} \cdots x_j$ is a path linking them. There are at least 2 vertices, so, $\kappa(P^m) \geq 1$ and $\lambda(P^m) \geq 1$. x_0 is adjacent only to x_1, so $\delta(P^m) \leq 1$, and by Proposition 1.4.2 we conclude

$$\kappa(P^m) = \lambda(P^m) = 1.$$

In C^n, given two non-equal and arbitrary points, there exist two internally-disjoint paths linking those points. Informally, these correspond to going around the cycle clockwise or counter-clockwise. Formally, if we label the cycle $C^n = x_0 x_1 x_2 \cdots x_n x_0$ and have points x_i and x_j with $i < j$, one path is $x_i x_{i+1} \cdots x_j$ (simply $x_i x_j$ in the case that they are adjacent) and the other is $x_j x_{j+1} \cdots x_n x_0 x_1 \cdots x_i$ (omitting x_{j+1} when $j = n$ and omitting x_1 when $i = 0$). Therefore, if one edge or vertex is arbitrarily deleted, it affects at most one of the two paths, so all points remain connected. There are at least 3 vertices, so, $\kappa(C^n) \geq 2$ and $\lambda(C^n) \geq 2$. Each vertex is adjacent to two other vertices, so $\delta(C^n) = 2$, and by Proposition 1.4.2 we conclude

$$\kappa(C^n) = \lambda(C^n) = 2.$$

(Note that a shorter proof is to say that deleting any vertex or edge gives a path!)

In K^n, suppose $j \leq n - 1$ vertices are deleted. Any two remaining vertices are adjacent, and therefore connected. Because there are n vertices, we conclude $\kappa(K^n) = n - 1$. This also bounds $\lambda(K^n) \geq n - 1$ by Proposition 1.4.2. Finally, each vertex has at most $n - 1$ neighbors, because there are only $n - 1$ other vertices, so $\delta(K^n) \leq n - 1$, and also by Proposition 1.4.2 this implies $\lambda(K^n) \leq n - 1$. Therefore

$$\kappa(K^n) = \lambda(K^n) = n - 1$$

In $K_{m,n}$ assume $m \leq n$ (because otherwise we look at $K_{n,m}$ which is the same graph). Since vertices in one partition have exactly n neighbors in vertices in the other have exactly m, and both partitions are non-empty (since m and n are positive), we know $\delta(K_{m,n}) = m$. By Proposition 1.4.2 this means $\kappa(K_{m,n}) \leq \lambda(K_{m,n}) \leq m$. Next, given two vertices in $K_{m,n}$, u and v, we will show that there is a path linking u and v even after deleting up to $m - 1$ vertices. If u and v are adjacent, this is clearly true, since the edge uv will remain. Otherwise, u and v are in the same partition, and have the other partition mutually-adjacent. The other partition contains at least m vertices, so if $m - 1$ vertices are deleted, at least one vertex, w, remains in the other partition. uwv is a path linking u and v after deleting the vertices, and since there are at least m vertices in one partition, there are at least m in the graph, so we conclude $\kappa(K_{m,n}) \geq m$. Therefore

$$\kappa(K_{m,n}) = \lambda(K_{m,n}) = m.$$

It is easy to bound the connectivities of the d-dimensional cube, Q^d above by its minimum degree, d (all vertices have degree d as shown in the first question). It remains to show $\kappa(Q^d) \geq d$ so that we can conclude (using Proposition 1.4.2) that $\kappa(Q^d) = \lambda(Q^d) = d$. We go by induction on d proving $\kappa(Q^d) \geq d$ with base case $d = 2$ solved earlier since $Q^2 = C^4$. For $d > 2$, Q^d is composed of two Q^{d-1} cubes (one with first coordinate 0 and the other with first coordinate 1) which we call A and one B. Given a set X of vertices with $|X| < d$, we consider two cases. In the first, X is a cut-set of, without loss of generality, A. So by the induction hypothesis X is a subset of the vertices of A (since we need all $d-1$ vertices to disconnect a Q^{d-1}), so B remains connected, and each remaining vertex of A is adjacent to a vertex in B, so the whole graph is connected. Otherwise, A and B both remain connected after removing X. In this case the graph is only disconnected if there is no path from A to B. But there were 2^{d-1} edges that cross from A to B, so one vertex from each pair is deleted. Because $2^{d-1} \geq d$ for $d \geq 3$, X cannot disconnect A from B, completing the proof that

$$\kappa(Q^d) = \lambda(Q^d) = d.$$

Diestel 1.21: Show that a tree without a vertex of degree 2 has more leaves than other vertices. Can you find a very short proof that does not use induction?

Let T be a tree and consider the average degree of T. By the Handshaking Lemma, we know $\sum_{v \in V} d(v) = 2|E|$. Thus, by Proposition 1.3.2, the average degree of T is

$$\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|} = \frac{2(|V|-1)}{|V|} < 2.$$

Let L be the set of leaves in V, and O be the set of other vertices. L and O partition V, so

$$\begin{aligned}
2 &> \frac{\sum_{v \in V} d(v)}{|V|} \\
&= \frac{\sum_{v \in L} d(v) + \sum_{v \in O} d(v)}{|V|} \\
&\geq \frac{\sum_{v \in L} 1 + \sum_{v \in O} 3}{|V|} && \text{Leaves have degree 1; } d(v) \geq 3 \text{ for each } v \in O \\
&= \frac{|L| + 3|O|}{|V|} \\
&= 1 + \frac{2|O|}{|V|} && \text{Noting } |L| + |O| = |V|
\end{aligned}$$

After some algebraic manipulation we find

$$|O| < \frac{|V|}{2},$$

and since

$$|L| = |V| - |O|,$$

we conclude

$$|L| > \frac{|V|}{2} > |O|.$$

Diestel 1.28: Show that every automorphism of a tree fixes a vertex or an edge.

We start by proving two lemmas.

Lemma 1: If x and y are central vertices of a tree T, then x and y are adjacent.

Proof: Assume for the sake of contradiction that x and y are both central vertices of G but are not adjacent. Let $x', y' \in V$ such that $\mathrm{dist}(x, x') = \mathrm{dist}(y, y') = \mathrm{rad}\ G$, as assured by the definition of central. Let $P = x, v_1, \dots, y$ be an x–y path. Note we are assured $x \neq v_1 \neq y$ as x and y are not adjacent.

Recall that any pair of vertices in a tree is connected by a unique path. We consider two cases:

Case 1: Suppose v_1 is not on both the x–x' path and the y–y' path. Without loss of generality, v_1 is not on the y–y' path. We expect $\mathrm{dist}(x, y') \leq \mathrm{rad}\ G$ by the definition of central, however the x–y' path goes through v_1 and y and thus has length at least $\mathrm{rad}\ G + 2$: a contradiction.

Case 2: Therefore, suppose v_1 is on both the x–x' path and the y–y' path. We have three internally disjoint paths: the first from x' to y, the second from y to x through v_1, and the last from x to y'. Then $\mathrm{dist}(v_1, y') < \mathrm{dist}(y, y')$ and $\mathrm{dist}(v_1, x') < \mathrm{dist}(x, x')$. Furthermore, for any vertex w, at least one of the x–w and y–w paths must pass through v_1 as it lies on the x–y path, so for every vertex v_1 has smaller distance than at least one of the proposed centers. But then v_1 is at distance less than $\mathrm{rad}\ G$ from every vertex: a contradiction.

As both cases lead to a contradiction, we conclude x and y must be adjacent.

Lemma 2: A tree has exactly one or two central vertices.

Proof: Let T be a tree. As T has finitely many vertices, it must have at least one central vertex as the minimum of a finite set exists. Suppose for the sake of contradiction that T has at least three central vertices, c_1, c_2 and c_3. Then by the first lemma, c_1 is adjacent to c_2, c_2 is adjacent to c_3, and c_3 is adjacent to c_1, forming a cycle, contradicting that T is a tree. We conclude T has at least one but fewer than three central vertices.

Finally, we prove the main theorem. Note that distance is preserved under automorphism. If ϕ is an automorphism of graph G, $x, y \in V(G)$, then the set of x–y paths is preserved under ϕ. If P is an x–y path, then the length of P is equal to the length of $\phi(P)$. Therefore, the minimum length x–y path has the same length as the minimum length $\phi(x)$–$\phi(y)$ path.

Thus central vertices must still be central under automorphism. If T has one central vertex, then it is fixed under any automorphism. If T has two central vertices, then they must be adjacent by the first lemma, so any automorphism will either fix them or swap them, in which case the edge connecting them is fixed.

7: A graph is *self-complementary* if it is isomorphic to its complement. Show that:

(a) The number of vertices in any self-complementary graph is congruent to 0 or 1 mod 4.

Let G be a self-complementary graph. As G is isomorphic to $\overline{G}$,

$$|E(G)| = |E(\overline{G})| = \left|\binom{V(G)}{2} \setminus E(G)\right| = \binom{|V(G)|}{2} - |E(G)|$$

from which we conclude $|E(G)| = \frac{1}{2}\binom{|V(G)|}{2} = \frac{(|V(G)|)(|V(G)|-1)}{4}$.

Note that at least one of $|V(G)|, |V(G)|-1$ must be odd, and thus the other must be a multiple of 4 in order for $|E(G)|$ to be integral. Thus either

$$4 \,\Big|\, |V(G)| \Rightarrow |V(G)| \equiv 0 \text{ mod } 4$$

or

$$4 \,\Big|\, |V(G)| - 1 \Rightarrow |V(G)| \equiv 1 \text{ mod } 4.$$

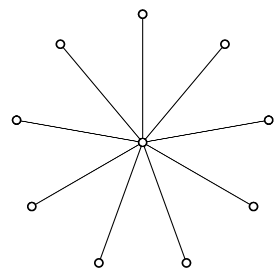
(b) Every self-complementary graph on $4k+1$ vertices has a vertex of degree $2k$.

Let f be an isomorphism from G to $\overline{G}$. Suppose $v \in V(G)$ has degree ℓ. Then $f(v)$ has degree $4k-\ell$. By the definition of graph complement, there must be $u \in V(G)$ with degree $4k-\ell$. When $\ell \neq 2k$, the pair of vertices has distinct degree and must represent two different vertices. As the graph has an odd number of vertices, not every pair can be distinct vertices. Thus there must be $v \in V(G)$ such that $f(v) = v$, in which case we necessarily have $d(v) = 2k$.

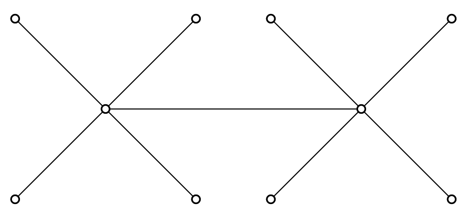
8: A tree is *homeomorphically irreducible* if it has no vertices of degree 2. Draw all non-isomorphic, homeomorphically irreducible trees on 10 vertices. Explain why all such trees are represented among your drawings.

From question 1.18, there must be more leaves than other vertices, so there are at most 4 non-leaves. Further, if we delete all of the leaves, the non-leaves remain and form a tree, so we can consider the number and structure of the non-leaves.

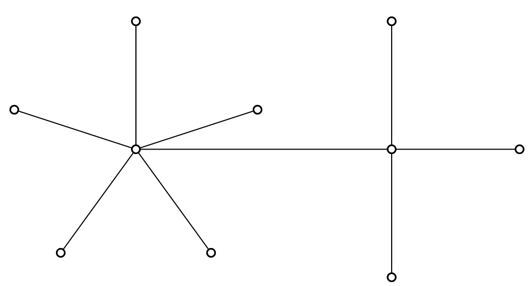
When there is 1 non-leaf, it must be that all other vertices are leaves adjacent to this vertex:



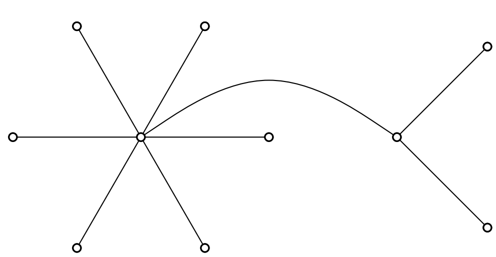
If there are two non-leaves, they must be adjacent to each other, and each must have at least two leaves since it cannot be degree 2. The remaining 4 leaves can be distributed in any way: half and half:



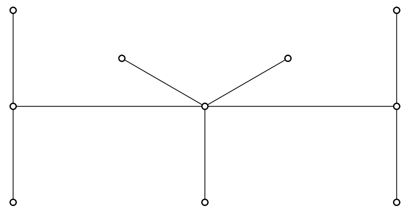
three and one:



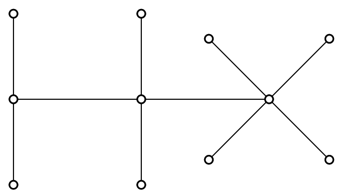
or four and zero:



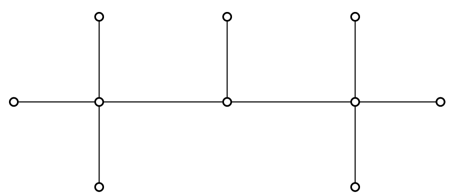
When there are 3 non-leaves, they must induce a P^2, and to assure that each has degree at least 3 the ends must have two leaves and the middle must have 1. Two leaves remain to be distributed in any way: all in the middle:



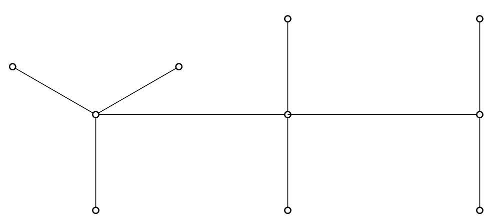
all on one end:



one at each end:



or one in the middle and one at an end:



Finally, if there are 4 non-leaves, those form either a P^3 or a star with 3 leaves. In both cases, the remaining vertices are forced: