4 | More Counting 2 Ways

Prove this by counting two ways:

$$\sum_{i=1}^{n} i = \binom{n+1}{2}$$

Note that the right hand side simplifies to something we've seen before:

$$\binom{n+1}{2} = \frac{(n+1)!}{2!(n+1-2)!} = \frac{(n+1)(n)}{2}$$

Again, on exam, follow these steps to count a set in two ways:

1.) Choose easy side to count.
2.) Define set S you're counting looking at that side.
    a. It can be "the set of k element subset of $[n]$" for $\binom{n}{k}$, or
    b. It can be "the set of ways to choose k XXX's from the set of n XXX's."
3.) Count the easy side.
4.) Count the hard side. Partition S into S_i if necessary. In that case, invoke rule of sum at the end.
5.) Don't need to prove that the partition is indeed a partition, but explain briefly in English.

Proof.

We will count the right hand side.

Let S be the set of 2 element subset of $[n+1]$.

By definition, $|S| = \binom{n+1}{2}$.

Since there's a summation, we'll do partitioning. Since there are 2 elements in each set in S, we have a choice to partition based on the smaller element or the larger element.

Let S_i be the set of 2 element subset of $[n+1]$ where the *largest* element in each set is $i+1$.

Clearly, since we're choosing the largest elements and the largest element is in between 2 and $n+1$,

$$S_1, S_2, \ldots, S_n$$

partition S. Furthermore, S_i can be formed as follows:

Step 1.) Pick the largest element.
Step 2.) Pick the smaller element (i choices).

Thus, $|S_i| = (1)\binom{i}{1} = i$

And by rule of sum, $|S| = \sum_{i=1}^{n} |S_i| = \sum_{i=1}^{n} i$.

Since LHS and RHS both count S, they are equal.

3. Prove the following by counting 2 ways when q is an integer greater than 1. This is a geometric sum formula.

$$\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q-1}$$

$$\Leftrightarrow \quad (q-1)\sum_{i=0}^{n-1} q^i = q^n - 1$$

$$\Leftrightarrow \quad 1 + (q-1)\sum_{i=0}^{n-1} q^i = q^n.$$

- Let S = the set of ways to assign q colors to $[n]$, where one of the colors is red.
- By n step process w/ each q choices, $|S| = q^n$ = RHS.

- Now, partition S into the following sets.

S_B = the set of ways to assign q colors to $[n]$ such that all are colored red. $\rightarrow$ 1 way to make S_B

$S_{i,c}$ = the set of ways to assign q colors to $[n]$ such that the maximum non-red element is $i+1$ with color c.

To form $S_{i,c}$: 1) Color $i+1$th element as c, everything above is red $\rightarrow$ 1 way

2) For all other i elements, choose color $\rightarrow$ q^i ways

Then $S_{i,c}$ where $0 \le i \le n-1$ and $c \neq$ red and S_B partition S.

Thus $|S| = 1 + (q-1)\sum_{i=0}^{n-1} q^i$. So LHS counts S ✓

*we will go over the solution on Monday 8-11pm at West Wing Cluster.

4. Prove the following by counting 2 ways.

$$\sum_{k=0}^{n}\binom{x+k}{k}=\binom{x+n+1}{n}$$

Let S = the set of n element subset of $[x+n+1]$

clearly, RHS counts this.

Let S_k = the set of n element subset of $[x+n+1]$ such that the smallest number that's not in the subset is $n-k+1$.

For example, $\{1, 2, 3, 5, 6, 8, \ldots, \}$

If this is an n element subset of $[x+n+1]$, since 4 is the smallest number that's not in the subset, this will go to

S_{n-3} because $4 = n-k+1$
$\Leftrightarrow k = n-3$

The smallest missing number possible is 1, so $1 \le n-k+1$
$\Leftrightarrow k \le n$

and the largest missing number possible is $n+1$ (because then $\{1,\ldots,n\}$ is in it, which is n element subset), so $n+1 \ge n-k+1$
$\Leftrightarrow k \ge 0$

Thus $S_0, \ldots, S_k$ partition S, and

S_k can be formed by
1) Put $n-k$ elements in set
2) From $\{n-k+2, \ldots, x+n+1\}$ ← this has $x+k$ elements.
Choose k more elements.

Step (1) has 1 way and (2) has $\binom{x+k}{k}$ ways. So $|S| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n}\binom{x+k}{k}$ ✓