Derangements:
So combinatorially, we want to partition $[n]$ into its cycle decomposition. This can be treated as just partitioning $[n]$ into bins of size at least 2 and order the elements of the bin.
We will first want to count the number of ways to partition $[n]$ into k cycles of size at least 2. Denote this as $f(n, k)$. Let's say we partition $[n]$ into k bins of size $a_1, a_2, \ldots, a_k$ (because we are treating the bins as distinguishable now, we will need to divide by $k!$ later to account for our overcounting) so that $a_i \geq 2$ and $\sum_{i=1}^{k} a_i = n$. Then there are $\binom{n}{a_1, a_2, \ldots, a_k}$ to choose the elements of each of the bins, and there are $(a_i - 1)!$ to arrange the elements of the bin into a cycle. So we have that $f(n, k)$ is equal to

$$\frac{1}{k!} \sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i \geq 2}} \binom{n}{a_1, a_2, \ldots, a_k} \prod_{i=1}^{k} (a_i - 1)!$$
$$= \frac{n!}{k!} \sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i \geq 2}} \prod_{i=1}^{k} \frac{(a_i - 1)!}{a_i!}$$

Now remember we wanted to determine $\frac{d_n}{n!}$. Note that this is

$$\frac{1}{n!} \sum_{k=0}^{\infty} f(n, k)$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i \geq 2}} \prod_{i=1}^{k} \frac{(a_i - 1)!}{a_i!}$$

Now we notice that the term inside the sum is a convolution. In particular, we have that $\sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i \geq 2}} \prod_{i=1}^{k} \frac{(a_i-1)!}{a_i!}$ is the coefficient of x^n in

$$\left(0 + 0x + \frac{1!}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{3!}{4!}x^4 + \cdots\right)^k$$
$$= \left(0 + 0x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\right)^l$$
$$= (-x - \ln(1 - x))^k$$

(as a footnote, from here it is quickly modifiable to obtain the EGF of stirling numbers of the first kind) Hence, the coefficient of x^n in the EGF of the derangement numbers is the coefficient of x^n in the sum

$$\sum_{k=0}^{\infty} \frac{1}{k!} (-x - \ln(1 - x))^k$$
$$= e^{-x-\ln(1-x)}$$
$$= \frac{e^{-x}}{e^{\ln(1-x)}}$$
$$= \frac{e^{-x}}{1 - x}$$

So of course the EGF of the derangement numbers is just $\frac{e^{-x}}{1-x}$.

Stirling numbers of the second kind:

Their EGF follows from a very similar process. We want to partition $[n]$ into k nonempty bins. Let's say the bins have size $a_1, \ldots, a_k$ all at least 1 (we again need to divide by $k!$ later). There are $\binom{n}{a_1, a_2, \ldots, a_k}$ ways to choose the elements of each bin. And since the elements of these bins are unordered, there is 1 way to fill the bin with the elements we chose. So this means that

$$S_2(n,k) = \frac{1}{k!} \sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i \geq 1}} \binom{n}{a_1, a_2, \ldots, a_k}$$

$$S_2(n,k) = \frac{n!}{k!} \sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i \geq 1}} \prod_{i=1}^{k} \frac{1}{a_i!}$$

$$\frac{S_2(n,k)}{n!} = \frac{1}{k!} \sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i \geq 1}} \prod_{i=1}^{k} \frac{1}{a_i!}$$

The RHS is again a convolution. In particular, it is the coefficient of x^n in the product of the generating functions

$$\begin{aligned} &\frac{1}{k!}\left(0 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots +\right)^k \\ =&\frac{1}{k!}\left(-1 + 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots +\right)^k \\ =&\frac{1}{k!}\left(-1 + e^x\right)^k \end{aligned}$$

So we have that the coefficient of x^n in the EGF of the stirling numbers of the second kind, $\frac{S_2(n,k)}{n!}$, is equal to the coefficient of x^n in the generating function $\frac{1}{k!}\left(e^x - 1\right)^k$. So of course the EGF of the stirling numbers of the second kind is $\frac{1}{k!}\left(e^x - 1\right)^k$.