1: Find a nonhamiltonian graph G with 10 vertices such that $G - v$ is hamiltonian for every vertex v of G.

The answer is the Petersen Graph (it is the only answer, but this is slightly more difficult to show). You've already shown on a previous homework, that there is no 3-edge-coloring of the Petersen Graph. A Hamilton cycle in the Petersen Graph would yield a contradicting 3-edge-coloring of it by alternating 2 colors on the edges of that cycle and assigning the third color to the remaining pairwise non-incident edges. Thus, the Petersen Graph is not hamiltonian.

The Petersen Graph is vertex transitive, so it suffices to show that the deletion of vertex from it leaves a graph with a 9-cycle. To that end, suppose that the Petersen Graph is given by the two disjoint 5-cycles, 123451 and $abcdea$, with additional edges $1a$, $2c$, $3e$, $4b$ and $5d$. The deletion of vertex 1 leaves the 9-cycle $ae32cd54ba$.

2, Diestel 10.1: Show that every tournament contains a (directed) Hamilton path.

We prove this by induction on the number of vertices in the tournament. The result is true for tournaments having 1 or 2 vertices, so now assume that T is a tournament with $n > 2$ vertices and that the result is true for all tournaments having fewer than n vertices. Select a vertex v and obtain, by the induction hypothesis, a Hamilton path from v_1 to v_{n-1} in $T - v$.

If there is an arc from v to v_1 or an arc from v_{n-1} to v, then we have the desired path, so we assume that there is an arc from v_1 to v and an arc from v to v_{n-1}. In this case, there is a largest index $j < n-1$ such that there is an arc from v_j to v. The path from v_1 to v_j to v to v_{j+1} to v_{n-1} is then the desired path.

3: Let G have $n > 1$ vertices and m edges. Prove that G has a bipartite subgraph with at least

$$\frac{2\lfloor n^2/4\rfloor m}{n(n-1)}$$

edges. (You should consider a random bipartition... but don't allow just any bipartition.)

Consider a random bipartition A and B of G where the size of A and B differ by at most 1. There are $\lfloor n^2/4\rfloor$ (a,b) pairs of vertices and each of these has probability $\frac{m}{\binom{n}{2}}$ to be an edge in G. Therefore:

$$\begin{aligned}
E[\text{edges in the bipartite subgraph}] &= \sum_{\text{(a,b) pairs}} P(\text{(a,b) is an edge of } G) \\
&= \sum_{\text{(a,b) pairs}} \frac{m}{\binom{n}{2}} \\
&= \frac{\lfloor n^2/4\rfloor m}{\binom{n}{2}} \\
&= \frac{2\lfloor n^2/4\rfloor m}{n(n-1)}
\end{aligned}$$

Therefore at least one of the bipartite subgraphs has at least $\frac{2\lfloor n^2/4\rfloor m}{n(n-1)}$ edges.

4, Diestel 11.6:

If $G \in G_{n,p}$ has properties $\mathcal{P}_1$ and $\mathcal{P}_2$ with high probability then $\lim_{n\to\infty} \mathbb{P}(G \notin \mathcal{P}_1) = 0$ and $\lim_{n\to\infty} \mathbb{P}(G \notin \mathcal{P}_2) = 0$. By the union bound, we know that

$$\begin{aligned}
\mathbb{P}(G \notin \mathcal{P}_1 \cap \mathcal{P}_2) &\leq \mathbb{P}(G \notin \mathcal{P}_1) + \mathbb{P}(G \notin \mathcal{P}_2) \\
\lim_{n\to\infty} \mathbb{P}(G \notin \mathcal{P}_1 \cap \mathcal{P}_2) &\leq \lim_{n\to\infty} \mathbb{P}(G \notin \mathcal{P}_1) + \mathbb{P}(G \notin \mathcal{P}_2) \\
\lim_{n\to\infty} \mathbb{P}(G \notin \mathcal{P}_1 \cap \mathcal{P}_2) &\leq 0
\end{aligned}$$

So the complimentary probability that $G \in \mathcal{P}_1 \cap \mathcal{P}_2$ goes to 1 as $n \to \infty$ so $G \in G_{n,p}$ is in $\mathcal{P}_1 \cap \mathcal{P}_2$ with high probability.

5, Diestel 11.8:

Consider a clique K. Let u, v be two vertices that are not in K. By $\mathcal{P}_{2,0}$, there exists y adjacent to u and v. By $\mathcal{P}_{2,1}$, there exists z adjacent to u and v but not to y. y and z cannot both be in K since they are not adjacent. Therefore any pair of vertices not in K are not separated and $G_{n,p}$ has no separating set which is a clique with high probability.

6, Diestel 11.10:

Let H be a graph with k vertices and m edges and $p(n)$ be a function such that $p(n) \to 0$ as $n \to \infty$. Let $U \subseteq G$ be a subgraph of G with exactly k vertices. Denote ϕ as the probability that H is isomorphic to U. For H to be isomorphic to U, H must have exactly the same edges as U. As there are k vertices,

$$\phi \;\geq\; p(n)^m(1-p(n))^{\binom{k}{2}-m}$$

Partition G into $\lfloor \frac{n}{k} \rfloor$ sets $U_1, \ldots, U_{\lfloor \frac{n}{k} \rfloor}$ of size k with the last set having "leftover" vertices. As edges in these sets occur independently, the probability that G does not have H as an induced subgraph is bounded by

$$\begin{aligned} \mathbb{P}(\forall U.G[U] \not\equiv H) &\leq \mathbb{P}(i \leq \lfloor \frac{n}{k} \rfloor .G[U_i] \not\equiv H) \\ &\leq (1-\phi)^{\lfloor \frac{n}{k} \rfloor} \\ &\leq e^{-\phi \lfloor \frac{n}{k} \rfloor} \end{aligned}$$

Thus it suffices to show that $e^{-\phi\lfloor \frac{n}{k} \rfloor} \to 0$ as $n \to \infty$. To do this we choose $p(n) = \frac{1}{\log(n+1)}$ and show that $\phi\lfloor \frac{n}{k} \rfloor \to \infty$.

$$\phi\lfloor \frac{n}{k} \rfloor \;=\; p(n)^m(1-p(n))^{\binom{k}{2}-m}\lfloor \frac{n}{k} \rfloor$$

For sufficiently large n, we have that

$$\begin{aligned} \phi\lfloor \frac{n}{k} \rfloor &\geq (p(n)^2)^{\binom{k}{2}}\lfloor \frac{n}{k} \rfloor \\ &\geq \frac{n}{k(\log(n))^{k^2}} \to \infty \end{aligned}$$

So $e^{-\phi\lfloor \frac{n}{k} \rfloor} \to 0$ and we have that $\mathbb{P}(\exists U.G[U] \equiv H) \to 1$ as $n \to \infty$ and so G has H as an induced subgraph with high probability.