Induction

1. Assume $\sin(x) \neq 0$. Prove the following for all natural number n.

$$\prod_{i=0}^{n-1} \cos(2^i x) = \frac{\sin(2^n x)}{2^n(\sin(x))}$$

Hint: $\sin(2x) = 2\sin(x)\cos(x)$.

Let $P(n) = " \prod_{i=0}^{n-1} \cos(2^i x) = \frac{\sin(2^n x)}{2^n(\sin x)} "$

Base $P(1)$ true b/c $\cos(2^0 x) = \frac{\sin(2^1 x)}{2^1(\sin x)}$. $\Leftrightarrow 2\sin x \cos x = \sin 2x$

I.H. Assume $P(k)$ true for some $k \in \mathbb{N}$

I.S

WTS $P(k+1) = " \prod_{i=0}^{k} \cos(2^i x) = \frac{\sin(2^{k+1} x)}{2^{k+1}(\sin x)} "$ true

By IH

$$\prod_{i=0}^{k-1} \cos(2^i x) = \frac{\sin(2^k x)}{2^k(\sin x)}$$

$$\Rightarrow \cos(2^k x) \prod_{i=0}^{k-1} \cos(2^i x) = \frac{\sin(2^k x)\cos(2^k x)}{2^k(\sin x)}$$

$$\Rightarrow \prod_{i=0}^{k} \cos(2^i x) = \left(\frac{2}{2}\right) \frac{\sin(2^k x)\cos(2^k x)}{2^k(\sin x)}$$

this becomes $\sin(2^{k+1} x)$

$$= \frac{\sin(2^{k+1} x)}{2^{k+1}(\sin x)}$$

✓