LOCAL WELL-POSEDNESS OF THE VISCOUS SURFACE WAVE PROBLEM WITHOUT SURFACE TENSION

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ABSTRACT. We consider a viscous fluid of finite depth below the air, occupying a three-dimensional domain bounded below by a fixed solid boundary and above by a free moving boundary. The domain is allowed to have a horizontal cross-section that is either periodic or infinite in extent. The fluid dynamics are governed by the gravity-driven incompressible Navier-Stokes equations, and the effect of surface tension is neglected on the free surface. This paper is the first in a series of three [19, 20] on the global well-posedness and decay of the viscous surface wave problem without surface tension. Here we develop a local well-posedness theory for the equations in the framework of the nonlinear energy method, which is based on the natural energy structure of the problem. Our proof involves several novel techniques, including: (1) optimal energy estimates in a "geometric" reformulation of the equations; (2) a well-posedness theory of the linearized Navier-Stokes equations in moving domains; (3) a time-dependent functional framework, which couples to a Galerkin method with a time-dependent basis.

1. Introduction

1.1. **Formulation of the equations in Eulerian coordinates.**

We consider a viscous, incompressible fluid evolving in a moving domain

$$\Omega(t) = \{y \in \Sigma \times \mathbb{R} \mid -b(y_1, y_2) < y_3 < \eta(y_1, y_2, t)\}. \tag{1.1}$$

Here we assume that either $\Sigma = \mathbb{R}^2$, or else $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the usual $1-$torus and $L_1, L_2 > 0$ the periodicity lengths. The lower boundary b is assumed to be fixed and given, but the upper boundary is a free surface that is the graph of the unknown function $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$. We assume that

$$\begin{cases} 0 < b \in C^\infty(\Sigma) & \text{if } \Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T}) \\ b \in (0,\infty) \text{ is constant} & \text{if } \Sigma = \mathbb{R}^2. \end{cases} \tag{1.2}$$

For each t, the fluid is described by its velocity and pressure functions $(u,p) : \Omega(t) \to \mathbb{R}^3 \times \mathbb{R}$. We require that (u,p,η) satisfy the gravity-driven incompressible Navier-Stokes equations in $\Omega(t)$ for $t > 0$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \\ \partial_t \eta = u_3 - u_1 \partial_{y_1}\eta - u_2 \partial_{y_2}\eta & \text{on } \{y_3 = \eta(y_1,y_2,t)\} \\ (pI - \mu\mathbb{D}(u))\nu = g\eta\nu & \text{on } \{y_3 = \eta(y_1,y_2,t)\} \\ u = 0 & \text{on } \{y_3 = -b(y_1,y_2)\} \end{cases} \tag{1.3}$$

for ν the outward-pointing unit normal on $\{y_3 = \eta\}$, I the 3×3 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, $g > 0$ the strength of gravity, and $\mu > 0$ the viscosity. The tensor $(pI - \mu\mathbb{D}(u))$ is known as the viscous stress tensor. The third equation in (1.3) implies that the free surface is advected with the fluid. Note that in (1.3) we have shifted the gravitational forcing to the boundary and eliminated the constant atmospheric pressure, p_{atm}, in the usual way by adjusting the actual pressure $\bar{p}$ according to $p = \bar{p} + gy_3 - p_{atm}$.

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The problem is augmented with initial data (u_0, η_0) satisfying certain compatibility conditions, which for brevity we will not write now. We will assume that $\eta_0 > -b$ on Σ. When $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$ we shall refer to the problem as either the "periodic problem" or the "periodic case," and when $\Sigma = \mathbb{R}^2$ we shall refer to it as either the "non-periodic problem" or the "infinite case."

Without loss of generality, we may assume that $\mu = g = 1$. Indeed, a standard scaling argument allows us to scale so that $\mu = g = 1$, at the price of multiplying b and the periodicity lengths L_1, L_2 by positive constants and rescaling b. This means that, up to renaming b, L_1, and L_2, we arrive at the above problem with $\mu = g = 1$.

The problem (1.3) possesses a natural physical energy. For sufficiently regular solutions to both the periodic and non-periodic problems, we have an energy evolution equation that expresses how the change in physical energy is related to the dissipation:

$$\frac{1}{2}\int_{\Omega(t)} |u(t)|^2 + \frac{1}{2}\int_{\Sigma} |\eta(t)|^2 + \frac{1}{2}\int_0^t \int_{\Omega(s)} |\mathbb{D}u(s)|^2 \, ds = \frac{1}{2}\int_{\Omega(0)} |u_0|^2 + \frac{1}{2}\int_{\Sigma} |\eta_0|^2 . \tag{1.4}$$

The first two integrals constitute the kinetic and potential energies, while the third constitutes the dissipation. The structure of this energy evolution equation is the basis of the energy method we will use to analyze (1.3).

1.2. **Geometric form of the equations.** In order to work in a fixed domain, we want to flatten the free surface via a coordinate transformation. We will not use a Lagrangian coordinate transformation, but rather a flattening transformation introduced by Beale in [7]. To this end, we consider the fixed equilibrium domain

$$\Omega := \{x \in \Sigma \times \mathbb{R} \mid -b(x_1, x_2) < x_3 < 0\} \tag{1.5}$$

for which we will write the coordinates as $x \in \Omega$. We will think of Σ as the upper boundary of Ω, and we will write $\Sigma_b := \{x_3 = -b(x_1, x_2)\}$ for the lower boundary. We continue to view η as a function on $\Sigma \times \mathbb{R}^+$. We then define

$$\bar{\eta} := \mathcal{P}\eta = \text{harmonic extension of } \eta \text{ into the lower half space,} \tag{1.6}$$

where $\mathcal{P}\eta$ is defined by (A.10) when $\Sigma = \mathbb{R}^2$ and by (A.18) when $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$. The harmonic extension $\bar{\eta}$ allows us to flatten the coordinate domain via the mapping

$$\Omega \ni x \mapsto (x_1, x_2, x_3 + \bar{\eta}(x,t)(1 + x_3/b(x_1, x_2))) = \Phi(x,t) = (y_1, y_2, y_3) \in \Omega(t). \tag{1.7}$$

Note that $\Phi(\Sigma, t) = \{y_3 = \eta(y_1, y_2, t)\}$ and $\Phi(\cdot, t)|_{\Sigma_b} = Id_{\Sigma_b}$, i.e. Φ maps Σ to the free surface and keeps the lower surface fixed. We have

$$\nabla\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix} \text{ and } \mathcal{A} := (\nabla\Phi^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix} \tag{1.8}$$

for

$$\begin{aligned} &A = \partial_1\bar{\eta}\tilde{b} - (x_3\bar{\eta}\partial_1 b)/b^2, \quad B = \partial_2\bar{\eta}\tilde{b} - (x_3\bar{\eta}\partial_2 b)/b^2, \\ &J = 1 + \bar{\eta}/b + \partial_3\bar{\eta}\tilde{b}, \quad K = J^{-1}, \\ &\tilde{b} = (1 + x_3/b). \end{aligned} \tag{1.9}$$

Here $J = \det \nabla\Phi$ is the Jacobian of the coordinate transformation.

If η is sufficiently small (in an appropriate Sobolev space), then the mapping Φ is a diffeomorphism. This allows us to transform the problem to one on the fixed spatial domain Ω for

$t \geq 0$. In the new coordinates, the PDE (1.3) becomes

$$\begin{cases} \partial_t u - \partial_t \bar{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathcal{A}} u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(p,u)\mathcal{N} = \eta \mathcal{N} & \text{on } \Sigma \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \\ u(x,0) = u_0(x), \eta(x',0) = \eta_0(x'). \end{cases} \tag{1.10}$$

Here we have written the differential operators $\nabla_{\mathcal{A}}$, $\operatorname{div}_{\mathcal{A}}$, and $\Delta_{\mathcal{A}}$ with their actions given by $(\nabla_{\mathcal{A}} f)_i := \mathcal{A}_{ij}\partial_j f$, $\operatorname{div}_{\mathcal{A}} X := \mathcal{A}_{ij}\partial_j X_i$, and $\Delta_{\mathcal{A}} f = \operatorname{div}_{\mathcal{A}} \nabla_{\mathcal{A}} f$ for appropriate f and X; for $u \cdot \nabla_{\mathcal{A}} u$ we mean $(u \cdot \nabla_{\mathcal{A}} u)_i := u_j \mathcal{A}_{jk} \partial_k u_i$. We have also written $\mathcal{N} := -\partial_1 \eta e_1 - \partial_2 \eta_2 e_2 + e_3$ for the non-unit normal to Σ, and we write $S_{\mathcal{A}}(p,u) = (pI - \mathbb{D}_{\mathcal{A}} u)$ for the stress tensor, where I the 3×3 identity matrix and $(\mathbb{D}_{\mathcal{A}} u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i$ is the symmetric $\mathcal{A}-$gradient. Note that if we extend $\operatorname{div}_{\mathcal{A}}$ to act on symmetric tensors in the natural way, then $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u$ for vector fields satisfying $\operatorname{div}_{\mathcal{A}} u = 0$.

Recall that $\mathcal{A}$ is determined by η through the relation (1.8). This means that all of the differential operators in (1.10) are connected to η, and hence to the geometry of the free surface. This geometric structure is essential to our analysis, as it allows us to control high-order derivatives that would otherwise be out of reach.

1.3. **Previous results.**

Local well-posedness for the problem (1.3) in a bounded domain, all of whose boundary is free, was proved by Solonnikov [27]. Local well-posedness for the problem in domains like ours was proved by Beale [6]. Both of these results employ parabolic regularity theory in a functional framework different from the one we use: Solonnikov worked in Hölder spaces, while Beale worked in L^2-based space-time Sobolev spaces. Abels [1] extended this local theory to the framework of L^p-based Sobolev spaces. Global well-posedness was proved in the periodic case by Hataya [21] and discussed in the infinite case by Sylvester [29] as well as Tani-Tanaka [30], all within a Beale-Solonnikov functional framework.

If the effect of surface tension is included at the free interface, then the free surface function gains regularity, stabilizing the problem. This led to a proof of small-data global well-posedness by Beale [7], as well as a proof by Beale-Nishida [8] that the global solutions with surface tension decay algebraically in time. In the periodic case, Nishida-Teramoto-Yoshihara [24] proved global well-posedness and exponential decay. Bae [5] proved global well-posedness with surface tension using energy methods rather than a Beale-Solonnikov framework. For a bounded mass of fluid with surface tension, local well-posedness was proved by Coutand-Shkoller [11].

Several authors have considered problems with two fluids and surface tension, where among other things, the free surface boundary conditions in (1.3) are replaced with jump conditions. Local well-posedness for this problem was proved by Xu-Zhang [35] for two fluid layers of finite depth and by Prüess-Simonett [25] for two layers of infinite depth. Denisova [15] proved local well-posedness with surface tension for a bubble of one fluid within another fluid.

Many authors have also considered one-fluid free boundary problems for inviscid fluids, which are modeled by setting $\mu = 0$ in (1.3) and replacing the no slip condition with the no penetration condition, $u \cdot \nu = 0$ on Σ_b. For this problem, it is often assumed that the fluid is initially curl-free, in which case this condition propagates in time and the fluid is said to be irrotational. The velocity field is then both curl-free and divergence-free for all time, and is therefore the gradient of a function that is harmonic in $\Omega(t)$. This allows for the reformulation of the problem as one only on the free surface, involving the Dirichlet-to-Neumann operator. Local well-posedness in this framework was established by Wu [31, 32] and Lannes [22], an almost-global well-posedness result was then proved by Wu [33] for the 2D problem, and global well-posedness was proved by Wu [34] and Germain-Masmoudi-Shatah [17] in 3D. Only the irrotational problem has been shown to admit global solutions in the inviscid case. Local well-posedness without the irrotationality assumption was proved with a modified surface formulation by Zhang-Zhang [36] and with the original formulation by Christodoulou-Lindblad [10], Lindblad [23], Coutand-Shkoller

[12], and Shatah-Zeng [26]. Note that in the viscous case, it is known that vorticity is generated at the free surface, even if the fluid is initially irrotational. As such, it is not possible to use the surface formulation of the problem.

1.4. **Main result.** As mentioned above, the standard method for constructing solutions in the existing literature is based on the parabolic regularity theory pioneered by Beale [6] for domains like ours and by Solonnikov [27] for bounded, non-periodic domains. The advantage of full parabolic regularity is that it enables one to treat viscous surface waves as a perturbation of the "flat surface" problem, which is obtained by setting $\eta = 0$, $\mathcal{A} = I$, $\mathcal{N} = e_3$, etc in (1.10). The actual problem (1.10) is then rewritten as the flat surface problem with nonlinear forcing terms that correspond to the difference between the two forms of the equations. The key to the existence theory of, say [6], is regularity in H^r with the choice of $r = 3 + \delta$ for $\delta \in (0, 1/2)$. According to the natural energy structure of the problem, (1.4), one might expect r to naturally be an integer. The extra gain of $\delta > 0$ regularity allows for enough control of the nonlinear forcing terms to produce a local solution to (1.10) from solutions to the flat surface problem and an iteration argument. As recognized early on by Beale himself, a disadvantage of Beale-Solonnikov theory is that the functional framework makes it difficult to extract time decay information.

In a pair of companion papers [19, 20], we prove a priori decay estimates that are developed through a high regularity energy method. This necessitates using the natural energy structure of the problem, (1.4), which in turn requires us to use positive integer Sobolev indices for u. The advantage of the natural energy structure is that it produces two distinct types of estimates: roughly speaking, $L^\infty([0,T]; L^2)$ "energy estimates" and $L^2([0,T]; H^1)$ "dissipation estimates." The interplay between the energy and the dissipation naturally leads to time decay information. The disadvantage of the energy structure is that our regularity index r must be an integer, so we cannot use the $\delta > 0$ gain that would allow us to treat the problem (1.10) as a perturbation of the flat surface problem.

The difficulty in proving local well-posedness in the natural energy structure is thus clear. We cannot use solutions to the standard flat surface problem to produce solutions to (1.10) via an iteration argument since the forcing terms cannot be controlled in the iteration. For example, we would have trouble controlling the interaction between the highest order temporal derivatives of p and $\operatorname{div} u$. Our solution, then, is to abandon the flat surface problem and prove local existence directly, using the geometric structure of (1.10). The geometric structure is crucial since it decreases the derivative count of the forcing terms, which then allows us to close an iteration argument using only the natural energy structure. The essential difficulty is that the geometric structure requires us to solve the Navier-Stokes equations in moving domains. In the presence of such a time-dependent geometric effect, even the construction of local-in-time solutions to the linear Navier-Stokes equations is highly delicate and has to be carried out from the beginning.

Before we state our local existence result, let us mention the issue of compatibility conditions for the initial data (u_0, η_0). We will work in a high-regularity context, essentially with regularity up to $2N$ temporal derivatives for $N \geq 3$ an integer. This requires us to use u_0 and η_0 to construct the initial data $\partial_t^j u(0)$ and $\partial_t^j \eta(0)$ for $j = 1, \dots, 2N$ and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-1$. These other data must then satisfy various conditions (essentially what one gets by applying ∂_t^j to (1.10) and then setting $t = 0$), which in turn require u_0 and η_0 to satisfy $2N$ compatibility conditions. We describe these conditions in detail in Section 5.2 and state them explicitly in (5.26), so for brevity we will not state them here.

In order to state our result, we must explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k \geq 0$ to be the usual Sobolev spaces. When we write norms we will suppress the H and Ω or Σ. When we write $\left\| \partial_t^j u \right\|_k$ and $\left\| \partial_t^j p \right\|_k$ we always mean that the space is $H^k(\Omega)$, and when we write $\left\| \partial_t^j \eta \right\|_k$ we always mean that the space is $H^k(\Sigma)$. In the following result we write $\left\| \cdot \right\|_{-1}$ for the norm in $({}_0H^1(\Omega))^*$, where ${}_0H^1(\Omega)$ is defined later in (2.1). Here it is not the case that $({}_0H^1(\Omega))^* = H^{-1}$ because of boundary conditions; we employ this abuse of notation in order to have indexed sums of norms include terms like $\|\cdot\|_{4N-2j+1}$ for $j = 2N+1$.

Theorem 1.1. *Let $N \geq 3$ be an integer. Assume that u_0 and η_0 satisfy the bounds $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N+1/2}^2 < \infty$ as well as the $(2N)^{th}$ compatibility conditions (5.26). There exist $0 < \delta_0, T_0 < 1$ so that if*

$$0 < T \leq T_0 \min\left\{1, \frac{1}{\|\eta_0\|_{4N+1/2}^2}\right\}, \tag{1.11}$$

and $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 \leq \delta_0$, then there exists a unique solution (u, p, η) to (1.10) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimates

$$\begin{aligned} \sum_{j=0}^{2N} \sup_{0\leq t\leq T} \left\|\partial_t^j u\right\|_{4N-2j}^2 + \sum_{j=0}^{2N} \sup_{0\leq t\leq T} \left\|\partial_t^j \eta\right\|_{4N-2j}^2 + \sum_{j=0}^{2N-1} \sup_{0\leq t\leq T} \left\|\partial_t^j p\right\|_{4N-2j-1}^2 \\ + \int_0^T \left(\sum_{j=0}^{2N+1} \left\|\partial_t^j u\right\|_{4N-2j+1}^2 + \sum_{j=0}^{2N} \left\|\partial_t^j p\right\|_{4N-2j}^2 \right) \\ + \int_0^T \left(\|\eta\|_{4N+1/2}^2 + \|\partial_t \eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \left\|\partial_t^j \eta\right\|_{4N-2j+5/2}^2 \right) \\ \leq C \left(\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 + T \|\eta_0\|_{4N+1/2}^2 \right) \end{aligned} \tag{1.12}$$

and

$$\sup_{0\leq t\leq T} \|\eta\|_{4N+1/2}^2 \leq C \left(\|u_0\|_{4N}^2 + (1+T) \|\eta_0\|_{4N+1/2}^2 \right) \tag{1.13}$$

for a universal constant $C > 0$. The solution is unique among functions that achieve the initial data and for which the sum of the first three sums in (1.12) is finite. Moroever, η is such that the mapping $\Phi(\cdot, t)$, defined by (1.7), is a C^{4N-2} diffeomorphism for each $t \in [0, T]$.

Remark 1.2. *Since the mapping $\Phi(\cdot, t)$ is a C^{4N-2} diffeomorphism, we may change coordinates to $y \in \Omega(t)$ to produce solutions to (1.3).*

The tools needed for the proof of Theorem 1.1 are developed throughout the rest of the paper, and the theorem is proved in Section 6.3. We will sketch here the main ideas of the proof.

Linear $\mathcal{A}-$Navier-Stokes

Our iteration procedure is based on a geometric variant of the linear Navier-Stokes problem. We consider η (and hence $\mathcal{A}, \mathcal{N}$, etc) as given and then solve the linear $\mathcal{A}-$Navier-Stokes equations for (u, p):

$$\begin{cases} \partial_t u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = F^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(p, u)\mathcal{N} = F^3 & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b, \end{cases} \tag{1.14}$$

with initial data u_0. Transforming this problem back to a moving domain $\Omega(t)$ using the mapping Φ defined in (1.7) shows that this problem is essentially equivalent (we have absorbed the correction to the time derivative into F^1, so it does not transform exactly) to solving the linear Navier-Stokes equations in a domain whose upper boundary is given by $\eta(t)$. In other words, we are really solving the usual linear problem in a moving domain.

Pressure as a Lagrange multiplier in time-dependent function spaces

It is well-known [28, 6, 11, 12] that for the usual linear Navier-Stokes equations, the pressure can be viewed as a Lagrange multiplier that arises by restricting the dynamics to the class of vectors satisfying $\operatorname{div} u = 0$. To adapt this idea to the problem (1.14), we must restrict to the class of vectors satisfying $\operatorname{div}_{\mathcal{A}} u = 0$, which is a time-dependent condition since η (and hence $\mathcal{A}$) depends on t. This leads us to build time-dependent variants of the usual Sobolev spaces $H^0 = L^2$ and H^1 so that we can make sense of this time-dependent collection of $\operatorname{div}_{\mathcal{A}}-$free vectors. For the purpose of estimates, we want the time-dependent norms on these spaces to all be comparable to the usual Sobolev norms; this can be achieved through a smallness assumption on η, which we quantify. With the spaces in hand, we then adapt a technique from [28] to introduce the pressure as a Lagrange multiplier for $\operatorname{div}_{\mathcal{A}}-$free dynamics.

Elliptic estimates for $\mathcal{A}-$problems

In order to get the regularity we need for solutions to the parabolic problem (1.14), we first need the corresponding elliptic regularity theory. We accomplish this by using (1.7) to transform these elliptic problems back into Eulerian coordinates so that the PDEs transform to ones with constant coefficients. We then apply standard estimates for elliptic equations and systems, proved in [3, 4], and then transform these estimates on the Eulerian domain back to estimates on Ω. The only problem with this process is that the Eulerian domain has a boundary whose regularity is dictated by η and is phrased in H^k norms rather than C^k norms, which are what appear in [3, 4]. We get around this problem by using a smoothing operator, a limiting argument, and the smallness of η.

Galerkin method with a time-dependent basis

We construct solutions to (1.14) by using a time-dependent Galerkin method. This requires a countable basis of our space of $\operatorname{div}_{\mathcal{A}}-$free vector fields. Since the requirement $\operatorname{div}_{\mathcal{A}} u = 0$ is time-dependent, any basis of this space must also be time-dependent. For each $t \in [0,T]$, the space we work in (basically H^2 with $\operatorname{div}_{\mathcal{A}} u = 0$) is separable, so the existence of a countable basis is not an issue. The technical difficulty is that, in order for the basis to be useful in the Galerkin method, we must be able to differentiate the basis elements in time, and we must be able to express these time derivatives in terms of finitely many basis elements. Fortunately, due to a clever observation of Beale in [7], we are able construct an explicit time-dependent isomorphism that maps the $\operatorname{div}-$free vector fields to the $\operatorname{div}_{\mathcal{A}}-$free fields. This allows us to construct the desired basis and push through the Galerkin method to produce "pressureless" weak solutions that are restricted to the collection of $\operatorname{div}_{\mathcal{A}}-$free fields. We then use our previous analysis to introduce the pressure as a Lagrange multiplier, which gives a weak solution to (1.14). We also use the Galerkin scheme to get higher regularity, showing that the solution is actually strong. The compatibility conditions serve as necessary conditions for controlling the temporal derivatives of the approximate solutions in the Galerkin scheme. The result of our strong existence theorem then allows us to iteratively deduce higher regularity, given that the forcing terms are more regular and higher-order compatibility conditions are satisfied.

Transport estimates

The problem (1.14) considers η as given and then produces (u,p). The second step in our iteration procedure is to take u as given and then solve $\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3$ on Σ. This is a standard transport equation, so solving it presents no real obstacle. The difficulty is that in our analysis of (1.14), we need control of $\sup_{0\le t\le T} \|\eta(t)\|^2_{4N+1/2}$, but owing to the transport structure, the only available estimate is, roughly speaking,

(1.15)
$$\sup_{0\le t\le T} \|\eta\|^2_{4N+1/2} \le C \exp\left(C \int_0^T \|Du(t)\|_{H^2(\Sigma)}\, dt \right) \left[\|\eta_0\|^2_{4N+1/2} + T \int_0^T \|u(t)\|^2_{4N+1}\, dt \right].$$

Without knowing a priori that u decays, the right side of this estimate has the potential to grow at the rate of $(1+T)e^{\sqrt{T}}$. Even if u decays rapidly, the right side can still grow like $(1+T)$. Of course, such a growth in time is disastrous for global stability analysis, but even in our local-existence iteration scheme, a delicate technique is required to accommodate such a growth without breaking the estimates of Theorem 1.1.

Closing the iteration with a two-tier energy scheme

Our iteration scheme then proceeds as described, using η^m to produce (u^{m+1}, p^{m+1}), and then using u^{m+1} to produce η^{m+1}. Iterating in this manner without losing control of our high-order energy estimates is rather delicate, and can only be completed by using sufficiently small initial data. The boundedness of the infinite sequence (u^m, p^m, η^m) in our high-order norms gives weak limits in the usual way, but because of the nature of our iteration scheme, we cannot guarantee a priori that the weak limits constitute a solution to (1.10). Instead of using high-order weak limits, we instead show that the sequence contracts in low-order norms, yielding strong convergence in low norms. We then combine the low-order strong convergence with the high-order weak convergence and an interpolation argument to deduce strong convergence in higher (but not all the way to the highest order) norms, which then suffices for passing to the limit $m \to \infty$ to produce a solution to (1.10).

1.5. **Utility in the global theory.** We believe that our local well-posedness result, Theorem 1.1, is interesting in its own right. It provides an alternative to the standard Beale-Solonnikov framework that is perhaps more natural due to the natural energy structure (1.4). The new ideas and techniques that we have introduced in order to work in this framework will likely be useful in many other problem.

However, we also need Theorem 1.1 as a crucial component in our global analysis of (1.3), which we carry out in [19] in the infinite case and in [20] in the periodic case. In both cases we develop novel a priori estimates that couple to the local theory to produce global-in-time solutions that decay to equilibrium at an algebraic rate. We call our a priori estimates a two-tier energy method because it couples the boundedness of certain high-regularity norms to the decay of certain low-regularity norms. The local theory we develop here both provides the tools for iteratively achieving global well-posedness and justifies all of the computations used in our two-tier a prior estimates. We do not believe that our a priori estimates would be compatible with a modification of the Beale-Solonnikov method due to differences in the functional framework.

Let us now informally state the theorems we prove in [19, 20].

Theorem 1.3. *The problem* (1.3) *is globally well-posed for sufficiently small initial data. In the infinite case, the solutions decay at a fixed algebraic rate. In the periodic case, by adjusting the smallness of the initial data, the solutions can be made to decay at arbitrarily fast algebraic rates. In other words, solutions in the periodic case decay almost exponentially.*

Remark 1.4. *The reader interested in a unified presentation of the present paper and the global decay results of* [19, 20] *may consult* [18].

Remark 1.5. *One can see a glimpse of the utility of our two-tier energy method already in the local theory. Indeed, the contraction argument we use to produce local solutions uses the boundedness of the high norms to close the contraction estimate for the low norms.*

1.6. **Definitions and terminology.** We now mention some of the definitions, bits of notation, and conventions that we will use throughout the paper.

Einstein summation and constants

We will employ the Einstein convention of summing over repeated indices for vector and tensor operations. Throughout the paper $C > 0$ will denote a generic constant that can depend on the parameters of the problem, N, and Ω, but does not depend on the data, etc. We refer to such constants as "universal." They are allowed to change from one inequality to the next. When a constant depends on a quantity z we will write $C = C(z)$ to indicate this. We will employ the notation $a \lesssim b$ to mean that $a \le Cb$ for a universal constant $C > 0$.

Derivatives and norms

We will write Df for the horizontal gradient of f, i.e. $Df = \partial_1 f e_1 + \partial_2 f e_2$, while ∇f will denote the usual full gradient. We write $H^k(\Omega)$ with $k \ge 0$ and and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for the usual Sobolev spaces. We will not need negative index spaces on Ω except $\|\cdot\|_{-1}$, which we take to mean the norm on $({}_0H^1(\Omega))^*$, where ${}_0H^1(\Omega)$ is defined later in (2.1). We employ this abuse of notation for the reasons discussed immediately before the statement of Theorem 1.1. We will typically write $H^0 = L^2$; the exception to this is where we use $L^2([0,T]; H^k)$ notation to indicate the space of square-integrable functions with values in H^k. For these spaces, we will further abuse notation by writing $L^2H^{-1} = L^2({}_0H^1(\Omega))^*$. This is meant to extend the abuse of notation $\|\cdot\|_{({}_0H^1(\Omega))^*} = \|\cdot\|_{-1}$.

To avoid notational clutter, we will avoid writing $H^k(\Omega)$ or $H^k(\Sigma)$ in our norms and typically write only $\|\cdot\|_k$. Since we will do this for functions defined on both Ω and Σ, this presents some ambiguity. We avoid this by adopting two conventions. First, we assume that functions have natural spaces on which they "live." For example, the functions u, p, and $\bar{\eta}$ live on Ω, while η itself lives on Σ. As we proceed in our analysis, we will introduce various auxiliary functions; the spaces they live on will always be clear from the context. Second, whenever the norm of a function is computed on a space different from the one in which it lives, we will explicitly write the space. This typically arises when computing norms of traces onto Σ of functions that live on Ω.

1.7. **Plan of paper.** Our proof of Theorem 1.1 employs an iteration that is based on the following linear problem for (u,p), where we think of η (and hence $\mathcal{A}, \mathcal{N}$, etc) as given,

$$\begin{cases} \partial_t u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = F^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(p,u)\mathcal{N} = F^3 & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b, \end{cases} \tag{1.16}$$

subject to the initial condition $u(0) = u_0$. Note that the first equation in (1.16) may be rewritten as $\partial_t u + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p,u) = F^1$.

In Section 2 we develop the machinery of time-dependent function spaces so that we can consider the class of $\operatorname{div}_{\mathcal{A}}$ −free vector fields. We use an orthogonal splitting of a space to introduce the pressure as a Lagrange multiplier. In Section 3 we record some elliptic estimates for the $\mathcal{A}$−Stokes problem and the $\mathcal{A}$−Poisson problem. In Section 4 we develop the local existence theory for (1.16) by using a time-dependent Galerkin scheme. We iterate this result to produce high-regularity solutions. In Section 5 we do some preliminary work for the nonlinear problem, constructing initial data, detailing the compatibility conditions, and constructing solutions to the transport equation with high-regularity estimates. In Section 6 we construct solutions to (1.10) through the use of iteration and contraction arguments, completing the proof of Theorem 1.1.

Throughout the paper we assume that $N \geq 3$ is an integer. We consider both the non-periodic and periodic cases simultaneously. When different analysis is needed for each case, we will indicate so. Otherwise, the argument we write works in both cases.

2. Functional setting

2.1. **Time-dependent function spaces.** We begin our analysis of (1.16) by introducing some function spaces. We write $H^k(\Omega)$ and $H^k(\Sigma)$ for the usual L^2-based Sobolev spaces of either scalar or vector-valued functions. Define

$$\begin{aligned} {}_0H^1(\Omega) &:= \{u \in H^1(\Omega) \mid u|_{\Sigma_b} = 0\}, \\ {}^0H^1(\Omega) &:= \{u \in H^1(\Omega) \mid u|_{\Sigma} = 0\}, \text{ and} \\ {}_0H^1_\sigma(\Omega) &:= \{u \in {}_0H^1(\Omega) \mid \operatorname{div} u = 0\}, \end{aligned} \tag{2.1}$$

with the obvious restriction that the last space is for vector-valued functions only.

For our time-dependent function spaces we will consider η (and hence $\mathcal{A}$, J, etc) as given; in our subsequent analysis η will always be sufficiently regular for all terms derived from η to make sense. We define a time-dependent inner-product on $L^2 = H^0$ by introducing

$$(u,v)_{\mathcal{H}^0} := \int_\Omega (u \cdot v) J(t) \tag{2.2}$$

with corresponding norm $\|u\|_{\mathcal{H}^0} := \sqrt{(u,u)_{\mathcal{H}^0}}$. Then we write $\mathcal{H}^0(t) := \{\|u\|_{\mathcal{H}^0} < \infty\}$. Similarly, we define a time-dependent inner-product on ${}_0H^1(\Omega)$ according to

$$(u,v)_{\mathcal{H}^1} := \int_\Omega \left(\mathbb{D}_{\mathcal{A}(t)} u : \mathbb{D}_{\mathcal{A}(t)} v\right) J(t), \tag{2.3}$$

and we define the corresponding norm by $\|u\|_{\mathcal{H}^1} = \sqrt{(u,u)_{\mathcal{H}^1}}$. Then we define

$$\mathcal{H}^1(t) := \{u \mid \|u\|_{\mathcal{H}^1} < \infty, u|_\Sigma = 0\} \text{ and } \mathcal{X}(t) := \{u \in \mathcal{H}^1(t) \mid \operatorname{div}_{\mathcal{A}(t)} u = 0\}. \tag{2.4}$$

We will also need the orthogonal decomposition $\mathcal{H}^0(t) = \mathcal{Y}(t) \oplus \mathcal{Y}(t)^\perp$, where

$$\mathcal{Y}(t)^\perp := \{\nabla_{\mathcal{A}(t)}\varphi \mid \varphi \in {}^0H^1(\Omega)\}. \tag{2.5}$$

A further discussion of the space $\mathcal{Y}(t)$ can be found later in Remark 3.4. In our use of these norms and spaces, we will often drop the (t) when there is no potential for confusion.

Finally, for $T > 0$ and $k = 0, 1$, we define inner-products on $L^2([0,T]; H^k(\Omega))$ by

$$(u,v)_{\mathcal{H}^k_T} = \int_0^T (u(t), v(t))_{\mathcal{H}^k}\, dt. \tag{2.6}$$

Write $\|u\|_{\mathcal{H}^k_T}$ for the corresponding norms and $\mathcal{H}^k_T$ for the corresponding spaces. We define the subspace of $\operatorname{div}_{\mathcal{A}}$-free vector fields as

$$\mathcal{X}_T := \{u \in \mathcal{H}^1_T \mid \operatorname{div}_{\mathcal{A}(t)} u(t) = 0 \text{ for a.e. } t \in [0,T]\}. \tag{2.7}$$

A priori we do not know that the spaces $\mathcal{H}^k(t)$ and $\mathcal{H}^k_T$ have the same topology as H^k and L^2H^k, respectively. This can be established under a smallness assumption on η.

Lemma 2.1. *There exists a universal $\varepsilon_0 > 0$ so that if*

$$\sup_{0 \le t \le T} \|\eta(t)\|_3 < \varepsilon_0, \tag{2.8}$$

then

$$\frac{1}{\sqrt{2}} \|u\|_k \le \|u\|_{\mathcal{H}^k} \le \sqrt{2}\,\|u\|_k \tag{2.9}$$

for $k = 0, 1$ and for all $t \in [0,T]$. As a consequence, for $k = 0, 1$,

$$\frac{1}{\sqrt{2}} \|u\|_{L^2H^k} \le \|u\|_{\mathcal{H}^k_T} \le \sqrt{2}\,\|u\|_{L^2H^k}. \tag{2.10}$$

Proof. Consider $\varepsilon \in (0, 1/2)$ with precise value to be chosen later. It is straightforward to verify, using Lemma A.5 in the non-periodic case and Lemma A.7 in the periodic case, that

$$\sup\{\|J-1\|_{L^\infty}, \|A\|_{L^\infty}, \|B\|_{L^\infty}\} \le C\,\|\eta\|_3. \tag{2.11}$$

Then we may choose $\varepsilon_0 = \varepsilon/C$ so that the right side of (2.11) is bounded by ε. Since $K = 1/J$, this implies that

$$\|K-1\|_{L^\infty} \le \frac{\varepsilon}{1-\varepsilon}, \|K\|_{L^\infty} \le \frac{1}{1-\varepsilon}, \tag{2.12}$$

and

$$\|I-\mathcal{A}\|_{L^\infty} \le \frac{3\varepsilon}{1-\varepsilon}, \|\mathcal{A}+I\|_{L^\infty} \le 2\sqrt{3} + \frac{3\varepsilon}{1-\varepsilon}. \tag{2.13}$$

In turn, this implies that

$$\|J\|_{L^\infty} \|I-\mathcal{A}\|_{L^\infty} \|I+\mathcal{A}\|_{L^\infty} \le \frac{3\varepsilon(1+\varepsilon)(2\sqrt{3} - (2\sqrt{3}-3)\varepsilon)}{(1-\varepsilon)^2} := g(\varepsilon). \tag{2.14}$$

Notice that g is a continuous, increasing function on $(0, 1/2)$ so that $g(0) = 0$. With the estimates (2.11) and (2.14) in hand, we can show that if ε is chosen sufficiently small, then (2.9) and (2.10) hold.

In the case $k = 0$, the estimate (2.9) follows directly from the estimate for J in (2.11):

$$\frac{1}{2}\int_\Omega |u|^2 \le (1-\varepsilon)\int_\Omega |u|^2 \le \int_\Omega J\,|u|^2 \le (1+\varepsilon)\int_\Omega |u|^2 \le 2\int_\Omega |u|^2. \tag{2.15}$$

To derive (2.9) when $k = 1$, we first rewrite

$$\int_\Omega J\,|\mathbb{D}_{\mathcal{A}} u|^2 = \int_\Omega J\,|\mathbb{D}u|^2 + \int_\Omega J(\mathbb{D}_{\mathcal{A}}u + \mathbb{D}u) : (\mathbb{D}_{\mathcal{A}}u - \mathbb{D}u). \tag{2.16}$$

To estimate the last term, we note that $|(\mathbb{D}_{\mathcal{A}} u \pm \mathbb{D} u)| \le 2\,|\mathcal{A} \pm I|\,|\nabla u|$, which implies that

$$\left| \int_\Omega J(\mathbb{D}_{\mathcal{A}} u + \mathbb{D}u) : (\mathbb{D}_{\mathcal{A}} u - \mathbb{D}u) \right| \le 4 \left\| J \right\|_{L^\infty} \left\| I - \mathcal{A} \right\|_{L^\infty} \left\| I + \mathcal{A} \right\|_{L^\infty} \int_\Omega |\nabla u|^2 \le 4 C_\Omega g(\varepsilon) \int_\Omega |\mathbb{D} u|^2 , \tag{2.17}$$

where C_Ω is the constant in Korn's inequality, Lemma A.10. We may then employ the bounds (2.11) and (2.17) in (2.16) to estimate

$$\int_\Omega |\mathbb{D}_{\mathcal{A}} u|^2 J \ge \int_\Omega J \,|\mathbb{D}u|^2 - 4C_\Omega g(\varepsilon) \int_\Omega |\mathbb{D}u|^2 \ge (1 - \varepsilon - 4C_\Omega g(\varepsilon)) \int_\Omega |\mathbb{D}u|^2 \tag{2.18}$$

and

$$\int_\Omega |\mathbb{D}_{\mathcal{A}} u|^2 J \le \int_\Omega J \,|\mathbb{D}u|^2 + 4C_\Omega g(\varepsilon) \int_\Omega |\mathbb{D}u|^2 \le (1 + \varepsilon + 4C_\Omega g(\varepsilon)) \int_\Omega |\mathbb{D}u|^2 . \tag{2.19}$$

Then (2.9) with $k = 1$ follows from (2.18)–(2.19) by choosing ε small enough so that $\varepsilon + 4C_\Omega g(\varepsilon) \le 1/2$. The estimates (2.10) follow by applying (2.9) for a.e. $t \in [0, T]$, squaring, and integrating over $t \in [0, T]$. □

Remark 2.2. *Throughout the rest of this paper, we will assume that* (2.8) *is satisfied so that* (2.9)–(2.10) *hold.*

Remark 2.3. *Because of the bound* (2.9) *and the usual Korn inequality on Ω, Lemma A.10, we have a corresponding Korn-type inequality in $\mathcal{H}^1(t)$: $\|u\|_{\mathcal{H}^0} \lesssim \|u\|_{\mathcal{H}^1}$. The standard trace embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$ and* (2.9) *imply that $\|u\|_{H^{1/2}(\Sigma)} \lesssim \|u\|_{\mathcal{H}^1}$ for all $t \in [0, T]$. Similarly, given $f \in H^{1/2}(\Sigma)$, we may construct an extension $\tilde{f} \in \mathcal{H}^1(t)$ so that $\|f\|_{\mathcal{H}^1} \lesssim \|f\|_{H^{1/2}(\Sigma)}$.*

We now prove a result about the differentiability of norms in our time-dependent spaces.

Lemma 2.4. *Suppose that $u \in \mathcal{H}^1_T$, $\partial_t u \in (\mathcal{H}^1_T)^*$. Then the mapping $t \mapsto \|u(t)\|^2_{\mathcal{H}^0(t)}$ is absolutely continuous, and*

$$\frac{d}{dt} \|u(t)\|^2_{\mathcal{H}^0} = 2 \langle \partial_t u(t), u(t) \rangle_{(\mathcal{H}^1)^*} + \int_\Omega |u(t)|^2 \partial_t J(t) \tag{2.20}$$

for a.e. $t \in [0, T]$. Moreover, $u \in C^0([0, T]; H^0(\Omega))$. If $v \in \mathcal{H}^1_T$, $\partial_t v \in (\mathcal{H}^1_T)^*$ as well, then

$$\frac{d}{dt} (u(t), v(t))_{\mathcal{H}^0} = \langle \partial_t u(t), v(t) \rangle_{(\mathcal{H}^1)^*} + \langle \partial_t v(t), u(t) \rangle_{(\mathcal{H}^1)^*} + \int_\Omega u(t) \cdot v(t) \partial_t J(t). \tag{2.21}$$

Proof. In light of Lemma 2.1, the time-dependent spaces $\mathcal{H}^0_T$, $\mathcal{H}^1_T$, $(\mathcal{H}^1_T)^*$ present no obstacle to the usual method of approximation by temporally smooth functions via convolution. This allows us to argue as in Theorem 3 in Section 5.9 of [16] to deduce (2.20) and the continuity $u \in C^0([0, T]; H^0(\Omega))$. The equality (2.21) follows by applying (2.20) to $u + v$ and canceling terms by using (2.20) with u and with v. □

Now we want to show the spaces ${}_0H^1(\Omega)$ and ${}_0H^1_\sigma(\Omega)$ are related to the spaces $\mathcal{H}^1(t)$ and $\mathcal{X}(t)$. To this end, we define the matrix

$$M := M(t) = K \nabla \Phi = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ AK & BK & 1 \end{pmatrix}. \tag{2.22}$$

Note that M is invertible, and $M^{-1} = J\mathcal{A}^T$. Since $J \neq 0$ and $\partial_j (J\mathcal{A}_{ij}) = 0$ for each $i = 1, 2, 3$,

$$p = \operatorname{div}_{\mathcal{A}} v \Leftrightarrow Jp = J \operatorname{div}_{\mathcal{A}} v = J\mathcal{A}_{ij} \partial_j v_i = \partial_j (J\mathcal{A}_{ij} v_i) = \partial_j (J\mathcal{A}^T v)_j = \partial_j (M^{-1} v)_j = \operatorname{div}(M^{-1} v). \tag{2.23}$$

The matrix $M(t)$ induces a linear operator $\mathcal{M}_t : u \mapsto \mathcal{M}_t(u) = M(t)u$ that possesses several nice properties, the most important of which is that div-free vector fields are mapped to $\operatorname{div}_{\mathcal{A}}$-free vector fields. We record these now.

Proposition 2.5. *For each $t \in [0,T]$, $\mathcal{M}_t$ is a bounded, linear isomorphism: from $H^k(\Omega)$ to $H^k(\Omega)$ for $k = 0,1,2$; from $L^2(\Omega)$ to $\mathcal{H}^0(t)$; from ${}_0H^1(\Omega)$ to $\mathcal{H}^1(t)$; and from ${}_0H^1_\sigma(\Omega)$ to $\mathcal{X}(t)$. In each case the norms of the operators $\mathcal{M}_t, \mathcal{M}_t^{-1}$ are bounded by a constant times $1 + \|\eta(t)\|_{9/2}$.*

Moreover, the mapping $\mathcal{M}$ given by $\mathcal{M}u(t) := \mathcal{M}_t u(t)$ is a bounded, linear isomorphism: from $L^2([0,T]; H^k(\Omega))$ to $L^2([0,T]; H^k(\Omega))$ for $k = 0,1,2$; from $L^2([0,T]; H^0(\Omega))$ to $\mathcal{H}^0_T$; from $L^2([0,T]; {}_0H^1(\Omega))$ to $\mathcal{H}^1_T$; and from $L^2([0,T]; {}_0H^1_\sigma(\Omega))$ to $\mathcal{X}_T$. In each case, the norms of the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ are bounded by a constant times the sum $1 + \sup_{0 \le t \le T} \|\eta(t)\|_{9/2}$.

Proof. For each $t \in [0,T]$, it is easy to see that

$$\|\mathcal{M}_t u\|_k \lesssim \|M(t)\|_{C^3} \|u\|_k \lesssim (1 + \|\eta(t)\|_{9/2}) \|u\|_k \tag{2.24}$$

for $k = 0,1,2$, which establishes that $\mathcal{M}_t$ is a bounded operator on H^k. Since $M(t)$ is an invertible matrix, $\mathcal{M}_t^{-1} v = M(t)^{-1} v = J \nabla \Phi(t) v$, which allows us to argue similarly to see that for $k = 0,1,2$, $\left\|\mathcal{M}_t^{-1} v\right\|_k \lesssim (1 + \|\eta(t)\|_{9/2}) \|v\|_k$. Hence $\mathcal{M}_t$ is an isomorphism of H^k to itself for $k = 0,1,2$. With this fact in hand, Lemma 2.1 implies that $\mathcal{M}_t$ is an isomorphism of $H^0(\Omega)$ to $\mathcal{H}^0(t)$ and of ${}_0H^1(\Omega)$ to $\mathcal{H}^1(t)$.

To prove that $\mathcal{M}_t$ is an isomorphism of ${}_0H^1_\sigma(\Omega)$ to $\mathcal{X}(t)$, we must only establish that $\operatorname{div} u = 0$ if and only if $\operatorname{div}_{\mathcal{A}}(Mu) = 0$. To see this we appeal to (2.23) with $p = 0$ to see that $0 = \operatorname{div}_{\mathcal{A}} v$ if and only if $0 = \operatorname{div}(M^{-1} v)$. Hence, writing $v = Mu$, we see that $\operatorname{div} u = 0$ if and only if $\operatorname{div}_{\mathcal{A}}(Mu) = 0$.

The mapping properties of the operator $\mathcal{M}$ on space-time functions may be established in a similar manner. □

2.2. **Pressure as a Lagrange multiplier.** It is well-known [28, 6, 12] that the space ${}_0H^1(\Omega)$ can be orthogonally decomposed as ${}_0H^1(\Omega) = {}_0H^1_\sigma(\Omega) \oplus R(Q)$, where $R(Q)$ is the range of the operator $Q : H^0(\Omega) \to {}_0H^1(\Omega)$, defined by the Riesz representation theorem via the relation

$$\int_\Omega p \operatorname{div} u = \int_\Omega \mathbb{D}(Qp) : \mathbb{D}u \text{ for all } u \in {}_0H^1(\Omega). \tag{2.25}$$

We now wish to establish a similar decomposition for our spaces $\mathcal{X}(t) \subset \mathcal{H}^1(t)$. Unfortunately, the mappings $\mathcal{M}_t$, while isomorphisms, are not isometries, so we cannot use the known result to decompose $\mathcal{H}^1(t)$. Instead, we must adapt the method of [28] to our time-dependent context.

For $p \in \mathcal{H}^0(t)$, we define the functional $\mathcal{Q}_t \in (\mathcal{H}^1(t))^*$ by $\mathcal{Q}_t(v) = (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0}$. By the Riesz representation theorem, there exists a unique $Q_t p \in \mathcal{H}^1(t)$ so that $\mathcal{Q}_t(v) = (Q_t p, v)_{\mathcal{H}^1}$ for all $v \in \mathcal{H}^1(t)$. This defines a linear operator $Q_t : \mathcal{H}^0(t) \to \mathcal{H}^1(t)$, which is bounded since we may take $v = Q_t p$ to bound

$$\begin{aligned} \|Q_t p\|^2_{\mathcal{H}^1} &= (Q_t p, Q_t p)_{\mathcal{H}^1} = \mathcal{Q}_t(v) = (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} \\ &\le \|p\|_{\mathcal{H}^0} \|\operatorname{div}_{\mathcal{A}} v\|_{\mathcal{H}^0} \le \|p\|_{\mathcal{H}^0} \|v\|_{\mathcal{H}^1} = \|p\|_{\mathcal{H}^0} \|Q_t p\|_{\mathcal{H}^1}, \end{aligned} \tag{2.26}$$

so that $\|Q_t p\|_{\mathcal{H}^1} \le \|p\|_{\mathcal{H}^0}$. In the previous inequality we have utilized the simple bound $\|\operatorname{div}_{\mathcal{A}} v\|_{\mathcal{H}^0} \le \|v\|_{\mathcal{H}^1}$, which follows from the fact that $\operatorname{div}_{\mathcal{A}} v = \operatorname{tr}(\mathbb{D}_{\mathcal{A}} u)/2$. In a straightforward manner, we may also define a bounded linear operator $Q : \mathcal{H}^0_T \to \mathcal{H}^1_T$ via the relation

$$(p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} = (Qp, v)_{\mathcal{H}^1_T} \text{ for all } v \in \mathcal{H}^1_T. \tag{2.27}$$

Arguing as above, we can show that Q satisfies $\|Qp\|_{\mathcal{H}^1_T} \le \|p\|_{\mathcal{H}^0_T}$.

In order to study the range of Q_t in $\mathcal{H}^1(t)$ and of Q in $\mathcal{H}^1_T$, we will first need a lemma on the solvability of the equation $\operatorname{div}_{\mathcal{A}} v = p$.

Lemma 2.6. *Let $p \in \mathcal{H}^0(t)$. Then there exists a $v \in \mathcal{H}^1(t)$ so that $\operatorname{div}_{\mathcal{A}} v = p$ and $\|v\|_{\mathcal{H}^1} \lesssim (1 + \|\eta(t)\|_{9/2}) \|p\|_{\mathcal{H}^0}$. If instead $p \in \mathcal{H}^0_T$, then there exists a $v \in \mathcal{H}^1_T$ so that $\operatorname{div}_{\mathcal{A}} v = p$ for a.e. $t \in [0,T]$, and $\|v\|_{\mathcal{H}^1_T} \lesssim (1 + \sup_{0 \le t \le T} \|\eta(t)\|_{9/2}) \|p\|_{\mathcal{H}^0_T}$.*

Proof. It is established in the proof of Lemma 3.3 of [6] that for any $q \in L^2(\Omega)$ the problem $\operatorname{div} u = q$ admits a solution $u \in {}_0H^1(\Omega)$ so that $\|u\|_1 \lesssim \|q\|_0$. The result in [6] concerns the non-periodic case, but its proof may be easily adapted to the periodic case as well. Choose $q = Jp$ so that

$$\|q\|_0^2 = \int_\Omega |q|^2 = \int_\Omega |p|^2 J^2 \le \|J\|_{L^\infty} \|p\|_{\mathcal{H}^0}^2 \le 2\|p\|_{\mathcal{H}^0}^2 . \tag{2.28}$$

Then by (2.23) we know that $v = M(t)u \in \mathcal{H}^1(t)$ satisfies $\operatorname{div}_{\mathcal{A}} v = p$, and Proposition 2.5 implies that

$$\|v\|_{\mathcal{H}^1} \lesssim (1 + \|\eta(t)\|_{9/2}) \|u\|_1 \lesssim (1 + \|\eta(t)\|_{9/2}) \|q\|_0 \lesssim (1 + \|\eta(t)\|_{9/2}) \|p\|_{\mathcal{H}^0} . \tag{2.29}$$

If $p \in \mathcal{H}^0_T$, then for a.e. $t \in [0,T]$, $p(t) \in \mathcal{H}^0(t)$, so we may apply the above analysis to find $v(t) \in \mathcal{H}^1(t)$ so that $\operatorname{div}_{\mathcal{A}} v(t) = p(t)$ and the bound (2.29) holds with $v = v(t)$ and $p = p(t)$. We may then square both sides and integrate over $t \in [0,T]$ to deduce that

$$\begin{aligned} \|v\|_{\mathcal{H}^1_T}^2 = \int_0^T \|v(t)\|_{\mathcal{H}^1}^2 \, dt &\lesssim \left(1 + \sup_{0 \le t \le T} \|\eta(t)\|_{9/2}^2\right) \int_0^T \|p(t)\|_{\mathcal{H}^0}^2 \, dt \\ &\lesssim \left(1 + \sup_{0 \le t \le T} \|\eta(t)\|_{9/2}^2\right) \|v\|_{\mathcal{H}^0_T}^2 . \end{aligned} \tag{2.30}$$

□

With this lemma in hand, we can show that $R(Q_t)$ is a closed subspace of $\mathcal{H}^1(t)$ and that $R(Q)$ is a closed subspace of $\mathcal{H}^1_T$.

Lemma 2.7. *$R(Q_t)$ is closed in $\mathcal{H}^1(t)$, and $R(Q)$ is closed in $\mathcal{H}^1_T$.*

Proof. For $p \in \mathcal{H}^0(t)$ let $v \in \mathcal{H}^1(t)$ be the solution to $\operatorname{div}_{\mathcal{A}} v = p$ provided by Lemma 2.6. Then

$$\begin{aligned} \|p\|_{\mathcal{H}^0}^2 = (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \mathcal{Q}_t(v) = (Q_t p, v)_{\mathcal{H}^1} \\ \le \|Q_t p\|_{\mathcal{H}^1} \|v\|_{\mathcal{H}^1} \lesssim \|Q_t p\|_{\mathcal{H}^1} (1 + \|\eta(t)\|_{9/2}) \|p\|_{\mathcal{H}^0} \end{aligned} \tag{2.31}$$

so that $\|Q_t p\|_{\mathcal{H}^1} \le \|p\|_{\mathcal{H}^0} \lesssim (1 + \|\eta(t)\|_{9/2}) \|Q_t p\|_{\mathcal{H}^1}$. Hence $R(Q_t)$ is closed in $\mathcal{H}^1(t)$. A similar analysis shows that $R(Q)$ is closed in $\mathcal{H}^1_T$. □

Now we can perform the orthogonal decomposition of $\mathcal{H}^1(t)$ and $\mathcal{H}^1_T$.

Lemma 2.8. *We have that $\mathcal{H}^1(t) = \mathcal{X}(t) \oplus R(Q_t)$, i.e. $\mathcal{X}(t)^\perp = R(Q_t)$. Also, $\mathcal{H}^1_T = \mathcal{X}_T \oplus R(Q)$, i.e. $\mathcal{X}_T^\perp = R(Q)$.*

Proof. By Lemma 2.7, $R(Q_t)$ is a closed subspace of $\mathcal{H}^1(t)$, and so it suffices to prove that $R(Q_t)^\perp = \mathcal{X}(t)$.

Let $v \in R(Q_t)^\perp$. Then for all $p \in \mathcal{H}^0(t)$, we know that

$$\int_\Omega p \operatorname{div}_{\mathcal{A}} v J = \mathcal{Q}_t(v) = (Q_t p, v)_{\mathcal{H}^1} = 0, \tag{2.32}$$

and hence $\operatorname{div}_{\mathcal{A}} v = 0$. This implies that $R(Q_t)^\perp \subseteq \mathcal{X}(t)$.

Now suppose that $v \in \mathcal{X}(t)$. Then $\operatorname{div}_{\mathcal{A}} v = 0$ implies that

$$0 = \int_\Omega p \operatorname{div}_{\mathcal{A}} v J = \mathcal{Q}_t(v) = (Q_t p, v)_{\mathcal{H}^1} \tag{2.33}$$

for all $p \in \mathcal{H}^0(t)$. Hence $v \in R(Q_t)^\perp$, and we see that $\mathcal{X}(t) \subseteq R(Q_t)^\perp$.

A similar argument shows that $\mathcal{H}^1_T = \mathcal{X}_T \oplus R(Q)$. □

This decomposition will eventually allow us to introduce the pressure function. This will be accomplished by use of the following result.

Proposition 2.9. *If $\Lambda_t \in (\mathcal{H}^1(t))^*$ is such that $\Lambda_t(v) = 0$ for all $v \in \mathcal{X}(t)$, then there exists a unique $p(t) \in \mathcal{H}^0(t)$ so that*

$$(p(t), \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \Lambda_t(v) \text{ for all } v \in \mathcal{H}^1(t) \tag{2.34}$$

and $\|p(t)\|_{\mathcal{H}^0} \lesssim (1 + \|\eta(t)\|_{9/2}) \|\Lambda_t\|_{(\mathcal{H}^1(t))^*}$.

If $\Lambda \in (\mathcal{H}^1_T)^*$ is such that $\Lambda(v) = 0$ for all $v \in \mathcal{X}_T$, then there exists a unique $p \in \mathcal{H}^0_T$ so that

$$(p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} = \Lambda(v) \text{ for all } v \in \mathcal{H}^1_T \tag{2.35}$$

and $\|p\|_{\mathcal{H}^0_T} \lesssim (1 + \sup_{0\le t\le T} \|\eta(t)\|_{9/2}) \|\Lambda\|_{(\mathcal{H}^1_T)^*}$.

Proof. If $\Lambda_t(v) = 0$ for all $v \in \mathcal{X}(t)$, then the Riesz representation theorem yields the existence of a unique $w \in \mathcal{X}(t)^\perp$ so that $\Lambda_t(v) = (w, v)_{\mathcal{H}^1}$ for all $v \in \mathcal{H}^1(t)$. By Lemma 2.8, $w = Q_t p(t)$ for some $p(t) \in \mathcal{H}^0(t)$. Then $\Lambda_t(v) = (Q_t p(t), v)_{\mathcal{H}^1} = (p(t), \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0}$ for all $v \in \mathcal{H}^1(t)$. By Lemma 2.6, we may find $v(t) \in \mathcal{H}^1(t)$ so that $\operatorname{div}_{\mathcal{A}} v(t) = p(t)$ and $\|v(t)\|_{\mathcal{H}^1} \lesssim (1 + \|\eta(t)\|_{9/2}) \|p(t)\|_{\mathcal{H}^0}$. Hence

$$\|p(t)\|^2_{\mathcal{H}^0} = (p(t), \operatorname{div}_{\mathcal{A}} v(t))_{\mathcal{H}^0} = \Lambda_t(v(t)) \le \|\Lambda_t\|_{(\mathcal{H}^1(t))^*} (1 + \|\eta(t)\|_{9/2}) \|p(t)\|_{\mathcal{H}^0}, \tag{2.36}$$

and the desired estimate holds. A similar argument proves the result for $\Lambda \in (\mathcal{H}^1_T)^*$ such that $\Lambda(v) = 0$ for all $v \in \mathcal{X}_T$. □

3. Elliptic estimates

3.1. **Preliminary estimates.** In studying the elliptic problems in the rest of this section we will utilize the fact that the equations can be transformed into constant coefficient equations on the domain $\Omega' = \Phi(\Omega)$. In order to properly utilize this transformation we must verify that composition with Φ generates an isomorphism of $H^k(\Omega')$ to $H^k(\Omega)$. This type of result is standard (see the appendix of [9] for a bounded domain, or Lemma 5.2 of [7] and Lemma 6.2 of [29] for domain $\mathbb{R}^n$), but the precise form we need is not readily available in the literature, so we record it now.

Lemma 3.1. *Let $\Psi : \Omega \to \Omega'$ be a C^1 diffeomorphism satisfying $\|1 - \det \nabla\Psi\|_{L^\infty} \le 1/2$ and $\nabla\Psi - I \in H^k(\Omega)$ for an integer $k \ge 3$. If $v \in H^m(\Omega')$, then $v\circ\Psi \in H^m(\Omega)$ for $m = 0, 1, \dots, k+1$, and*

$$\|v \circ \Psi\|_{H^m(\Omega)} \lesssim C(\|\nabla\Psi - I\|_{H^k(\Omega)}) \|v\|_{H^m(\Omega')} \tag{3.1}$$

for $C(\|\nabla\Psi - I\|_{H^k(\Omega)})$ a constant depending on $\|\nabla\Psi - I\|_{H^k(\Omega)}$. Similarly, for $u \in H^m(\Omega)$, $u \circ \Psi^{-1} \in H^m(\Omega')$ for $m = 0, 1, \dots, k + 1$, and

$$\left\|u \circ \Psi^{-1}\right\|_{H^m(\Omega')} \lesssim C(\|\nabla\Psi - I\|_{H^k(\Omega)}) \|u\|_{H^m(\Omega)}. \tag{3.2}$$

Let $\Sigma' = \Psi(\Sigma)$ denote the upper boundary of Ω'. If $v \in H^{m-1/2}(\Sigma')$ for $m = 1, \dots, k - 1$, then $v \circ \Psi \in H^{m-1/2}(\Sigma)$ and

$$\|v \circ \Psi\|_{H^{m-1/2}(\Sigma)} \lesssim C(\|\nabla\Psi - I\|_{H^k(\Omega)}) \|v\|_{H^{m-1/2}(\Sigma')}. \tag{3.3}$$

If $u \in H^{m-1/2}(\Sigma)$ for $m = 1, \dots, k - 1$, then $v \circ \Psi^{-1} \in H^{m-1/2}(\Sigma')$ and

$$\left\|u \circ \Psi^{-1}\right\|_{H^{m-1/2}(\Sigma')} \lesssim C(\|\nabla\Psi - I\|_{H^k(\Omega)}) \|u\|_{H^{m-1/2}(\Sigma)}. \tag{3.4}$$

Proof. The proof of (3.1)–(3.2) is similar to the proofs of the results in [9, 7, 29] mentioned above, so we present only a sketch. We first prove that for $m = 0, 1, 2$, it holds that

$$\|v \circ \Psi\|_{H^m(\Omega)} \lesssim C(\|\nabla\Psi - I\|_{H^{k-1}(\Omega)}) \|v\|_{H^m(\Omega')}. \tag{3.5}$$

Such a bound follows easily from the size of k and the bound on $\det \nabla\Psi$. We then proceed inductively for $m = 3, \dots, k + 1$. Suppose the bound (3.5) holds for $m = 0, 1, 2, \dots, m_0$ for $2 \le m_0 \le k$. To show that it holds for $m_0 + 1$ we write x for coordinates in Ω and y for coordinates in Ω' and note that

$$\frac{\partial}{\partial x_i}(v \circ \Psi)(x) = \frac{\partial v}{\partial y_j} \circ \Psi(x) \cdot \frac{\partial \Psi_j}{\partial x_i}(x) = \frac{\partial v}{\partial y_i} \circ \Psi(x) + \frac{\partial v}{\partial y_j} \circ \Psi(x) \cdot \left(\frac{\partial \Psi_j}{\partial x_i}(x) - I_{ij}\right). \tag{3.6}$$

By the induction hypothesis, if $v \in H^{m_0+1}$, then

$$\frac{\partial v}{\partial y_j} \circ \Psi \in H^{m_0} \text{ for all } j = 1, 2, 3, \tag{3.7}$$

and since we have the multiplicative embedding $H^{m_0} \cdot H^k \hookrightarrow H^{m_0}$ for $m_0 \geq 2$ and $k \geq 3$, we deduce that

$$\frac{\partial}{\partial x_i}(v \circ \Psi) \in H^{m_0} \text{ for all } i = 1, 2, 3, \tag{3.8}$$

and hence that $v \circ \Psi \in H^{m_0+1}$. Moreover, an estimate of the form (3.5) holds. By induction, we deduce that (3.1) holds. The result (3.2) follow similarly, utilizing the fact that $\nabla \Psi^{-1}(y) = (\nabla \Psi)^{-1} \circ \Psi^{-1}(y)$.

We now turn to the proof of (3.3)–(3.4). First note that since $\Psi \in H^{k+1}_{loc}$, we have that Σ' is locally the graph of a $C^{k-1,1/2}$ function. As such (cf. [2]), there exists a bounded extension operator $E : H^{m-1/2}(\Sigma') \to H^m(\Omega')$ for $m = 1, \dots, k-1$ with the norm of the operator depending on $C(\|\nabla \Psi - I\|_{H^k(\Omega)})$. For $v \in H^{m-1/2}(\Sigma')$, let $V = Ev \in H^m(\Omega')$. By (3.1), we have that $V \circ \Psi \in H^m(\Omega)$, and by the usual trace theory, $v \circ \Psi = V \circ \Psi|_\Sigma \in H^{m-1/2}(\Sigma)$. Moreover,

$$\begin{aligned} \|v \circ \Psi\|_{H^{m-1/2}(\Sigma)} &\lesssim \|V \circ \Psi\|_{H^m(\Omega)} \lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|Ev\|_{H^m(\Omega')} \\ &\lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|v\|_{H^{m-1/2}(\Sigma')}, \end{aligned} \tag{3.9}$$

which is (3.3). The bound (3.4) follows similarly. □

Remark 3.2. *It is easy to show, using Lemma A.7 in the periodic case and Lemma A.5 in the non-periodic case, that if $\|\eta\|^2_{k+1/2}$ is sufficiently small for $k \geq 3$, then the mapping Φ defined by (1.7) is a C^1 diffeomorphism that satisfies the hypotheses of Lemma 3.1.*

We will also need the following $H^{-1/2}$ boundary estimates for functions satisfying $u, \operatorname{div}_{\mathcal{A}} u \in \mathcal{H}^0(t)$.

Lemma 3.3. *If $v \in \mathcal{H}^0(t)$ and $\operatorname{div}_{\mathcal{A}} v \in \mathcal{H}^0(t)$, then $v \cdot \mathcal{N} \in H^{-1/2}(\Sigma)$, $v \cdot \nu \in H^{-1/2}(\Sigma_b)$ (with ν the unit normal on Σ_b), and*

$$\|v \cdot \mathcal{N}\|_{H^{-1/2}(\Sigma)} + \|v \cdot \nu\|_{H^{-1/2}(\Sigma_b)} \lesssim \|v\|_{\mathcal{H}^0} + \|\operatorname{div}_{\mathcal{A}} v\|_{\mathcal{H}^0}. \tag{3.10}$$

Proof. We will only prove the result on Σ; the result on Σ_b may be derived in a similar manner, using the fact that $J\mathcal{A}\nu = \nu$ on Σ_b.

Let $\varphi \in H^{1/2}(\Sigma)$ be a scalar function, and let $\tilde{\varphi} \in {}_0H^1(\Omega)$ be a bounded extension. If we define the vector field $w = \tilde{\varphi} e_1$, then a straightforward computation reveals that

$$2 \int_\Omega |\nabla_{\mathcal{A}} \tilde{\varphi}|^2 J \leq \|w\|^2_{\mathcal{H}^1} \text{ and that } \|w\|^2_{{}_0H^1(\Omega)} \leq 4 \int_\Omega |\nabla \tilde{\varphi}|^2, \tag{3.11}$$

which, when combined with Lemma 2.1, implies that $\|\tilde{\varphi}\|_{\mathcal{H}^0} + \|\nabla_{\mathcal{A}} \tilde{\varphi}\|_{\mathcal{H}^0} \lesssim \|\varphi\|_{H^{1/2}(\Sigma)}$. Then

$$\begin{aligned} \int_\Sigma \varphi v \cdot \mathcal{N} &= \int_\Sigma J \mathcal{A}_{ij} v_i \varphi (e_j \cdot e_3) = \int_\Omega \operatorname{div}_{\mathcal{A}}(v \tilde{\varphi}) J = \int_\Omega \tilde{\varphi} \operatorname{div}_{\mathcal{A}} v J + v \cdot \nabla_{\mathcal{A}} \tilde{\varphi} J \\ &\leq \|\tilde{\varphi}\|_{\mathcal{H}^0} \|\operatorname{div}_{\mathcal{A}} v\|_{\mathcal{H}^0} + \|v\|_{\mathcal{H}^0} \|\nabla_{\mathcal{A}} \tilde{\varphi}\|_{\mathcal{H}^0} \lesssim \|\varphi\|_{H^{1/2}(\Sigma)} (\|v\|_{\mathcal{H}^0} + \|\operatorname{div}_{\mathcal{A}} v\|_{\mathcal{H}^0}). \end{aligned} \tag{3.12}$$

The desired bound follows from this inequality by taking the supremum over all φ so that $\|\varphi\|_{H^{1/2}(\Sigma)} \leq 1$. □

Remark 3.4. *Recall the space $\mathcal{Y}(t) \subset \mathcal{H}^0(t)$, defined by (2.5). It can be shown that if $v \in \mathcal{Y}(t)$, then $\operatorname{div}_{\mathcal{A}} v = 0$ in the weak sense, so that Lemma 3.3 implies that $v \cdot \mathcal{N} \in H^{-1/2}(\Sigma)$ and $v \cdot \nu \in H^{-1/2}(\Sigma_b)$. Moreover, since the elements of $\mathcal{Y}(t)$ are orthogonal to each $\nabla_{\mathcal{A}} \varphi$ for $\varphi \in {}^0H^1(\Omega)$, we find that $v \cdot \nu = 0$ on Σ_b.*

3.2. **The $\mathcal{A}$−Stokes problem.** In order to derive the regularity for our solutions to (1.16), we will first need to study the regularity of the corresponding stationary problem

$$\begin{cases} -\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = F^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = F^2 & \text{in } \Omega \\ S_{\mathcal{A}}(p,u)\mathcal{N} = F^3 & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.13}$$

Since this problem is stationary, we will temporarily ignore the time dependence of $\eta, \mathcal{A}$, etc.

We are interested in the regularity theory for strong solutions to (3.13), but before discussing that, we shall mention the weak formulation. Our method of solution is similar to that of [28, 6, 12]; we utilize Proposition 2.9 to introduce p after first solving a pressureless problem. Suppose $F^1 \in (\mathcal{H}^1)^*$, $F^2 \in \mathcal{H}^0$, $F^3 \in H^{-1/2}(\Sigma)$. We say $(u,p) \in \mathcal{H}^1 \times \mathcal{H}^0$ is a weak solution to (3.13) if $\operatorname{div}_{\mathcal{A}} u = F^2$ a.e. in Ω, and

$$\frac{1}{2}(u,v)_{\mathcal{H}^1} - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \langle F^1, v\rangle_{(\mathcal{H}^1)^*} - \langle F^3, v\rangle_{-1/2} \text{ for all } v \in \mathcal{H}^1, \tag{3.14}$$

where $\langle \cdot,\cdot\rangle_{(\mathcal{H}^1)^*}$ denotes the dual pairing in $\mathcal{H}^1$ and $\langle \cdot,\cdot\rangle_{-1/2}$ denotes the dual pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$.

Proposition 3.5. *Suppose $F^1 \in (\mathcal{H}^1)^*$, $F^2 \in \mathcal{H}^0$, $F^3 \in H^{-1/2}(\Sigma)$. Then there is exists a unique weak solution $(u,p) \in \mathcal{H}^1 \times \mathcal{H}^0$ to* (3.14).

Proof. By Lemma 2.6, there exists a $\bar{u} \in \mathcal{H}^1$ so that $\operatorname{div}_{\mathcal{A}} \bar{u} = F^2$. We may then switch unknowns to $w = u - \bar{u}$ so that the weak formulation for w is $\operatorname{div}_{\mathcal{A}} w = 0$ and

$$\frac{1}{2}(w,v)_{\mathcal{H}^1} - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = -\frac{1}{2}(\bar{u},v)_{\mathcal{H}^1} + \langle F^1, v\rangle_{(\mathcal{H}^1)^*} - \langle F^3, v\rangle_{-1/2} \text{ for all } v \in \mathcal{H}^1. \tag{3.15}$$

To solve for w without p we restrict the test functions to $v \in \mathcal{X}$ so that the second term on the left vanishes. A straightforward application of the Riesz representation theorem then provides a unique $w \in \mathcal{X}$ satisfying

$$\frac{1}{2}(w,v)_{\mathcal{H}^1} = -\frac{1}{2}(\bar{u},v)_{\mathcal{H}^1} + \langle F^1, v\rangle_{(\mathcal{H}^1)^*} - \langle F^3, v\rangle_{-1/2} \text{ for all } v \in \mathcal{X}. \tag{3.16}$$

To introduce the pressure, p, we define $\Lambda \in (\mathcal{H}^1)^*$ as the difference between the left and right sides of (3.16). Then $\Lambda(v) = 0$ for all $v \in \mathcal{X}$, so by Proposition 2.9, there exists a unique $p \in \mathcal{H}^0$ satisfying $(p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \Lambda(v)$ for all $v \in \mathcal{H}^1$, which is equivalent to (3.15). □

The regularity gain available for solutions to (3.13) is limited by the regularity of the coefficients of the operators $\Delta_{\mathcal{A}}, \nabla_{\mathcal{A}}, \operatorname{div}_{\mathcal{A}}$, and hence by the regularity of η. In the next result we establish the strong solvability of (3.13) and present some elliptic estimates, but we do not yet seek the optimal regularity.

Lemma 3.6. *Suppose that $\eta \in H^{k+1/2}(\Sigma)$ for $k \ge 3$ is as small as in Remark 3.2 so that the mapping Φ defined by* (1.7) *is a C^1 diffeomorphism of Ω to $\Omega' = \Phi(\Omega)$. If $F^1 \in H^0(\Omega)$, $F^2 \in H^1(\Omega)$, and $F^3 \in H^{1/2}(\Sigma)$, then the problem* (3.13) *admits a unique strong solution $(u,p) \in H^2(\Omega) \times H^1(\Omega)$, i.e. u,p satisfy* (3.13) *a.e. in Ω, Σ, and Σ_b. Moreover, for $r = 2,\dots,k-1$ we have the estimate*

$$\|u\|_r + \|p\|_{r-1} \lesssim C(\eta)\left(\left\|F^1\right\|_{r-2} + \left\|F^2\right\|_{r-1} + \left\|F^3\right\|_{r-3/2} \right), \tag{3.17}$$

whenever the right hand side is finite, where $C(\eta)$ is a constant depending on $\|\eta\|_{k+1/2}$.

Proof. We transform the problem (3.13) to one on $\Omega' = \Phi(\Omega)$ by introducing the unknowns v,q according to $u = v \circ \Phi$, $p = q \circ \Phi$. Then v,q should be solutions to the usual Stokes problem on

$\Omega' = \{-b(y_1,y_2) \le y_3 \le \eta(y_1,y_2)\}$ with upper boundary $\Sigma' = \{y_3 = \eta\}$:

$$\begin{cases} -\Delta v + \nabla q = G^1 = F^1 \circ \Phi^{-1} & \text{in } \Omega' \\ \operatorname{div} v = G^2 = F^2 \circ \Phi^{-1} & \text{in } \Omega' \\ (qI - \mathbb{D}v)\mathcal{N} = G^3 = F^3 \circ \Phi^{-1} & \text{on } \Sigma' \\ v = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.18}$$

Note that, according to Lemma 3.1, $G^1 \in H^0(\Omega')$, $G^2 \in H^1(\Omega')$, and $G^3 \in H^{1/2}(\Sigma')$. We claim that there exist unique $v \in H^2(\Omega')$, $q \in H^1(\Omega')$, solving problem (3.18) with

$$\|v\|_{H^2(\Omega')} + \|q\|_{H^1(\Omega')} \lesssim C(\eta)\left(\left\|G^1\right\|_{H^0(\Omega')} + \left\|G^2\right\|_{H^1(\Omega')} + \left\|G^3\right\|_{H^{1/2}(\Sigma')} \right), \tag{3.19}$$

for $C(\eta)$ a constant depending on $\|\eta\|_{k+1/2}$. Let us assume for the moment that the claim is true; we first show how (3.17) follows from the claim, and then turn to its proof.

To go from $H^2 \times H^1$ to higher regularity, we appeal to the theory of elliptic systems with complementary boundary conditions, developed in [4]. It is well-known that the Stokes system (3.18) is such an elliptic system. Theorem 10.5 of [4] provides estimates in bounded domains, but we may argue as in Lemma 3.3 of [6] to transform the localized estimates into estimates in all of Ω', provided that the boundary Σ' is sufficiently smooth. In order for estimates of the form (3.17) to hold for $r = 2, \dots, k-1$, [4] requires that Σ' be C^{k-1}, which is satisfied since $\eta \in C^{k-1,1/2}(\Sigma)$. Hence, for $r = 2, \dots, k-1$,

$$\|v\|_{H^r(\Omega)'} + \|q\|_{H^{r-1}(\Omega')} \lesssim C(\eta)\left(\left\|G^1\right\|_{H^{r-2}(\Omega')} + \left\|G^2\right\|_{H^{r-1}(\Omega)'} + \left\|G^3\right\|_{H^{r-3/2}(\Sigma')} \right), \tag{3.20}$$

for $C(\eta)$ a constant depending on $\|\eta\|_{k+1/2}$, whenever the right side is finite.

We now transform back to Ω with $u = v \circ \Phi$, $p = q \circ \Phi$. It is readily verified that u, p are strong solutions of (3.13). Since Φ satisfies $\nabla\Phi - I \in H^k$, Lemma 3.1 and (3.20) imply that

$$\|u\|_r + \|p\|_{r-1} \lesssim C(\eta)\left(\left\|F^1\right\|_{r-2} + \left\|F^2\right\|_{r-1} + \left\|F^3\right\|_{r-3/2} \right). \tag{3.21}$$

for $r = 2, \dots, k-1$ whenever the right side is finite. This is (3.17).

We now turn to the proof of the above claim, which employs ideas from [6]. To demonstrate the existence of $H^2 \times H^1$ solutions of (3.18), we first consider the special case in which $G^2 = 0$, $G^3 = 0$, and $G^1 \in H^0(\Omega')$ is arbitrary. In this case, we may argue as in Lemma 3.3 of [6] (which in turn invokes [28]) to deduce the existence of a unique solution to (3.18) satisfying (3.19) with $G^2 = 0$, $G^3 = 0$.

To handle the case of non-vanishing G^2 and G^3, we construct some special auxiliary functions that allow us to reduce to the special case. First, there exists a $v^1 \in H^2(\Omega') \cap {}_0H^1(\Omega')$ so that $\operatorname{div} v^1 = G^2 \in H^1(\Omega')$ and

$$\left\|v^1\right\|_{H^2(\Omega')} \lesssim \left\|G^2\right\|_{H^1(\Omega')}. \tag{3.22}$$

The existence of v^1 may be established as in Lemma 3.3 and Section 4 of [6]. To deal with the boundary term G^3 we first need some projections. For a vector field $X : \Sigma' \to \mathbb{R}^3$ let us write ΠX for the vector field so that $\Pi X(y)$ is the orthogonal projection of $X(y)$ onto the space of vectors orthogonal to $\mathcal{N}(y)$, and let us write $\Pi^\perp X(y)$ for the orthogonal projection onto the line generated by $\mathcal{N}(y)$. Our second special function is $v^2 \in H^2(\Omega') \cap {}_0H^1_\sigma(\Omega')$ that satisfies $\Pi(-\mathbb{D}v^2\mathcal{N}) = \Pi(G^3 + \mathbb{D}v^1\mathcal{N})$ and

$$\left\|v^2\right\|_{H^2(\Omega')} \lesssim C(\eta)\left(\left\|G^3 + \mathbb{D}v^1\mathcal{N}\right\|_{H^{1/2}(\Sigma')} \right) \lesssim C(\eta)\left(\left\|G^2\right\|_{H^1(\Omega')} + \left\|G^3\right\|_{H^{1/2}(\Sigma')} \right). \tag{3.23}$$

The construction of v^2 may be carried out through a simple modification of the proof of Lemma 4.2 in [6], working in Sobolev spaces defined on Ω' rather than $\Omega' \times (0,T)$. The third special function is $q^1 \in H^1(\Omega')$ that satisfies $q|_{\Sigma'} = \Pi^\perp(G^3 + \mathbb{D}v^1\mathcal{N})$ and

$$\left\|q^1\right\|_{H^1(\Omega')} \lesssim C(\eta)\left(\left\|G^3 + \mathbb{D}v^1\mathcal{N}\right\|_{H^{1/2}(\Sigma')} \right) \lesssim C(\eta)\left(\left\|G^2\right\|_{H^1(\Omega')} + \left\|G^3\right\|_{H^{1/2}(\Sigma')} \right). \tag{3.24}$$

The existence of q^1 follows from the usual trace and extension theory since $G^3 + \mathbb{D}v^1\mathcal{N} \in H^{1/2}(\Sigma')$.

Now, with v^1, v^2 and q^1 in hand, we reduce the solvability of (3.18) with the estimate (3.19) to the special case discussed above. The construction of these special functions guarantees that $w = v - v^1 - v^2$, $Q = q - q^1$ should satisfy

$$\begin{cases} -\Delta w + \nabla Q = G^1 + \Delta v^1 + \Delta v^2 - \nabla p^2 \in H^0(\Omega') & \text{in } \Omega' \\ \operatorname{div} w = 0 & \text{in } \Omega' \\ (QI - \mathbb{D}w)\mathcal{N} = 0 & \text{on } \Sigma' \\ w = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.25}$$

As above, there exist unique w, Q solving this so that

$$\|w\|_{H^2(\Omega')} + \|Q\|_{H^1(\Omega')} \lesssim C(\eta) \left\|G^1 + \Delta v^1 + \Delta v^2 - \nabla p^2\right\|_{H^0(\Omega')}. \tag{3.26}$$

The existence of unique v, q solving (3.18) is immediate, and the estimate (3.19) follows by combining (3.26) with (3.22)–(3.24), finishing the proof of the claim.

□

It turns out that we can achieve somewhat more of a regularity gain than is mentioned in Lemma 3.6 by making a smallness assumption on η. The smallness allows us to view the problem (3.13) as a perturbation of the Stokes problem on Ω. For this problem there is no constraint to regularity gain since the coefficients are constant and the boundary is smooth. This allows us to shift the constraint of regularity gain to the regularity of η in $H^{k+1/2}$ rather than in C^{k-1}. We note that although we require $\eta \in H^{k+1/2}$, the smallness assumption is written in terms of $\|\eta\|_{k-1/2}$.

Proposition 3.7. *Let $k \geq 4$ be an integer and suppose that $\eta \in H^{k+1/2}$. There exists $\varepsilon_0 > 0$ so that if $\|\eta\|_{k-1/2} \leq \varepsilon_0$, then solutions to* (3.13) *satisfy*

$$\|u\|_r + \|p\|_{r-1} \leq C\left(\left\|F^1\right\|_{r-2} + \left\|F^2\right\|_{r-1} + \left\|F^3\right\|_{r-3/2}\right) \tag{3.27}$$

for $r = 2, \dots, k$, whenever the right side is finite. Here C is a constant that does not depend on η.

In the case $r = k+1$, solutions to (3.13) *satisfy*

$$\begin{aligned} \|u\|_{k+1} + \|p\|_k \leq C\left(\left\|F^1\right\|_{k-1} + \left\|F^2\right\|_k + \left\|F^3\right\|_{k-1/2}\right) \\ + C\|\eta\|_{k+1/2}\left(\left\|F^1\right\|_2 + \left\|F^2\right\|_3 + \left\|F^3\right\|_{5/2}\right). \end{aligned} \tag{3.28}$$

Proof. In the case that $\Sigma = \mathbb{R}^2$, we let $\rho \in C_c^\infty(\mathbb{R}^2)$ be such that $\operatorname{supp}(\rho) \subset B(0,2)$ and $\rho(x) = 1$ for $x \in B(0,1)$. For $m \in \mathbb{N}$ define η^m by $\mathcal{F}\eta^m(\xi) = \rho(\xi/m)\mathcal{F}\eta(\xi)$, where $\mathcal{F}$ denotes the Fourier transform. Clearly, for each m, $\eta^m \in H^j(\Sigma)$ for all $j \geq 0$, and also $\eta^m \to \eta$ in $H^{k-1/2}(\Sigma)$ (and in $H^{k+1/2}(\Sigma)$ if $\eta \in H^{k+1/2}(\Sigma)$) as $m \to \infty$. In the periodic case, we similarly define η^m by throwing away high frequencies: $\mathcal{F}\eta^m(n) = 0$ for $|n| \geq m$. In this case η^m has the same convergence properties as before. Let $\mathcal{A}^m$ and $\mathcal{N}^m$ be defined in terms of η^m. Initially let ε_0 be small enough so that η^m is as small as in Remark 3.2. This allows the mapping Φ^m defined by η^m to be a C^1 diffeomorphism.

Consider the problem (3.13) with $\mathcal{A}$ and $\mathcal{N}$ replaced with $\mathcal{A}^m$ and $\mathcal{N}^m$. Since $\eta^m \in H^{k+5/2}(\Sigma)$, we may apply Lemma 3.6 to deduce the existence of a unique pair (u^m, p^m) that solve (3.13) (with $\mathcal{A}^m, \mathcal{N}^m$) and that satisfy

$$\|u^m\|_r + \|p^m\|_{r-1} \lesssim C(\|\eta^m\|_{k+5/2})\left(\left\|F^1\right\|_{r-2} + \left\|F^2\right\|_{r-1} + \left\|F^3\right\|_{r-3/2}\right) \tag{3.29}$$

for $r = 2, \dots, k+1$, whenever the right hand side is finite. We rewrite the equations (3.13) as a perturbation of the usual Stokes equations on Ω:

$$
\begin{cases} -\Delta u^m + \nabla p^m = F^1 + G^{1,m} & \text{in } \Omega \\ \operatorname{div} u^m = F^2 + G^{2,m} & \text{in } \Omega \\ (p^m I - \mathbb{D} u^m) e_3 = F^3 + G^{3,m} & \text{on } \Sigma \\ u^m = 0 & \text{on } \Sigma_b. \end{cases} \tag{3.30}
$$

Suppose that $\|\eta^m\|_{k+1/2} \le 1$, which implies that $\|\eta^m\|_{k+1/2}^\ell \le \|\eta^m\|_{k+1/2}$ for any $\ell \ge 1$. This fact and a straightforward calculation reveal that

$$
\begin{aligned} \left\| G^{1,m} \right\|_{r-2} &\le C \|\eta^m\|_{k-1/2} \left(\|u^m\|_r + \|p^m\|_{r-1} \right), \\ \left\| G^{2,m} \right\|_{r-1} &\le C \|\eta^m\|_{k-1/2} \|u^m\|_r , \end{aligned} \tag{3.31}
$$

and

$$
\begin{aligned} \left\| G^{3,m} \right\|_{H^{r-3/2}(\Sigma)} &\le C \|\eta^m\|_{k-1/2} \left(\|u^m\|_{H^{r-1/2}(\Sigma)} + \|p^m\|_{H^{r-3/2}(\Sigma)} \right) \\ &\le C \|\eta^m\|_{k-1/2} \left(\|u^m\|_r + \|p^m\|_{r-1} \right) \end{aligned} \tag{3.32}
$$

for $r = 2, \dots, k$ and a constant $C > 0$ independent of η and m. In the case $r = k+1$ a minor variant of this argument shows that

$$
\begin{aligned} \left\| G^{1,m} \right\|_{k-1} + \left\| G^{2,m} \right\|_{k} + \left\| G^{3,m} \right\|_{H^{k-1/2}(\Sigma)} &\le C \|\eta^m\|_{k-1/2} \left(\|u^m\|_{k-1} + \|p^m\|_k \right) \\ &\quad + C \|\eta^m\|_{k+1/2} \|u^m\|_{7/2} \end{aligned} \tag{3.33}
$$

for C independent of η and m. The key to this variant is that nowhere in the terms $G^{i,m}$ do there occur products of the highest derivative count of both η^m and u^m (or p^m). Note that the right sides of (3.31), (3.32), and (3.33) are finite by virtue of the estimate (3.29).

Since the boundaries Σ and Σ_b are smooth and the problem (3.30) has constant coefficients, we may argue as in Lemma 3.6, employing the elliptic estimates of [4] as done in Lemma 3.3 of [6], to arrive at the estimate

$$
\|u^m\|_r + \|p^m\|_{r-1} \le C \left(\left\| F^1 + G^{1,m} \right\|_{r-2} + \left\| F^2 + G^{2,m} \right\|_{r-1} + \left\| F^3 + G^{3,m} \right\|_{r-3/2} \right) \tag{3.34}
$$

for $r = 2, \dots, k+1$ and for $C > 0$ independent of η and m. We may then combine (3.31)–(3.32) with (3.34) to find that, if $\|\eta^m\|_{k-1/2} \le 1$, then

$$
\begin{aligned} \|u^m\|_r + \|p^m\|_{r-1} &\le C \left(\left\| F^1 \right\|_{r-2} + \left\| F^2 \right\|_{r-1} + \left\| F^3 \right\|_{r-3/2} \right) \\ &\quad + C \|\eta^m\|_{k-1/2} \left(\|u^m\|_r + \|p^m\|_{r-1} \right) + \delta_{r,k+1} C \|\eta^m\|_{k+1/2} \|u^m\|_{7/2} . \end{aligned} \tag{3.35}
$$

On the right side of (3.35) we have written $\delta_{r,k+1}$ for the quantity that vanishes when $r \neq k+1$ and is unity when $r = k+1$.

We now derive the estimate (3.27). Since $\eta^m \to \eta$ in $H^{k-1/2}$ we may assume that m is sufficiently large so that $\|\eta^m\|_{k-1/2} \le 2 \|\eta\|_{k-1/2}$. Then if

$$
\|\eta\|_{k-1/2} \le \min \left\{ \frac{1}{4C}, \frac{1}{2} \right\} := \varepsilon_0 \tag{3.36}
$$

for $C > 0$ the constant appearing on the right side of (3.35), the bound (3.35) may be rearranged to get

$$
\|u^m\|_r + \|p^m\|_{r-1} \le 2C \left(\left\| F^1 \right\|_{r-2} + \left\| F^2 \right\|_{r-1} + \left\| F^3 \right\|_{r-3/2} \right), \tag{3.37}
$$

for $r = 2, \dots, k$ when the right side is finite.

The bound (3.37) implies that the sequence $\{u^m, p^m\}$ is uniformly bounded in $H^r \times H^{r-1}$, so up to the extraction of a subsequence, $u^m \rightharpoonup u^0$ weakly in $H^r(\Omega)$ and $p^m \rightharpoonup p^0$ weakly in $H^{r-1}(\Omega)$. Since $\eta^m \to \eta$ in $H^{k-1/2}(\Sigma)$, we also have that $\mathcal{A}^m - \mathcal{A} \to 0$, $J^m - J \to 0$ in

$H^{k-1}(\Omega)$, and $\mathcal{N}^m - \mathcal{N} \to 0$ in $H^{k-3/2}(\Sigma)$. We multiply the equation $\operatorname{div}_{\mathcal{A}} u^m = F^2$ by $J^m w$ for $w \in C_c^\infty(\Omega)$ to see that

$$\int_\Omega F^2 w J^m = \int_\Omega \operatorname{div}_{\mathcal{A}^m}(u^m) w J^m \\ = -\int_\Omega u^m \cdot \nabla_{\mathcal{A}^m} w J^m \to -\int_\Omega u^0 \cdot \nabla_{\mathcal{A}} w J = \int_\Omega \operatorname{div}_{\mathcal{A}}(u^0) w J, \tag{3.38}$$

from which we deduce that $\operatorname{div}_{\mathcal{A}}(u^0) = F^2$. Then we multiply the first equation in (3.13) (with u^m, etc) by wJ^m for $w \in {}_0H^1(\Omega)$ and integrate by parts to see that

$$\int_\Omega \frac{1}{2}\mathbb{D}_{\mathcal{A}^m} u^m : \mathbb{D}_{\mathcal{A}^m} w J^m - p^m \operatorname{div}_{\mathcal{A}^m}(w) J^m = \int_\Omega F^1 \cdot w J^m - \int_\Sigma F^3 \cdot w. \tag{3.39}$$

Passing to the limit $m \to \infty$, we deduce that

$$\int_\Omega \frac{1}{2}\mathbb{D}_{\mathcal{A}} u^0 : \mathbb{D}_{\mathcal{A}} w J - p^0 \operatorname{div}_{\mathcal{A}} w J = \int_\Omega F^1 \cdot w J - \int_\Sigma F^3 \cdot w, \tag{3.40}$$

which reveals, upon integrating by parts again, that u^0, p^0 satisfy (3.13). Since u, p are the unique solutions to (3.13), we have that $u = u^0$, $p = p^0$. This, weak lower semi-continuity, and the bound (3.37) imply (3.27).

Now we derive the estimate (3.28), supposing that $F^1 \in H^{k-1}$, $F^2 \in H^k$, and $F^3 \in H^{k-1/2}$. The bound (3.37) with $r = 4$ implies that

$$\|u^m\|_4 \le 2C\left(\|F^1\|_2 + \|F^2\|_3 + \|F^3\|_{5/2}\right) < \infty. \tag{3.41}$$

Since $\eta^m \to \eta$ in $H^{k+1/2}$, we are free to assume that m is sufficiently large so that $\|\eta^m\|_{k+1/2} \le 2\|\eta\|_{k+1/2}$. Then if $\|\eta\|_{k-1/2} \le \varepsilon_0$ we may use (3.35) and (3.41) to deduce that

$$\|u^m\|_{k+1} + \|p^m\|_k \le 2C\left(\|F^1\|_{k-1} + \|F^2\|_k + \|F^3\|_{k-1/2}\right) \\ + 4C\|\eta\|_{k+1/2}\left(\|F^1\|_2 + \|F^2\|_3 + \|F^3\|_{5/2}\right). \tag{3.42}$$

We may then argue as above to extract weak limits, show that the limits equal u and p, and then deduce that the bound (3.42) holds with u^m and p^m replaced by u and p. This is (3.28).

□

3.3. **The $\mathcal{A}-$Poisson problem.** Next we consider the scalar elliptic problem

$$\begin{cases} \Delta_{\mathcal{A}} p = f^1 & \text{in } \Omega \\ p = f^2 & \text{on } \Sigma \\ \nabla_{\mathcal{A}} p \cdot \nu = f^3 & \text{on } \Sigma_b, \end{cases} \tag{3.43}$$

where ν is the outward-pointing normal on Σ_b. We will eventually discuss the strong solvability of this problem, but first we consider the weak formulation of the problem. We define a scalar $\mathcal{H}^1$ in a natural way through the norm

$$\|f\|_{\mathcal{H}^1}^2 = \int_\Omega J\,|\nabla_{\mathcal{A}} f|^2. \tag{3.44}$$

Note that $\|f\|_{\mathcal{H}^1}^2 = \left\|\sqrt{2} f e_1\right\|_{\mathcal{H}^1}$, where the right side is the $\mathcal{H}^1$ norm for vectors. Then Lemma 2.1 shows that this scalar norm generates the same topology as the usual scalar H^1 norm.

For the weak formulation we suppose $f^1 \in ({}^0H^1(\Omega))^*$, $f^2 \in H^{1/2}(\Sigma)$, and $f^3 \in H^{-1/2}(\Sigma_b)$. Let $\bar{p} \in H^1(\Omega)$ be an extension of f^2 so that $\operatorname{supp}(\bar{p}) \subset \{-(\inf b)/2 < x_3 \le 0\}$. We switch unknowns to $q = p - \bar{p}$. Then we can define a weak formulation of (3.43) by finding a $q \in {}^0H^1(\Omega)$ so that

$$(q, \varphi)_{\mathcal{H}^1} = -(\bar{p}, \varphi)_{\mathcal{H}^1} - \langle f^1, \varphi\rangle_* + \langle f^3, \varphi\rangle_{-1/2} \text{ for all } \varphi \in {}^0H^1(\Omega), \tag{3.45}$$

where $\langle\cdot,\cdot\rangle_*$ is the dual pairing with ${}^0H^1(\Omega)$ and $\langle\cdot,\cdot\rangle_{-1/2}$ is the dual pairing with $H^{1/2}(\Sigma_b)$. The existence and uniqueness of a solution to (3.45) follows from standard arguments, and the resulting $p = q + \bar{p} \in H^1(\Omega)$ satisfies

$$\|p\|_{\mathcal{H}^1}^2 \lesssim \left(\left\|f^1\right\|_{({}^0H^1(\Omega))^*}^2 + \left\|f^2\right\|_{H^{1/2}(\Sigma)}^2 + \left\|f^3\right\|_{H^{-1/2}(\Sigma_b)}^2 \right). \tag{3.46}$$

In the event that the action of f^1 is given in a more specific fashion, we will rewrite the PDE (3.43) to accommodate the structure of f^1. To make this precise, suppose that the action of f^1 on an element $\varphi \in {}^0H^1(\Omega)$ is given by

$$\langle f^1, \varphi\rangle_* = (g_0, \varphi)_{\mathcal{H}^0} + (G, \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} \tag{3.47}$$

for $(g_0, G) \in H^0(\Omega;\mathbb{R}) \times H^0(\Omega;\mathbb{R}^3)$ with $\|g_0\|_0^2 + \|G\|_0^2 = \left\|f^1\right\|_{({}^0H^1(\Omega))^*}^2$ (standard arguments show that it is always possible to uniquely write f^1 in this way). Then (3.45) may be rewritten as

$$(\nabla_{\mathcal{A}}p + G, \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} = -(g_0, \varphi)_{\mathcal{H}^0} + \langle f^3, \varphi\rangle_{-1/2} \text{ for all } \varphi \in {}^0H^1(\Omega). \tag{3.48}$$

We may take $\varphi \in C_c^\infty(\Omega)$ in this equality and integrate by parts to see that $\operatorname{div}_{\mathcal{A}}(\nabla_{\mathcal{A}}p + G) = g_0 \in \mathcal{H}^0$, which allows us to deduce from Lemma 3.3 that $(\nabla_{\mathcal{A}}p + G)\cdot\nu \in H^{-1/2}(\Sigma_b)$. This serves as motivation for us to say that p is a weak solution to the PDE

$$\begin{cases} \operatorname{div}_{\mathcal{A}}(\nabla_{\mathcal{A}}p + G) = g_0 \in H^0(\Omega) \\ p = f^2 \in H^{1/2}(\Sigma) \\ (\nabla_{\mathcal{A}}p + G)\cdot\nu = f^3 \in H^{-1/2}(\Sigma_b). \end{cases} \tag{3.49}$$

This way of writing the weak solution will be utilized later in Theorem 4.3. Note that when $f^1 \in H^0(\Omega)$, there is no need to make this distinction since then $G = 0$ and $f^1 = g_0$.

Our next result on this problem is the analogue of Lemma 3.6; it establishes the strong solvability of (3.43) and some regularity.

Lemma 3.8. *Suppose that $\eta \in H^{k+1/2}(\Sigma)$ for $k \geq 3$ is as small as in Remark 3.2 so that the mapping Φ defined by* (1.7) *is a C^1 diffeomorphism of Ω to $\Omega' = \Phi(\Omega)$. If $f^1 \in H^0(\Omega)$, $f^2 \in H^{3/2}(\Sigma)$, and $f^3 \in H^{1/2}(\Sigma_b)$, then the problem* (3.43) *admits a unique strong solution $p \in H^2(\Omega)$. Moreover, for $r = 2,\dots,k-1$ we have the estimate*

$$\|p\|_r \lesssim C(\eta)\left(\left\|f^1\right\|_{r-2} + \left\|f^2\right\|_{r-1/2} + \left\|f^3\right\|_{r-3/2} \right), \tag{3.50}$$

whenever the right hand side is finite, where $C(\eta)$ is a constant depending on $\|\eta\|_{k+1/2}$.

Proof. If $f^2 \in H^{r-1/2}(\Sigma)$ for $r = 2,\dots,k-1$, there exists a $\psi \in H^r(\Omega)$ so that $\psi|_\Sigma = f^2$, $\operatorname{supp}(\psi) \subset \{-(\inf b)/2 < x_3 \leq 0\}$, and $\|\psi\|_r \lesssim \left\|f^2\right\|_{r-1/2}$. Writing $p = q + \psi$, the problem (3.43) may be rewritten for the unknown q as

$$\begin{cases} \Delta_{\mathcal{A}}q = f^1 + g^1 & \text{in } \Omega \\ q = 0 & \text{on } \Sigma \\ \nabla_{\mathcal{A}}q\cdot\nu = f^3 & \text{on } \Sigma_b, \end{cases} \tag{3.51}$$

where $g^1 = -\Delta_{\mathcal{A}}\psi \in H^{r-2}$.

The problem (3.51) may be solved as in Lemma 3.6 by transforming to the domain Ω', where the problem for $Q = q\circ\Phi^{-1}$ becomes $\Delta Q = (f^1 + g^1)\circ\Phi^{-1}$ in Ω' with boundary conditions $Q = 0$ on Σ' and $\nabla Q\cdot\nu = f^3\circ\Phi^{-1}$ on Σ_b. The existence of a unique solution to this problem is established in the non-periodic case in Lemma 2.8 of [6], and estimates of the form (3.50) for Q hold by virtue of the elliptic estimates in [3], adapted to Ω' as in [6]. This method may be adapted easily to the periodic case as well. Then the existence and uniqueness of a solution to (3.43) satisfying (3.50) follows by transforming to $q = Q\circ\Phi$ on Ω for a solution to (3.51) and then applying Lemma 3.1. □

Our next result is the analogue of Proposition 3.7 for the problem (3.43). For our purposes, we only need a regularity gain up to k, and this is less important than the estimate in terms of a constant independent of η. Notice again that the smallness assumption is stated in $H^{k-1/2}$ even though we require $\eta \in H^{k+1/2}$.

Proposition 3.9. *Let $k \geq 4$ be an integer and suppose that $\eta \in H^{k+1/2}$. There exists $\varepsilon_0 > 0$ so that if $\|\eta\|_{k-1/2} \leq \varepsilon_0$, then solutions to* (3.43) *satisfy*

$$\|p\|_r \leq C \left(\left\|f^1\right\|_{r-2} + \left\|f^2\right\|_{r-1/2} + \left\|f^3\right\|_{r-3/2} \right) \tag{3.52}$$

for $r = 2, \ldots, k$, whenever the right side is finite. Here C is a constant that does not depend on η.

Proof. The proof is similar to that of Proposition 3.7. We smooth η to get η^m and solve (3.43) with $\mathcal{A}$ replaced with $\mathcal{A}^m$. Then we rewrite the problem as a perturbation of the Poisson problem

$$\begin{cases} \Delta p^m = f^1 + g^{1,m} & \text{in } \Omega \\ p^m = f^2 & \text{on } \Sigma \\ \nabla p^m \cdot \nu = f^3 + g^{3,m} \text{on } \Sigma_b. \end{cases} \tag{3.53}$$

The constants in the elliptic estimates for this problem do not depend on η^m, and we may estimate $g^{i,m}$ in terms of p^m. Then if $\|\eta\|_{k-1/2} \leq \varepsilon_0$ for some ε_0 sufficiently small, we can absorb the highest Sobolev norms on the right side of the elliptic estimate into the left side, and we deduce (3.52) for p^m. Then we pass to the limit $m \to \infty$.

□

4. Solving the time-dependent problem (1.16)

4.1. **The weak solution.** In our analysis of problem (1.16) we will employ two notions of solution: weak and strong. The definition of a weak solution to (1.16) is motivated by assuming the existence of a smooth solution to (1.16), multiplying by Jv for $v \in \mathcal{H}^1_T$, integrating over Ω by parts, and then in time from 0 to T to see that

$$(\partial_t u, v)_{\mathcal{H}^0_T} + \frac{1}{2} (u, v)_{\mathcal{H}^1_T} - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} = \left(F^1, v\right)_{\mathcal{H}^0_T} - \left(F^3, v\right)_{0,\Sigma,T} \tag{4.1}$$

for $\left(F^3, v\right)_{0,\Sigma,T} = \int_0^T \int_\Sigma F^3 \cdot v$. Suppose that

$$F^1 \in (\mathcal{H}^1_T)^*, F^3 \in L^2([0,T]; H^{-1/2}(\Sigma)), \text{ and } u_0 \in \mathcal{Y}(0), \tag{4.2}$$

where $\mathcal{Y}(0)$ is defined by 2.5. Then our definition of a weak solution of (1.16) requires only that a relaxed form of (4.1) holds. In particular, we say that (u,p) is a weak solution of (1.16) if

$$\begin{cases} u \in \mathcal{X}_T, \partial_t u \in (\mathcal{H}^1_T)^*, p \in \mathcal{H}^0_T, \\ \langle \partial_t u, v \rangle_* + \frac{1}{2} (u, v)_{\mathcal{H}^1_T} - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} = \langle F^1, v \rangle_* - \langle F^3, v \rangle_{-1/2} \quad \text{for every } v \in \mathcal{H}^1_T, \\ u(0) = u_0, \end{cases} \tag{4.3}$$

where $\langle \cdot, \cdot \rangle_*$ denotes the dual pairing between $(\mathcal{H}^1_T)^*$ and $\mathcal{H}^1_T$, and $\langle \cdot, \cdot \rangle_{-1/2}$ denotes the dual pairing between $L^2([0,T]; H^{-1/2}(\Sigma))$ and $L^2([0,T]; H^{1/2}(\Sigma))$. The third condition in (4.3) only makes sense in light of Lemma 2.4.

If we were to restrict our class of test functions in (4.3) to $v \in \mathcal{X}_T$, then the term $(p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T}$ would vanish, and we would be left with a "pressureless" weak formulation of the problem involving only the velocity field. This leads us to define a weak formulation without the pressure. Suppose the data satisfy (4.2). Then u is a pressureless weak solution of (1.16) if

$$\begin{cases} u \in \mathcal{X}_T, \partial_t u \in (\mathcal{H}^1_T)^*, \\ \langle \partial_t u, \psi \rangle_* + \frac{1}{2} (u, \psi)_{\mathcal{H}^1_T} = \langle F^1, \psi \rangle_* - \langle F^3, \psi \rangle_{-1/2} \quad \text{for every } \psi \in \mathcal{X}_T, \\ u(0) = u_0. \end{cases} \tag{4.4}$$

A more natural assumption for this formulation would be to require $\partial_t u \in (\mathcal{X}_T)^*$. However, since $\mathcal{X}_T \subset \mathcal{H}^1_T$, the usual theory of Hilbert spaces provides a unique operator $E : (\mathcal{X}_T)^* \to (\mathcal{H}^1_T)^*$ with the property that $Ef|_{\mathcal{X}_T} = f$ and $\|Ef\|_{(\mathcal{H}^1_T)^*} = \|f\|_{(\mathcal{X}_T)^*}$ for all $f \in (\mathcal{X}_T)^*$. Using this E, we regard $\partial_t u \in (\mathcal{X}_T)^*$ as an element of $(\mathcal{H}^1_T)^*$ in a natural way, which allows us to require that $\partial_t u \in (\mathcal{H}^1_T)^*$.

Since our aim is to construct solutions to (1.16) with high regularity, we will not need to directly construct weak solutions to (4.4) or (4.3). Rather, weak solutions to problems of this type will arise as a byproduct of our construction of strong solutions of (1.16). As such, for our purposes, it will suffice to ignore the issue of existence and only record a couple results on the properties of weak solutions.

We now record a result on some integral equalities and bounds satisfied by solutions of (4.4).

Lemma 4.1. *Suppose that u is a weak solution of* (4.4). *Then for a.e. $t \in [0,T]$,*

$$\frac{1}{2}\|u(t)\|^2_{\mathcal{H}^0(t)} + \frac{1}{2}\int_0^t \|u(s)\|^2_{\mathcal{H}^1(s)}\,ds = \frac{1}{2}\|u(0)\|^2_{\mathcal{H}^0(0)} + \int_0^t \langle F^1(s), u(s)\rangle_{(\mathcal{H}^1(s))^*} ds - \int_0^t \langle F^3(s), u(s)\rangle_{H^{-1/2}(\Sigma)} ds + \frac{1}{2}\int_0^t\int_\Omega |u(s)|^2\,\partial_t J(s) ds. \tag{4.5}$$

Also

$$\sup_{0\le t\le T}\|u(t)\|^2_{\mathcal{H}^0(t)} + \|u\|^2_{\mathcal{H}^1_T} \lesssim \exp\left(C_0(\eta)T\right)\left(\|u(0)\|^2_{\mathcal{H}^0(0)} + \left\|F^1\right\|^2_{(\mathcal{H}^1_T)^*} + \left\|F^3\right\|^2_{L^2H^{-1/2}}\right), \tag{4.6}$$

where $C_0(\eta) := \sup_{0\le t\le T}\|\partial_t JK\|_{L^\infty}$.

Proof. The identity (4.5) follows directly from (4.4) and Lemma 2.4 by using the test function $\psi = u\chi_{[0,t]} \in \mathcal{X}_T$, where $\chi_{[0,t]}$ is a temporal indicator function equal to unity on the interval $[0,t]$.

From (4.5) it is straightforward to derive the inequality

$$\frac{1}{2}\|u(t)\|^2_{\mathcal{H}^0(t)} + \frac{1}{2}\|u\|^2_{\mathcal{H}^1_t} \le \frac{1}{2}\|u(0)\|^2_{\mathcal{H}^0(0)} + \left\|F^1\right\|_{(\mathcal{H}^1_t)^*}\|u\|_{\mathcal{H}^1_t} + \left\|F^3\right\|_{L^2([0,t];H^{-1/2})}\|u\|_{L^2([0,t];H^{1/2})} + \frac{C_0(\eta)}{2}\|u\|^2_{\mathcal{H}^0_t}, \tag{4.7}$$

where we have written

$$\|u\|^2_{\mathcal{H}^k_t} = \int_0^t \|u(s)\|^2_{\mathcal{H}^k(s)}\,ds \text{ for } k = 0,1, \tag{4.8}$$

and similarly defined $\left\|F^1\right\|_{(\mathcal{H}^1_t)^*}$. Note that, according to Remark 2.3, we have the estimate $\|u\|_{H^{1/2}(\Sigma)} \le C\|u\|_{\mathcal{H}^1}$ for a constant C independent of η. This, inequality (4.7), and Cauchy's inequality then imply that

$$\frac{1}{2}\|u(t)\|^2_{\mathcal{H}^0(t)} + \frac{1}{8}\|u\|^2_{\mathcal{H}^1_t} \le \frac{1}{2}\|u(0)\|^2_{\mathcal{H}^0(0)} + 2\left\|F^1\right\|^2_{(\mathcal{H}^1_t)^*} + 2C\left\|F^3\right\|^2_{L^2([0,t];H^{-1/2})} + \frac{C_0(\eta)}{2}\|u\|^2_{\mathcal{H}^0_t}. \tag{4.9}$$

Then (4.6) follows from the differential inequality (4.9) and Gronwall's lemma. □

We can now parlay the results of Lemma 4.1 into uniqueness results for weak solutions to (4.4) and (4.3).

Proposition 4.2. *Weak solutions to* (4.4) *are unique. Also, weak solutions* (u,p) *to* (4.3) *are unique.*

Proof. If u^1 and u^2 are both weak solutions to (4.4), then $w = u^1 - u^2$ is a weak solution with $F^1 = 0$, $F^3 = 0$, and $w(0) = u^1(0) - u^2(0) = 0$. Then the bound (4.6) of Lemma 4.1 implies that $w = 0$; hence solutions to (4.4) are unique.

Now, if (u,p) are a weak solution to (4.3), then we can restrict to test functions $\psi \in \mathcal{X}_T$ to find that u is a weak solution to (4.4). As such, u is unique. To see that p is unique we define $\Lambda \in (\mathcal{H}^1_T)^*$ via

$$\Lambda(v) = \langle \partial_t u, v\rangle_* + \frac{1}{2}(u,v)_{\mathcal{H}^1_T} - \langle F^1, v\rangle_* + \langle F^3, v\rangle_{-1/2}. \tag{4.10}$$

Since u is a weak solution to (4.4), we have that $\Lambda(v) = 0$ for all $v \in \mathcal{X}_T$. Proposition 2.9 them implies that there exists a unique $q \in \mathcal{H}^0_T$ so that $(q, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} = \Lambda(v)$ for all $v \in \mathcal{H}^1_T$. It follows that $q = p$ and that p is unique.

□

4.2. **The strong solution.** Now we turn to the construction of strong solutions to (1.16). We will make stronger assumptions on the data F^1, F^3, u_0 than we made in the weak formulation (4.2). In particular, we will assume that the forcing functions satisfy

$$\begin{aligned} &F^1 \in L^2([0,T]; H^1(\Omega)), \partial_t F^1 \in L^2([0,T]; ({}_0H^1(\Omega))^*), \\ &F^3 \in L^2([0,T]; H^{3/2}(\Sigma)), \partial_t F^3 \in L^2([0,T]; H^{-1/2}(\Sigma)), \\ &F^1(0) \in H^0(\Omega), F^3(0) \in H^{1/2}(\Sigma). \end{aligned} \tag{4.11}$$

Note that, owing to Lemma 2.4, (4.11) implies that $F^1 \in C^0([0,T]; H^0(\Omega))$ and that $F^3 \in C^0([0,T]; H^{1/2}(\Sigma))$. The initial data will also be taken to be more regular; we take $u_0 \in H^2(\Omega) \cap \mathcal{X}(0)$.

The solution that we construct will satisfy (1.16) in the strong sense, but we will also show that $(D_t u, \partial_t p)$ satisfy an equation of the form (1.16) in the weak sense of (4.3). Here we define

$$D_t u := \partial_t u - Ru \text{ for } R := \partial_t M M^{-1} \tag{4.12}$$

with M the matrix defined by (2.22). We employ the operator D_t because it preserves the $\operatorname{div}_{\mathcal{A}}$ −free condition. Before turning to the result, we define the quantity

$$\mathcal{K}(\eta) := \sup_{0\le t\le T} \left(\|\eta\|^2_{9/2} + \|\partial_t \eta\|^2_{7/2} + \left\|\partial_t^2 \eta\right\|^2_{5/2} \right). \tag{4.13}$$

We also define an orthogonal projection onto the tangent space of the surface $\{x_3 = \eta_0\}$ according to

$$\Pi_0 v = v - (v\cdot \mathcal{N}_0)\mathcal{N}_0 \left|\mathcal{N}_0\right|^{-2} \tag{4.14}$$

for $\mathcal{N}_0 = (-\partial_1\eta_0, -\partial_2\eta_0, 1)$. By construction, $\Pi_0 v = 0$ if and only if $v \parallel \mathcal{N}_0$.

Theorem 4.3. *Suppose that F^1, F^3 satisfy* (4.11)*, that $u_0 \in H^2(\Omega)\cap\mathcal{X}(0)$, and that u_0, $F^3(0)$ satisfy the compatibility condition*

$$\Pi_0\left(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u_0 \mathcal{N}_0\right) = 0, \textit{where } \mathcal{N}_0 = (-\partial_1\eta_0, -\partial_2\eta_0, 1), \tag{4.15}$$

and Π_0 is the projection defined by (4.14)*. Further suppose that $\mathcal{K}(\eta)$ is less than the smaller of ε_0 from Lemma 2.1 and ε_0 from Proposition 3.7 (in particular, this requires $\mathcal{K}(\eta) \le 1$). Then there exists a unique strong solution (u,p) to* (1.16) *so that*

$$\begin{aligned} &u \in \mathcal{X}_T \cap C^0([0,T]; H^2(\Omega)) \cap L^2([0,T]; H^3(\Omega)), \\ &\partial_t u \in C^0([0,T]; H^0(\Omega)) \cap L^2([0,T]; H^1(\Omega)), \partial_t^2 u \in (\mathcal{H}^1_T)^*, \\ &p \in C^0([0,T]; H^1(\Omega)) \cap L^2([0,T]; H^2(\Omega)), \partial_t p \in L^2([0,T]; H^0(\Omega)). \end{aligned} \tag{4.16}$$

The solution satisfies the estimate

$$\begin{aligned} &\|u\|^2_{L^\infty H^2} + \|u\|^2_{L^2H^3} + \|\partial_t u\|^2_{L^\infty H^0} + \|\partial_t u\|^2_{L^2H^1} + \left\|\partial_t^2 u\right\|^2_{(\mathcal{H}^1_T)^*} + \|p\|^2_{L^\infty H^1} + \|\partial_t p\|^2_{L^2H^0} \\ &\lesssim (1+\mathcal{K}(\eta))\exp\left(C(1+\mathcal{K}(\eta))T\right)\Big(\|u_0\|^2_2 + \left\|F^1(0)\right\|^2_0 + \left\|F^3(0)\right\|^2_{1/2} \\ &\qquad + \left\|F^1\right\|^2_{L^2H^1} + \left\|\partial_t F^1\right\|^2_{L^2({}_0H^1(\Omega))^*} + \left\|F^3\right\|^2_{L^2H^{3/2}} + \left\|\partial_t F^3\right\|^2_{L^2H^{-1/2}}\Big), \end{aligned} \tag{4.17}$$

where C is a constant independent of η. The initial pressure, $p(0) \in H^1(\Omega)$, is determined in terms of $u_0, F^1(0), F^3(0)$ as the weak solution to

$$\begin{cases} \operatorname{div}_{\mathcal{A}_0}(\nabla_{\mathcal{A}_0} p(0) - F^1(0)) = -\operatorname{div}_{\mathcal{A}_0}(R(0)u_0) \in H^0(\Omega) \\ p(0) = (F^3(0) + \mathbb{D}_{\mathcal{A}_0} u_0 \mathcal{N}_0) \cdot \mathcal{N}_0 \, |\mathcal{N}_0|^{-2} \in H^{1/2}(\Sigma) \\ (\nabla_{\mathcal{A}_0} p(0) - F^1(0)) \cdot \nu = \Delta_{\mathcal{A}_0} u_0 \cdot \nu \in H^{-1/2}(\Sigma_b) \end{cases} \tag{4.18}$$

in the sense of (3.49)*. Also, $D_t u(0) = \partial_t u(0) - R(0) u_0$ satisfies*

$$D_t u(0) = \Delta_{\mathcal{A}_0} u_0 - \nabla_{\mathcal{A}_0} p(0) + F^1(0) - R(0) u_0 \in \mathcal{Y}(0), \tag{4.19}$$

where $\mathcal{Y}(0)$ is defined by (2.5)*.*

Moreover, $(D_t u, \partial_t p)$ satisfy

$$\begin{cases} \partial_t(D_t u) - \Delta_{\mathcal{A}}(D_t u) + \nabla_{\mathcal{A}}(\partial_t p) = D_t F^1 + G^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}}(D_t u) = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(\partial_t p, D_t u)\mathcal{N} = \partial_t F^3 + G^3 & \text{on } \Sigma \\ D_t u = 0 & \text{on } \Sigma_b, \end{cases} \tag{4.20}$$

in the weak sense of (4.3)*, where G^1, G^3 are defined by*

$$G^1 = -(R + \partial_t J K)\Delta_{\mathcal{A}} u - \partial_t R u + (\partial_t J K + R - R^T)\nabla_{\mathcal{A}} p + \operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru) + R\mathbb{D}_{\mathcal{A}} u + \mathbb{D}_{\partial_t \mathcal{A}} u) \tag{4.21}$$

with R^T denoting the matrix transpose of R, and

$$G^3 = \mathbb{D}_{\mathcal{A}}(Ru)\mathcal{N} - (pI - \mathbb{D}_{\mathcal{A}} u)\partial_t \mathcal{N} + \mathbb{D}_{\partial_t \mathcal{A}} u \mathcal{N}. \tag{4.22}$$

Here the inclusions (4.16) *guarantee that G^1 and G^3 satisfy the same inclusions as F^1, F^3 listed in* (4.11)*, whereas* (4.18) *guarantees that the initial data $D_t u(0) \in \mathcal{Y}(0)$.*

Proof. The result will be established by first solving a pressureless problem and then introducing the pressure via Proposition 2.9. For the pressureless problem we will make use of the Galerkin method. We divide the proof into several steps.

Step 1 – The Galerkin setup

In order to utilize the Galerkin method, we must first construct a countable basis of $H^2(\Omega) \cap \mathcal{X}(t)$ for each $t \in [0,T]$. Since the requirement $\operatorname{div}_{\mathcal{A}} v = 0$ is time-dependent, any basis of this space must also be time-dependent. For each $t \in [0,T]$, the space $H^2(\Omega) \cap \mathcal{X}(t)$ is separable, so the existence of a countable basis is not an issue. The technical difficulty is that, in order for the basis to be useful in the Galerkin method, we must be able differentiate the basis elements in time, and we must be able to express these time derivatives in terms of finitely many basis elements. Fortunately, it is possible to overcome this difficulty by employing the matrix $M(t)$, defined by (2.22).

Since $H^2(\Omega) \cap {}_0H^1_\sigma(\Omega)$ is separable, it possesses a countable basis $\{w^j\}_{j=1}^\infty$. Note that this basis is not time-dependent. Define $\psi^j = \psi^j(t) := M(t) w^j$. According to Proposition 2.5, $\psi^j(t) \in H^2(\Omega) \cap \mathcal{X}(t)$, and $\{\psi^j(t)\}_{j=1}^\infty$ is a basis of $H^2(\Omega) \cap \mathcal{X}(t)$ for each $t \in [0,T]$. Moreover,

$$\partial_t \psi^j(t) = \partial_t M(t) w^j = \partial_t M(t) M^{-1}(t) M(t) w^j = \partial_t M(t) M^{-1}(t) \psi^j(t) := R(t)\psi^j(t), \tag{4.23}$$

which allows us to express $\partial_t \psi^j$ in terms of ψ^j. For any integer $m \ge 1$ we define the finite dimensional space $\mathcal{X}_m(t) := \operatorname{span}\{\psi^1(t), \dots, \psi^m(t)\} \subset H^2(\Omega) \cap \mathcal{X}(t)$, and we write $\mathcal{P}^m_t : H^2(\Omega) \to \mathcal{X}_m(t)$ for the $H^2(\Omega)$ orthogonal projection onto $\mathcal{X}_m(t)$. Clearly, for each $v \in H^2(\Omega) \cap \mathcal{X}(t)$ we have that $\mathcal{P}^m_t v \to v$ as $m \to \infty$.

The next ingredient needed for the Galerkin method is the orthogonal projection onto the tangent space of the surface $\{x_3 = \eta(0)\}$, Π_0, defined by (4.14). This projection will be used to compensate for the fact that our finite-dimensional Galerkin approximation of the initial data u_0 may fail to satisfy the compatibility conditions (4.15).

Step 2 – Solving the Galerkin problem

For our Galerkin problem we will first construct a solution to the pressureless problem as follows. For each $m \ge 1$ we define an approximate solution

$$u^m(t) = a^m_j(t)\psi^j(t), \text{with } a^m_j : [0,T] \to \mathbb{R} \text{ for } j = 1, \dots, m, \tag{4.24}$$

where as usual we use the Einstein convention of summation of the repeated index j. We want to choose the coefficients a_j^m so that

$$(\partial_t u^m, \psi)_{\mathcal{H}^0} + \frac{1}{2}(u^m, \psi)_{\mathcal{H}^1} = \left(F^1, \psi\right)_{\mathcal{H}^0} - \left(F^3 - \Pi_0(F^3(0) + \mathbb{D}_{\mathcal{A}_0}(\mathcal{P}_0^m u_0)\mathcal{N}_0), \psi\right)_{0,\Sigma} \tag{4.25}$$

for each $\psi \in \mathcal{X}_m(t)$, where we have written $(\cdot,\cdot)_{0,\Sigma}$ for the usual $H^0(\Sigma)$ inner-product, and where Π_0 and $\mathcal{P}_0^m$ are defined in the previous step. We supplement the equation (4.25) with the initial condition

$$u^m(0) = \mathcal{P}_0^m u_0 \in \mathcal{X}_m(0). \tag{4.26}$$

Appealing to (4.23), we find that $\partial_t u^m(t) = \dot{a}_j^m(t)\psi^j(t) + R(t)u^m(t)$, and hence (4.25) is equivalent to the system of ODEs for a_j^m given by

$$\begin{aligned}\dot{a}_j^m \left(\psi^j, \psi^k\right)_{\mathcal{H}^0} + a_j^m \left(\left(R(t)\psi^j, \psi^k\right)_{\mathcal{H}^0} + \frac{1}{2}\left(\psi^j, \psi^k\right)_{\mathcal{H}^1}\right) \\ = \left(F^1, \psi^k\right)_{\mathcal{H}^0} - \left(F^3 - \Pi_0(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0), \psi^k\right)_{0,\Sigma}\end{aligned} \tag{4.27}$$

for $j,k = 1,\cdots,m$. The $m \times m$ matrix with j,k entry $\left(\psi^j, \psi^k\right)_{\mathcal{H}^0}$ is invertible, the coefficients of the linear system (4.27) are $C^1([0,T])$, and the forcing term is $C^0([0,T])$, so the usual well-posedness theory of ODEs guarantees the existence of $a_j^m \in C^1([0,T])$, a unique solution to (4.27) that satisfies the initial conditions induced by (4.26). This, in turn, provides the desired solution, u^m, to (4.25)–(4.26). Since F^1, F^3 satisfy (4.11), the equation (4.27) may be differentiated in time to see that actually $a_j^m \in C^{1,1}([0,T])$, with a_j^m twice differentiable a.e. in $[0,T]$.

Note that throughout the rest of the proof, we use constants C and the symbol $\lesssim$ with the assumption that the constants do not depend on m.

Step 3 – Energy estimates for u^m

Since $u^m(t) \in \mathcal{X}_m(t)$, we may use $\psi = u^m$ as a test function in (4.25). Doing so, employing Remark 2.3, and using the fact that Π_0 is an orthogonal projection, we may derive the bound

$$\begin{aligned}\partial_t \frac{1}{2}\|u^m\|_{\mathcal{H}^0}^2 + \frac{1}{2}\|u^m\|_{\mathcal{H}^1}^2 \le C\left\|F^1\right\|_{\mathcal{H}^0}\|u^m\|_{\mathcal{H}^1} - \frac{1}{2}\int_\Omega |u^m|^2 \partial_t J \\ + C\|u^m\|_{\mathcal{H}^1}\left(\left\|F^3\right\|_{H^{1/2}(\Sigma)} + \left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\right\|_{H^0(\Sigma)}\right).\end{aligned} \tag{4.28}$$

We may then apply Cauchy's inequality to (4.28) to find that

$$\begin{aligned}\partial_t \frac{1}{2}\|u^m\|_{\mathcal{H}^0}^2 + \frac{1}{8}\|u^m\|_{\mathcal{H}^1}^2 \le C\left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\right\|_{H^0(\Sigma)}^2 \\ + C\left(\left\|F^1\right\|_{\mathcal{H}^0}^2 + \left\|F^3\right\|_{H^{1/2}(\Sigma)}^2\right) + C_0(\eta)\frac{1}{2}\|u^m\|_{\mathcal{H}^0}^2\end{aligned} \tag{4.29}$$

for $C_0(\eta) := 1 + \sup_{0\le t\le T}\|\partial_t JK\|_{L^\infty}$. Note that since $\mathcal{P}_0^m$ is the $H^2(\Omega)$ orthogonal projection, we may use Proposition 2.1 to bound

$$\|u^m(0)\|_{\mathcal{H}^0} \le 2\|u^m(0)\|_0 \le 2\|u^m(0)\|_2 = 2\|\mathcal{P}_0^m u_0\|_2 \le 2\|u_0\|_2. \tag{4.30}$$

Now we can apply Gronwall's lemma to the differential inequality (4.29) and utilize (4.30) to deduce energy estimates for u^m:

$$\begin{aligned}\sup_{0\le t\le T}\|u^m\|_{\mathcal{H}^0}^2 + \|u^m\|_{\mathcal{H}_T^1}^2 \le \sup_{0\le t\le T}\|u^m\|_{\mathcal{H}^0}^2 + \int_0^T \exp\left(C_0(\eta)(T-s)\right)\|u^m(s)\|_{\mathcal{H}^1}^2\,ds \\ \lesssim \exp\left(C_0(\eta)T\right)\left(\left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\right\|_{H^0(\Sigma)}^2 + \|u_0\|_2^2 + \left\|F^1\right\|_{\mathcal{H}_T^0}^2 + \left\|F^3\right\|_{L^2H^{1/2}}^2\right).\end{aligned} \tag{4.31}$$

Step 4 – Estimate of $\|\partial_t u^m(0)\|_{\mathcal{H}^0}$

We will eventually derive energy estimates for $\partial_t u^m$ similar to those derived in the previous step for u^m, but first we must be able to estimate $\|\partial_t u^m(0)\|_{\mathcal{H}^0}$. If $u \in H^2(\Omega) \cap \mathcal{X}(t)$, $\psi \in \mathcal{H}^1$, then an integration by parts reveals that

$$\frac{1}{2}(u,\psi)_{\mathcal{H}^1} = \int_\Omega -\Delta_{\mathcal{A}} u \cdot \psi J + \int_\Sigma (\mathbb{D}_{\mathcal{A}} u \mathcal{N}) \cdot \psi = (-\Delta_{\mathcal{A}} u, \psi)_{\mathcal{H}^0} + (\mathbb{D}_{\mathcal{A}} u \mathcal{N}, \psi)_{0,\Sigma}. \tag{4.32}$$

Evaluating (4.25) at $t = 0$ and employing (4.32), we find that

$$(\partial_t u^m(0), \psi)_{\mathcal{H}^0} = \left(\Delta_{\mathcal{A}_0} u^m(0) + F^1(0), \psi\right)_{\mathcal{H}^0} - \left(\Pi_0^\perp(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0), \psi\right)_{0,\Sigma} \tag{4.33}$$

for all $\psi \in \mathcal{X}_m(0)$, where we have written $\Pi_0^\perp = I - \Pi_0$ for the orthogonal projection onto the line generated by $\mathcal{N}_0$.

For $\psi \in \mathcal{X}_m(0)$, we must estimate the last term in (4.33) in terms of $\|\psi\|_{\mathcal{H}^0}$. This is possible due to the appearance of $\Pi_0^\perp$ and Lemma 3.3. Indeed, we know that

$$\Pi_0^\perp(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0) = (F^3(0)\cdot\mathcal{N}_0 + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0 \cdot \mathcal{N}_0)\frac{\mathcal{N}_0}{|\mathcal{N}_0|^2}, \tag{4.34}$$

which implies, since $|\mathcal{N}_0|^2 \ge 1$ and $\operatorname{div}_{\mathcal{A}_0} \psi = 0$, that

$$\begin{aligned} \left|\left(\Pi_0^\perp(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0), \psi\right)_{0,\Sigma}\right| &\le |\mathcal{N}_0|^2 \left|\left(\Pi_0^\perp(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0), \psi\right)_{0,\Sigma}\right| \\ &= \left|\left(F^3(0)\cdot\mathcal{N}_0 + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\cdot\mathcal{N}_0, \psi\cdot\mathcal{N}_0\right)_{0,\Sigma}\right| \\ &\le \|\psi\cdot\mathcal{N}_0\|_{H^{-1/2}(\Sigma)} \left\|(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0)\cdot\mathcal{N}_0)\right\|_{H^{1/2}(\Sigma)} \\ &\lesssim C_1(\eta)\|\psi\|_{\mathcal{H}^0}\left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\right\|_{H^{1/2}(\Sigma)}. \end{aligned} \tag{4.35}$$

In the last inequality we have used Lemmas 3.3 and A.1, and we have written $C_1(\eta) := \|\mathcal{N}_0\|_{C^1(\Sigma)}$.

By virtue of (4.23), we have that

$$\partial_t u^m(t) - R(t)u^m(t) = \dot{a}_j^m(t)\psi^j(t) \in \mathcal{X}_m(t), \tag{4.36}$$

so that $\psi = \partial_t u^m(0) - R(0)u^m(0) \in \mathcal{X}_m(0)$ is a valid choice of a test function in (4.33). We plug this ψ into (4.33), rearrange, and employ the bound (4.35) to see that

$$\begin{aligned} \|\partial_t u^m(0)\|_{\mathcal{H}^0}^2 &\le \|R(0)u^m(0)\|_{\mathcal{H}^0}\|\partial_t u^m(0)\|_{\mathcal{H}^0} \\ &+ \|\partial_t u^m(0) - R(0)u^m(0)\|_{\mathcal{H}^0}\left\|\Delta_{\mathcal{A}_0} u^m(0) + F^1(0)\right\|_{\mathcal{H}^0} \\ &+ CC_1(\eta)\|\partial_t u^m(0) - R(0)u^m(0)\|_{\mathcal{H}^0}\left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\right\|_{H^{1/2}(\Sigma)}. \end{aligned} \tag{4.37}$$

A simple computation and (4.30) imply that $\|\Delta_{\mathcal{A}_0} u^m(0)\|_{\mathcal{H}^0} \lesssim \|\mathcal{A}_0\|_{C^1}^2 \|u_0\|_2$. This allows us to use Cauchy's inequality and (4.30) to derive from (4.37) the bound

$$\|\partial_t u^m(0)\|_{\mathcal{H}^0}^2 \lesssim C_2(\eta)\left(\|u_0\|_2^2 + \left\|F^1(0)\right\|_{\mathcal{H}^0}^2 + \left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\right\|_{H^{1/2}(\Sigma)}^2\right) \tag{4.38}$$

for $C_2(\eta) := 1 + \|R(0)\|_{L^\infty}^2 + \|\mathcal{A}_0\|_{C^1}^2 + C_1(\eta)^2$. This is our desired estimate of $\|\partial_t u^m(0)\|_{\mathcal{H}^0}$.

Step 5 – Energy estimates for $\partial_t u^m$

We now turn to estimates for $\partial_t u^m$ of a similar form to those we already derived for u^m. Suppose for now that $\psi(t) = b_j^m(t)\psi^j$ for $b_j^m \in C^{0,1}([0,T])$, $j = 1, \cdots, m$; it is easily verified, as in (4.36), that $\partial_t\psi - R(t)\psi \in \mathcal{X}_m(t)$ as well. We now use this ψ in (4.25), temporally differentiate the resulting equation, and then subtract from this the equation (4.25) with test function $\partial_t \psi - R\psi$; this eliminates the appearance of $\partial_t \psi$ and leaves us with the equality

$$\begin{aligned}(4.39)\quad & \langle \partial_t^2 u^m, \psi \rangle_{(\mathcal{H}^1)^*} + \frac{1}{2} (\partial_t u^m, \psi)_{\mathcal{H}^1} = \langle \partial_t F^1, \psi \rangle_{(\mathcal{H}^1)^*} - \left(\partial_t F^3, \psi\right)_{0,\Sigma} - \left(F^3, R\psi\right)_{0,\Sigma} \\ & + \left(F^1, (\partial_t JK + R)\psi\right)_{\mathcal{H}^0} - (\partial_t u^m, (\partial_t JK + R)\psi)_{\mathcal{H}^0} - \frac{1}{2} (u^m, R\psi)_{\mathcal{H}^1} \\ & - \frac{1}{2} \int_\Omega (\partial_t JK \mathbb{D}_\mathcal{A} u^m : \mathbb{D}_\mathcal{A} \psi + \mathbb{D}_{\partial_t \mathcal{A}} u^m : \mathbb{D}_\mathcal{A} \psi + \mathbb{D}_\mathcal{A} u^m : \mathbb{D}_{\partial_t \mathcal{A}} \psi) J.\end{aligned}$$

Note here that the terms involving $\langle \cdot, \cdot \rangle_{(\mathcal{H}^1)^*}$ appear when we temporally differentiate because of Lemma 2.4.

According to (4.36) and the fact that a_j^m is twice differentiable a.e., we may use $\psi = \partial_t u^m(t) - R(t) u^m(t) \in \mathcal{X}_m(t)$ as a test function in (4.39). Plugging in this ψ and arguing as in the previous steps by employing Remark 2.3, Cauchy's inequality, and trace embeddings, we may deduce from (4.39) that

$$\begin{aligned}(4.40)\quad & \partial_t \left(\frac{1}{2} \|\partial_t u^m\|_{\mathcal{H}^0}^2 - (\partial_t u^m, R u^m)_{\mathcal{H}^0} \right) + \frac{1}{8} \|\partial_t u^m\|_{\mathcal{H}^1}^2 \le C C_3(\eta) \|u^m\|_{\mathcal{H}^1}^2 \\ & + C_0(\eta) \left(\frac{1}{2} \|\partial_t u^m\|_{\mathcal{H}^0}^2 - (\partial_t u^m, R u^m)_{\mathcal{H}^0} \right) + C \left(\left\|F^1\right\|_{\mathcal{H}^0}^2 + \left\|\partial_t F^1\right\|_{(\mathcal{H}^1)^*}^2 \right) \\ & + C \left(\left\|F^3\right\|_{H^{1/2}(\Sigma)}^2 + \left\|\partial_t F^3\right\|_{H^{-1/2}(\Sigma)}^2 \right)\end{aligned}$$

for $C_0(\eta)$ as defined above and

$$\begin{aligned}(4.41)\quad C_3(\eta) := & \sup_{0 \le t \le T} \left[1 + \|R\|_{C^1}^2 + \|\partial_t R\|_{L^\infty}^2 + \|\partial_t \mathcal{A}\|_{L^\infty}^2 + \left(1 + \|\mathcal{A}\|_{L^\infty}^2\right) \left(1 + \|\partial_t JK\|_{L^\infty}^2\right) \right] \\ & \times \sup_{0 \le t \le T} \left[1 + \|R\|_{C_1}^2 \right].\end{aligned}$$

Then (4.40), Gronwall's lemma, and a further application of Cauchy's inequality imply that

$$\begin{aligned}(4.42)\quad & \sup_{0 \le t \le T} \|\partial_t u^m\|_{\mathcal{H}^0}^2 + \|\partial_t u^m\|_{\mathcal{H}^1_T}^2 \lesssim \exp(C_0(\eta) T) \left(\|\partial_t u^m(0)\|_{\mathcal{H}^0}^2 + C_2(\eta) \|u^m(0)\|_{\mathcal{H}^0}^2 \right) \\ & + C_3(\eta) \left(\sup_{0 \le t \le T} \|u^m\|_{\mathcal{H}^0}^2 + \int_0^T \exp(C_0(\eta)(T-s)) \|u^m(s)\|_{\mathcal{H}^1}^2 \, ds \right) \\ & + \exp(C_0(\eta) T) \left(\left\|F^1\right\|_{\mathcal{H}^0_T}^2 + \left\|\partial_t F^1\right\|_{(\mathcal{H}^1_T)^*}^2 + \left\|F^3\right\|_{L^2 H^{1/2}}^2 + \left\|\partial_t F^3\right\|_{L^2 H^{-1/2}}^2 \right).\end{aligned}$$

Now we combine (4.42) with the estimates (4.30), (4.31), and (4.38) to deduce our energy estimates for $\partial_t u^m$:

$$\begin{aligned}(4.43)\quad & \sup_{0 \le t \le T} \|\partial_t u^m\|_{\mathcal{H}^0}^2 + \|\partial_t u^m\|_{\mathcal{H}^1_T}^2 \\ & \lesssim (C_2(\eta) + C_3(\eta)) \exp(C_0(\eta) T) \left(\|u_0\|_2^2 + \left\|F^1(0)\right\|_{\mathcal{H}^0}^2 + \left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0) \mathcal{N}_0\right\|_{H^{1/2}(\Sigma)}^2 \right) \\ & + \exp(C_0(\eta) T) \left[C_3(\eta) \left(\left\|F^1\right\|_{\mathcal{H}^0_T}^2 + \left\|F^3\right\|_{L^2 H^{1/2}}^2 \right) + \left\|\partial_t F^1\right\|_{(\mathcal{H}^1_T)^*}^2 + \left\|\partial_t F^3\right\|_{L^2 H^{-1/2}}^2 \right].\end{aligned}$$

Step 6 – Improved energy estimate for u^m

We can now improve our energy estimates for u^m by using $\psi = \partial_t u^m(t) - R(t) u^m(t) \in \mathcal{X}_m(t)$ as a test function in (4.25). Plugging this in and rearranging yields the equality

$$\begin{aligned}(4.44)\quad & \partial_t \frac{1}{4} \|u^m\|_{\mathcal{H}^1}^2 + \|\partial_t u^m\|_{\mathcal{H}^0}^2 = (\partial_t u^m, R u^m)_{\mathcal{H}^0} + \frac{1}{2} (u^m, R u^m)_{\mathcal{H}^1} + \left(F^1, \partial_t u^m - R u^m\right)_{\mathcal{H}^0} \\ & - \left(F^3 - \Pi_0(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0) \mathcal{N}_0), \partial_t u^m - R u^m\right)_{0,\Sigma} \\ & + \frac{1}{2} \int_\Omega \left(\mathbb{D}_\mathcal{A} u^m : \mathbb{D}_{\partial_t \mathcal{A}} u^m + \partial_t JK \frac{|\mathbb{D}_\mathcal{A} u^m|^2}{2} \right) J.\end{aligned}$$

We may then argue as before to use (4.44) to derive the inequality

$$\partial_t \frac{1}{4} \|u^m\|_{\mathcal{H}^1}^2 + \|\partial_t u^m\|_{\mathcal{H}^0}^2 \le C \left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0\right\|_{H^{1/2}(\Sigma)}^2 + C\left(\left\|F^1\right\|_{\mathcal{H}^0}^2 + \left\|F^3\right\|_{H^{1/2}(\Sigma)}^2\right) + C\left(\|\partial_t u^m\|_{\mathcal{H}^1}^2 + C_3(\eta)\|u^m\|_{\mathcal{H}^1}^2\right). \tag{4.45}$$

We could regard (4.45) as a differential inequality for $\|u^m\|_{\mathcal{H}^1}^2$ and apply Gronwall's lemma as before, but this is not necessary since we already control $\|u^m\|_{\mathcal{H}^1_T}^2$ and $\|\partial_t u^m\|_{\mathcal{H}^1_T}^2$. Indeed, we may simply integrate (4.45) in time to deduce an improved energy estimate for u^m:

$$\begin{aligned} &\sup_{0\le t\le T} \|u^m\|_{\mathcal{H}^1}^2 + \|\partial_t u^m\|_{\mathcal{H}^0_T}^2 \\ &\lesssim (C_2(\eta)+C_3(\eta))\exp(C_0(\eta)T)\left(\|u_0\|_2^2 + \left\|F^1(0)\right\|_{\mathcal{H}^0}^2 + \left\|F^3(0)+\mathbb{D}_{\mathcal{A}_0}u^m(0)\mathcal{N}_0\right\|_{H^{1/2}(\Sigma)}^2\right) \\ &\quad + \exp(C_0(\eta)T)\left[C_3(\eta)\left(\left\|F^1\right\|_{\mathcal{H}^0_T}^2 + \left\|F^3\right\|_{L^2H^{1/2}}^2\right) + \left\|\partial_t F^1\right\|_{(\mathcal{H}^1_T)^*}^2 + \left\|\partial_t F^3\right\|_{L^2H^{-1/2}}^2\right]. \end{aligned} \tag{4.46}$$

Step 7 – Estimating terms in (4.43), (4.46)

In order to use (4.43) and (4.46) as uniform bounds, we must first remove the appearance of $u^m(0)$ on the right side of the estimates. For this we use Lemma A.2, the embedding $H^2(\Omega) \hookrightarrow H^{3/2}(\Sigma)$, and the bound $\|u^m(0)\|_2 \le \|u_0\|_2$ to find that

$$\left\|F^3(0) + \mathbb{D}_{\mathcal{A}_0}u^m(0)\mathcal{N}_0\right\|_{H^{1/2}(\Sigma)}^2 \lesssim C_4(\eta)\left(\left\|F^3(0)\right\|_{H^{1/2}(\Sigma)}^2 + \|u_0\|_2^2\right) \tag{4.47}$$

for $C_4(\eta) := 1 + \|\mathcal{N}_0\|_{C^1(\Sigma)}^2 \|\mathcal{A}_0\|_{C^1}^2$.

We now seek to estimate the constants $C_i(\eta)$, $i = 0, \ldots, 4$ in terms of the quantity $\mathcal{K}(\eta)$. A simple computation shows that

$$C_0(\eta) + (C_2(\eta)+C_3(\eta))(1+C_4(\eta)) \le \sup_{0\le t\le T} \mathcal{Q}_1(\|\bar{\eta}\|_{C^2}^2, \|\partial_t\bar{\eta}\|_{C^2}^2, \left\|\partial_t^2\bar{\eta}\right\|_{C^1}^2), \tag{4.48}$$

where $\mathcal{Q}_1$ is a polynomial in three variables. According to Lemma A.5 in the non-periodic case and Lemma A.7 in the periodic case, we have the estimate $\left\|\partial_t^j \bar{\eta}\right\|_{C^k}^2 \lesssim \left\|\partial_t^j \eta\right\|_{k+3/2}^2$ for $j, k \ge 0$. This, (4.48), and the fact that $\mathcal{K}(\eta) \le 1$ then imply that

$$C_0(\eta) + (C_2(\eta)+C_3(\eta))(1+C_4(\eta)) \le \mathcal{Q}_1(\mathcal{K}(\eta),\mathcal{K}(\eta),\mathcal{K}(\eta)) \le C(1+\mathcal{K}(\eta)) \tag{4.49}$$

for a constant C independent of η.

Step 8 – Passing to the limit

We now utilize the energy estimates (4.43) and (4.46) in conjunction with (4.47) to pass to the limit $m \to \infty$. According to these energy estimates and Lemma 2.1, we have that the sequence $\{u^m\}$ is uniformly bounded in $L^\infty H^1$ and $\{\partial_t u^m\}$ is uniformly bounded in $L^\infty H^0 \cap L^2 H^1$. Up to the extraction of a subsequence, we then know that

$$u^m \overset{*}{\rightharpoonup} u \text{ weakly-}* \text{ in } L^\infty H^1, \partial_t u^m \overset{*}{\rightharpoonup} \partial_t u \text{ in } L^\infty H^0, \text{ and } \partial_t u^m \rightharpoonup \partial_t u \text{ weakly in } L^2 H^1. \tag{4.50}$$

By lower semi-continuity and (4.49), the energy estimates imply that the quantity

$$\|u\|_{L^\infty H^1}^2 + \|\partial_t u\|_{L^\infty H^0}^2 + \|\partial_t u\|_{L^2H^1}^2 \tag{4.51}$$

is bounded above by the right hand side of (4.17).

Because of these convergence results, we can integrate (4.39) in time from 0 to T and send $m \to \infty$ to deduce that $\partial_t^2 u^m \rightharpoonup \partial_t^2 u$ weakly in $L^2({}_0H^1(\Omega))^*$, with the action of $\partial_t^2 u$ on an element $\psi \in L^2{}_0H^1(\Omega)$ defined by replacing u^m with u everywhere in (4.39). It is more natural to regard $\partial_t^2 u \in (\mathcal{X}_T)^*$ since the action of $\partial_t^2 u$ is defined with test functions in $\mathcal{X}_T$, but the reasoning presented after (4.4) is applicable to $\partial_t^2 u$, so we may regard $\partial_t^2 u \in L^2{}_0H^1(\Omega)$ without ambiguity. From the equation resulting from passing to the limit in (4.39), it is straightforward to show that $\left\|\partial_t^2 u\right\|_{(\mathcal{H}^1_T)^*}^2$ is bounded by the right hand side of (4.17). This bound then shows that $\partial_t u \in C^0 L^2$.

Step 9 – The strong solution

Due to the convergence established in the last step, we may pass to the limit in (4.25) for a.e. $t \in [0,T]$. Since $u^m(0) \to u_0$ in H^2 and $u_0, F^3(0)$ satisfy the compatibility condition (4.15), we have that

$$\left\| \Pi_0(F^3(0) + \mathbb{D}_{\mathcal{A}_0} u^m(0)\mathcal{N}_0) \right\|_{H^{1/2}(\Sigma)} \to 0 \tag{4.52}$$

In the limit, (4.25) implies that for a.e. t,

$$(\partial_t u, \psi)_{\mathcal{H}^0} + \frac{1}{2}(u,\psi)_{\mathcal{H}^1} = \left(F^1, \psi\right)_{\mathcal{H}^0} - \left(F^3, \psi\right)_{0,\Sigma} \text{ for every } \psi \in \mathcal{X}(t). \tag{4.53}$$

Now we introduce the pressure. Define the functional $\Lambda_t \in (\mathcal{H}^1(t))^*$ so that $\Lambda_t(v)$ equals the difference between the left and right sides of (4.53), with ψ replaced by $v \in \mathcal{H}^1(t)$. Then $\Lambda_t(v) = 0$ for all $v \in \mathcal{X}(t)$, so by Proposition 2.9 there exists a unique $p(t) \in \mathcal{H}^0(t)$ so that $(p(t), \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \Lambda_t(v)$ for all $v \in \mathcal{H}^1(t)$. This is equivalent to

$$(\partial_t u, v)_{\mathcal{H}^0} + \frac{1}{2}(u,v)_{\mathcal{H}^1} - (p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0} = \left(F^1, v\right)_{\mathcal{H}^0} - \left(F^3, v\right)_{0,\Sigma} \text{ for every } v \in \mathcal{H}^1(t), \tag{4.54}$$

which in particular implies that (u,p) is the unique weak solution to (1.16) in the sense of (4.3).

For a.e. $t \in [0,T]$, $(u(t),p(t))$ is the unique weak solution to the elliptic problem (3.13) in the sense of (3.14), with F^1 replaced by $F^1(t) - \partial_t u(t)$, $F^2 = 0$, and F^3 replaced by $F^3(t)$. Since $F^1(t) - \partial_t u(t) \in H^0(\Omega)$ and $F^3(t) \in H^{1/2}(\Sigma)$, Lemma 3.6 implies that this elliptic problem admits a unique strong solution, which must coincide with the weak solution. We may then apply Proposition 3.7 and Lemma 2.1 for the bound

$$\|u(t)\|_r^2 + \|p(t)\|_{r-1}^2 \lesssim \left(\|\partial_t u(t)\|_{\mathcal{H}^{r-2}}^2 + \left\|F^1(t)\right\|_{r-2}^2 + \left\|F^3(t)\right\|_{H^{r-3/2}(\Sigma)}^2 \right) \tag{4.55}$$

when $r = 2,3$. When $r = 2$ we take the supremum of (4.55) over $t \in [0,T]$, and when $r = 3$ we integrate over $[0,T]$; the resulting inequalities imply that $u \in L^\infty H^2 \cap L^2 H^3$ and $p \in L^\infty H^1 \cap L^2 H^2$ with estimates as in (4.17). This, in turn, implies that (u,p) is a strong solution to (1.16).

Since we already know that $u \in L^2 H^3$ and $\partial_t u \in L^2 H^1$, Lemma A.3 implies that $u \in C^0 H^2$. Then since $F^1 - \partial_t u \in C^0 H^0$ and $\mathbb{D}_{\mathcal{A}} u \mathcal{N} + F^3 \in C^0 H^{1/2}(\Sigma)$, we know that $\nabla_{\mathcal{A}} p \in C^0 H^0$ and $p \in C^0 H^{1/2}(\Sigma)$ as well, from which we see, via Poincaré's inequality (Lemma A.9), that $p \in C^0 H^1$. With these continuity results established, we can compute $p(0)$ and $\partial_t u(0)$. We start with the Dirichlet condition for $p(0)$ on Σ, the second equation in (4.18). Since $p \in C^0 H^1(\Omega)$, $u \in C^0 H^2(\Omega)$, and $F^3 \in C^0 H^{1/2}(\Sigma)$, the boundary condition $S_{\mathcal{A}}(p,u)\mathcal{N} = F^3$, which holds in $H^{1/2}(\Sigma)$ for each $t > 0$, can be evaluated at $t = 0$. Then the Dirichlet condition for $p(0)$ on Σ in (4.18) is easily deduced by solving $S_{\mathcal{A}_0}(p(0), u_0)\mathcal{N}_0 = F^3(0)$ for $p(0)$.

Now we derive the PDE satisfied by $p(0)$ and compute $\partial_t u(0)$. Let $\varphi \in H^2(\Omega)$ be a scalar function satisfying $\varphi|_\Sigma = 0$ and $\nabla \varphi|_{\Sigma_b} = 0$. Then $\nabla_{\mathcal{A}} \varphi = \mathcal{A} \nabla \varphi \in \mathcal{H}^1(t)$, and we may choose $v = \nabla_{\mathcal{A}} \varphi$ as a test function in (4.54). Since $\partial_t u - Ru \in \mathcal{X}(t)$, we can integrate by parts to see that

$$\begin{aligned} (\partial_t u, \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} &= (\partial_t u - Ru, \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} + (Ru, \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} = (Ru, \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} \text{ and} \\ (p, \operatorname{div}_{\mathcal{A}} \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} &= (-\nabla_{\mathcal{A}} p, \nabla_{\mathcal{A}}\varphi)_{\mathcal{H}^0} + (p, \nabla_{\mathcal{A}}\varphi \cdot \mathcal{N})_{0,\Sigma}. \end{aligned} \tag{4.56}$$

This, (4.32), (4.54), and (1.16) imply that

$$\left(Ru + \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u - F^1, \nabla_{\mathcal{A}}\varphi\right)_{\mathcal{H}^0} = 0 \text{ for all such } \varphi. \tag{4.57}$$

By the established continuity properties, we may set $t = 0$ in (4.57), again integrate by parts, and employ a density argument to see that

$$\left(\nabla_{\mathcal{A}_0} p(0) - F^1(0), \nabla_{\mathcal{A}_0}\varphi\right)_{\mathcal{H}^0} = -\left(-\operatorname{div}_{\mathcal{A}_0}(R(0)u_0), \varphi\right)_{\mathcal{H}^0} + \langle \Delta_{\mathcal{A}_0} u_0 \cdot \nu, \varphi \rangle_{-1/2} \tag{4.58}$$

for all $\varphi \in {}^0H^1(\Omega)$. This establishes that $p(0)$ is the weak solution to (4.18). According to (3.46) we then have that $p(0) \in H^1(\Omega)$. This and (4.54) allow us to solve for $\partial_t u(0)$ as in (4.19), and then (4.57) implies that $\partial_t u(0) - R(0)u_0 \in \mathcal{Y}(0)$ since then $D_t u(0) \perp \nabla_{\mathcal{A}(0)}\varphi$ for every $\varphi \in {}^0H^1(\Omega)$.

Step 10 – The weak solution satisfied by $D_t u = \partial_t u - Ru$

Now we seek to use (4.39) to determine the PDE satisfied by $D_t u$. As mentioned above, we may integrate (4.39) in time from 0 to T and pass to the limit $m \to \infty$. For any $\psi \in \mathcal{X}_T$ we have $R\psi \in \mathcal{H}^1_T$, so that we may replace all of the terms $R\psi$ in the resulting equation by using $v = R\psi$ in (4.54); this yields the equality

$$(4.59)\quad \begin{aligned} \langle \partial_t^2 u, \psi\rangle_* + \frac{1}{2}(\partial_t u, \psi)_{\mathcal{H}^1_T} &= \langle \partial_t F^1, \psi\rangle_* - \langle \partial_t F^3, \psi\rangle_{-1/2} \\ &+ \left(\partial_t JKF^1, \psi\right)_{\mathcal{H}^0_T} - (\partial_t JK \partial_t u, \psi)_{\mathcal{H}^0_T} - (p, \operatorname{div}_{\mathcal{A}}(R\psi))_{\mathcal{H}^0_T} \\ &- \frac{1}{2}\int_0^T \int_\Omega (\partial_t JK \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}}\psi + \mathbb{D}_{\partial_t \mathcal{A}} u : \mathbb{D}_{\mathcal{A}}\psi + \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\partial_t\mathcal{A}}\psi)\, J \end{aligned}$$

for all $\psi \in \mathcal{X}_T$. In (4.59) we have employed the same duality notation as in (4.3).

Now we define $\Lambda \in (\mathcal{H}^1_T)^*$ with $\Lambda(v)$ equal to the difference between the left and right sides of (4.59) with ψ replaced with v. As above, we may use Proposition 2.9 to find a unique $q \in \mathcal{H}^0_T$ so that $\Lambda(v) = (q, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T}$ for all $v \in \mathcal{H}^1_T$. Simple computations reveal that $\partial_t (J\mathcal{A}_{ij}) = -J\mathcal{A}_{kj}R_{ki}$ and $J\mathcal{A}_{kj}\partial_j R_{ki} = \partial_j(J\mathcal{A}_{kj}R_{ki}) = -\partial_t\partial_j(J\mathcal{A}_{ij}) = 0$, which imply that

$$(4.60)\quad \begin{aligned} (p, \operatorname{div}_{\partial_t\mathcal{A}} v + \partial_t JK \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} &= \int_0^T \int_\Omega p\partial_j v_i \partial_t (J\mathcal{A}_{ij}) = \int_0^T\int_\Omega p\partial_j v_i J\mathcal{A}_{kj}R_{ki} \\ = \int_0^T\int_\Omega pJ\mathcal{A}_{kj}\partial_j(R_{ki}v_i) - pJ\mathcal{A}_{kj}v_i\partial_j R_{ki} &= \int_0^T\int_\Omega pJ\mathcal{A}_{kj}\partial_j(R_{ki}v_i) = (p, \operatorname{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0_T}. \end{aligned}$$

This, in turn, implies that the equation $\Lambda(v) = (q, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T}$ is the same as that which would result from computing the temporal distributional derivative of (4.54); we deduce that $q = \partial_t p$ and that

$$(4.61)\quad \begin{aligned} \langle \partial_t^2 u, v\rangle_* + \frac{1}{2}(\partial_t u, v)_{\mathcal{H}^1_T} - (\partial_t p, \operatorname{div}_{\mathcal{A}} v)_{\mathcal{H}^0_T} &= \langle \partial_t F^1, v\rangle_* - \langle \partial_t F^3, v\rangle_{-1/2} \\ &+ \left(\partial_t JKF^1, v\right)_{\mathcal{H}^0_T} - (\partial_t JK\partial_t u, v)_{\mathcal{H}^0_T} - (p, \operatorname{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0_T} \\ &- \frac{1}{2}\int_0^T\int_\Omega (\partial_t JK\mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}} v + \mathbb{D}_{\partial_t\mathcal{A}} u : \mathbb{D}_{\mathcal{A}} v + \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\partial_t\mathcal{A}} v)\, J \end{aligned}$$

for all $v \in \mathcal{H}^1_T$. As before, we may deduce from (4.61) the bound for $\|\partial_t p\|^2_{L^2L^2}$ stated in (4.17).

We now rewrite the terms in (4.61) to derive the PDE for $D_t u, \partial_t p$. A straightforward computation shows that on Σ, $R^T\mathcal{N} = \partial_t\mathcal{N}$, so that we may integrate by parts for the equality

$$(4.62)\quad (p, \operatorname{div}_{\mathcal{A}}(Rv))_{\mathcal{H}^0_T} = -\left(R^T\nabla_{\mathcal{A}} p, v\right)_{\mathcal{H}^0_T} + \langle pR^T\mathcal{N}, v\rangle_{-1/2} = -\left(R^T\nabla_{\mathcal{A}} p, v\right)_{\mathcal{H}^0_T} + \langle p\partial_t\mathcal{N}, v\rangle_{-1/2},$$

where R^T is the matrix transpose of R. Another integration by parts yields

$$(4.63)\quad \begin{aligned} &-\frac{1}{2}\int_0^T\int_\Omega (\partial_t JK\mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}} v + \mathbb{D}_{\partial_t\mathcal{A}} u : \mathbb{D}_{\mathcal{A}} v + \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\partial_t\mathcal{A}} v)\, J \\ &\qquad = (\operatorname{div}_{\mathcal{A}}(R\mathbb{D}_{\mathcal{A}} u + \mathbb{D}_{\partial_t\mathcal{A}} u), v)_{\mathcal{H}^0_T} - \langle \mathbb{D}_{\mathcal{A}} u\partial_t\mathcal{N} + \mathbb{D}_{\partial_t\mathcal{A}} u\mathcal{N}, v\rangle_{-1/2}. \end{aligned}$$

We replace the appearance of $\partial_t^2 u$ with $\partial_t D_t u$ via

$$(4.64)\quad \langle \partial_t^2 u, v\rangle_* = \langle \partial_t D_t u, v\rangle_* + (R\partial_t u, v)_{\mathcal{H}^0_T} + (\partial_t R u, v)_{\mathcal{H}^0_T}.$$

Since (u,p) are a strong solution to (1.16), we may multiply by $(R^T + \partial_t JK)v$ and integrate to see that

$$(4.65)\quad \left((\partial_t JK + R)(F^1 - \partial_t u), v\right)_{\mathcal{H}^0_T} = ((\partial_t JK + R)(-\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p), v)_{\mathcal{H}^0_T}.$$

We may then combine (4.61)–(4.65) with the fact that $D_t u = \partial_t u - Ru \in \mathcal{X}_T$ to deduce that $(D_t u, \partial_t p)$ are weak solutions of (4.20) with $D_t u(0) \in \mathcal{Y}(0)$ given by (4.19). Here, the inclusions $G^1 \in (\mathcal{H}^1_T)^*$ and $G^3 \in L^2([0,T]; H^{-1/2}(\Sigma))$ are easily established from the above bounds on u, p. $\square$

4.3. **Higher regularity.** Recall that, as discussed in the introduction, we abuse notation by writing $L^2H^{-1} = L^2({}_0H^1(\Omega))^*$. In order to state our higher regularity results for the problem (1.16), we must be able to define the forcing terms and initial data for the problem that results from temporally differentiating (1.16) several times. To this end, we first define some mappings. Given F^1, F^3, v, q we define the vector fields $\mathfrak{G}^0, \mathfrak{G}^1$ on Ω and $\mathfrak{G}^3$ on Σ by

$$
\begin{aligned}
\mathfrak{G}^0(F^1, v, q) &= \Delta_{\mathcal{A}} v - \nabla_{\mathcal{A}} q + F^1 - Rv, \\
\mathfrak{G}^1(v, q) &= -(R + \partial_t JK)\Delta_{\mathcal{A}} v - \partial_t R v + (\partial_t JK + R - R^T)\nabla_{\mathcal{A}} q \\
&\quad + \operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Rv) + R\mathbb{D}_{\mathcal{A}} v + \mathbb{D}_{\partial_t \mathcal{A}} v), \text{ and} \\
\mathfrak{G}^3(v, q) &= \mathbb{D}_{\mathcal{A}}(Rv)\mathcal{N} - (qI - \mathbb{D}_{\mathcal{A}} v)\partial_t \mathcal{N} + \mathbb{D}_{\partial_t \mathcal{A}} v \mathcal{N},
\end{aligned}
\tag{4.66}
$$

and we define the functions $\mathfrak{f}^1$ on Ω, $\mathfrak{f}^2$ on Σ, and $\mathfrak{f}^3$ on Σ_b according to

$$
\begin{aligned}
\mathfrak{f}^1(F^1, v) &= \operatorname{div}_{\mathcal{A}}(F^1 - Rv), \\
\mathfrak{f}^2(F^3, v) &= (F^3 + \mathbb{D}_{\mathcal{A}} v \mathcal{N}) \cdot \mathcal{N} \left|\mathcal{N}\right|^{-2}, \text{ and} \\
\mathfrak{f}^3(F^1, v) &= (F^1 + \Delta_{\mathcal{A}} v) \cdot \nu.
\end{aligned}
\tag{4.67}
$$

In the definitions of $\mathfrak{G}^i$ and $\mathfrak{f}^i$ we assume that $\mathcal{A}, \mathcal{N}, R$ (recall that R is defined by (4.12)), etc are evaluated at the same t as F^1, F^3, v, q. These mappings allow us to define the forcing terms as follows. Write $F^{1,0} = F^1$ and $F^{3,0} = F^3$. When F^1, F^3, u, and p are sufficiently regular for the following to make sense, we then recursively define the vectors

$$
\begin{aligned}
F^{1,j} &:= D_t F^{1,j-1} + \mathfrak{G}^1(D_t^{j-1} u, \partial_t^{j-1} p) = D_t^j F^1 + \sum_{\ell=0}^{j-1} D_t^\ell \mathfrak{G}^1(D_t^{j-\ell-1} u, \partial_t^{j-\ell-1} p), \\
F^{3,j} &:= \partial_t F^{3,j-1} + \mathfrak{G}^3(D_t^{j-1} u, \partial_t^{j-1} p) = \partial_t^j F^3 + \sum_{\ell=0}^{j-1} \partial_t^\ell \mathfrak{G}^3(D_t^{j-\ell-1} u, \partial_t^{j-\ell-1} p)
\end{aligned}
\tag{4.68}
$$

on Ω and Σ, respectively, for $j = 1, \dots, 2N$.

Now we define various sums of norms of F^1, F^3, and η that will appear in our estimates. Define the quantities

$$
\begin{aligned}
\mathfrak{F}(F^1, F^3) &:= \sum_{j=0}^{2N} \left\| \partial_t^j F^1 \right\|_{L^2 H^{4N-2j-1}}^2 + \left\| \partial_t^j F^3 \right\|_{L^2 H^{4N-2j-1/2}}^2 \\
&\quad + \sum_{j=0}^{2N-1} \left\| \partial_t^j F^1 \right\|_{L^\infty H^{4N-2j-2}}^2 + \left\| \partial_t^j F^3 \right\|_{L^\infty H^{4N-2j-3/2}}^2, \\
\mathfrak{F}_0(F^1, F^3) &:= \sum_{j=0}^{2N-1} \left\| \partial_t^j F^1(0) \right\|_{4N-2j-2}^2 + \left\| \partial_t^j F^3(0) \right\|_{4N-2j-3/2}^2.
\end{aligned}
\tag{4.69}
$$

For brevity, we will only write $\mathfrak{F}$ for $\mathfrak{F}(F^1, F^3)$ and $\mathfrak{F}_0$ for $\mathfrak{F}_0(F^1, F^3)$ throughout the rest of this section. Lemmas A.3 and 2.4 imply that if $\mathfrak{F} < \infty$, then

$$
\partial_t^j F^1 \in C^0([0,T]; H^{4N-2j-2}(\Omega)) \text{ and } \partial_t^j F^3 \in C^0([0,T]; H^{4N-2j-3/2}(\Sigma))
\tag{4.70}
$$

for $j = 0, \dots, 2N-1$. The same lemmas also imply that the sum of the $L^\infty H^k$ norms in the definition of $\mathfrak{F}$ can be bounded by a constant that depends on T times the sum of the $L^2 H^{k+1}$ norms. To avoid the introduction of a constant that depends on T, we will retain the L^∞ terms. For η we define

$$
\begin{aligned}
\mathfrak{D}(\eta) &:= \|\eta\|_{L^2 H^{4N+1/2}}^2 + \|\partial_t \eta\|_{L^2 H^{4N-1/2}}^2 + \sum_{j=2}^{2N+1} \left\| \partial_t^j \eta \right\|_{L^2 H^{4N-2j+5/2}}^2, \\
\mathfrak{E}(\eta) &:= \sum_{j=0}^{2N} \left\| \partial_t^j \eta \right\|_{L^\infty H^{4N-2j}}^2, \text{ and } \mathfrak{K}(\eta) := \mathfrak{E}(\eta) + \mathfrak{D}(\eta)
\end{aligned}
\tag{4.71}
$$

as well as

$$\mathfrak{E}_0(\eta) := \|\eta(0)\|_{4N}^2 + \|\partial_t\eta(0)\|_{4N-1}^2 + \sum_{j=2}^{2N} \left\|\partial_t^j\eta(0)\right\|_{4N-2j+3/2}^2. \tag{4.72}$$

Again, Lemma A.3 implies that $\eta \in C^0([0,T]; H^{4N}(\Sigma))$, $\partial_t\eta \in C^0([0,T]; H^{4N-1}(\Sigma))$, and $\partial_t^j\eta \in C^0([0,T]; H^{4N-2j+3/2}(\Sigma))$ for $j = 2,\dots,2N$. Throughout the rest of this section we will assume that $\mathfrak{K}(\eta), \mathfrak{E}_0(\eta) \le 1$, which implies that $\mathcal{Q}(\mathfrak{K}(\eta)) \lesssim 1 + \mathfrak{K}(\eta)$ and $\mathcal{Q}(\mathfrak{E}_0(\eta)) \lesssim 1 + \mathfrak{E}_0(\eta)$ for any polynomial $\mathcal{Q}$. Note that $\mathcal{K}(\eta) \le \mathfrak{E}(\eta) \le \mathfrak{K}(\eta)$, where $\mathcal{K}(\eta)$ is defined by (4.13); also, we have that $\|\eta_0\|_{4N-1/2}^2 \le \mathfrak{E}_0(\eta)$.

We now record an estimate of the $F^{i,j}$ in terms of $\mathfrak{F}, \mathfrak{K}(\eta)$ and certain norms of u, p.

Lemma 4.4. *For $m = 1,\dots,2N-1$ and $j = 1,\dots,m$, the following estimates hold whenever the right hand sides are finite:*

$$\begin{aligned} \left\|F^{1,j}\right\|_{L^2H^{2m-2j+1}}^2 + \left\|F^{3,j}\right\|_{L^2H^{2m-2j+3/2}}^2 \lesssim (1+\mathfrak{K}(\eta)) \Bigg(\mathfrak{F} + \sum_{\ell=0}^{j-1} \left\|\partial_t^\ell u\right\|_{L^\infty H^{2m-2\ell}}^2 \\ + \sum_{\ell=0}^{j-1} \left\|\partial_t^\ell p\right\|_{L^\infty H^{2m-2\ell-1}}^2 + \sum_{\ell=0}^{j-1} \left\|\partial_t^\ell u\right\|_{L^2H^{2m-2\ell+1}}^2 + \left\|\partial_t^\ell p\right\|_{L^2H^{2m-2\ell}}^2 \Bigg), \end{aligned} \tag{4.73}$$

$$\begin{aligned} &\left\|F^{1,j}\right\|_{L^\infty H^{2m-2j}}^2 + \left\|F^{3,j}\right\|_{L^\infty H^{2m-2j+1/2}}^2 \\ &\qquad \lesssim (1+\mathfrak{K}(\eta)) \left(\mathfrak{F} + \sum_{\ell=0}^{j-1} \left\|\partial_t^\ell u\right\|_{L^\infty H^{2m-2\ell}}^2 + \left\|\partial_t^\ell p\right\|_{L^\infty H^{2m-2\ell-1}}^2 \right), \end{aligned} \tag{4.74}$$

and

$$\begin{aligned} \left\|\partial_t F^{1,m}\right\|_{L^2H^{-1}}^2 + \left\|\partial_t F^{3,m}\right\|_{L^2H^{-1/2}}^2 \lesssim (1+\mathfrak{K}(\eta)) \Bigg(\mathfrak{F} + \sum_{\ell=0}^{m} \left\|\partial_t^\ell u\right\|_{L^\infty H^{2m-2\ell}}^2 \\ + \sum_{\ell=0}^{m-1} \left\|\partial_t^\ell p\right\|_{L^\infty H^{2m-2\ell-1}}^2 + \sum_{\ell=0}^{m} \left\|\partial_t^\ell u\right\|_{L^2H^{2m-2\ell+1}}^2 + \left\|\partial_t^\ell p\right\|_{L^2H^{2m-2\ell}}^2 \Bigg). \end{aligned} \tag{4.75}$$

Similarly, for $j = 1,\dots,2N-1$,

$$\begin{aligned} &\left\|F^{1,j}(0)\right\|_{4N-2j-2}^2 + \left\|F^{3,j}(0)\right\|_{4N-2j-3/2}^2 \\ &\qquad \lesssim (1+\mathfrak{E}_0(\eta)) \left(\mathfrak{F}_0 + \sum_{\ell=0}^{j-1} \left\|\partial_t^\ell u(0)\right\|_{4N-2\ell}^2 + \left\|\partial_t^\ell p(0)\right\|_{4N-2\ell-1}^2 \right). \end{aligned} \tag{4.76}$$

Proof. The estimates follow from simple but lengthy computations, invoking standard arguments. As such, we present only a sketch of how to derive the estimates (4.73). The estimates (4.74)–(4.76) follow from similar arguments.

To derive the estimate (4.73), we use the definition of $F^{1,j}, F^{3,j}$ given by (4.68) and expand all terms using the Leibniz rule and the definition D_t to rewrite $F^{i,j}$ as a sum of products of two terms: one involving products of various derivatives of $\bar{\eta}$, and one linear in derivatives of u, p, F^1, or F^3. For a.e. $t \in [0,T]$ we then estimate the the norm ($H^{2m-2j+1}$ and $H^{2m-2j+3/2}$, respectively) of the resulting products by using the usual algebraic properties of Sobolev spaces (i.e. Lemma A.1) in conjunction with the Sobolev embeddings. The resulting inequalities may then be integrated in time from 0 to T to find an inequality of the form

$$\left\|F^{1,j}\right\|_{L^2H^{2m-2j+1}}^2 + \left\|F^{3,j}\right\|_{L^2H^{2m-2j+3/2}}^2 \lesssim \mathcal{Q}(\mathfrak{E}(\eta))(\mathfrak{D}(\eta)Y_\infty + Y_2), \tag{4.77}$$

where $\mathcal{Q}(\cdot)$ is a polynomial,

$$(4.78)\quad Y_\infty = \sum_{j=0}^{2N-1} \left\| \partial_t^j F^1 \right\|_{L^\infty H^{4N-2j-2}}^2 + \left\| \partial_t^j F^3 \right\|_{L^\infty H^{4N-2j-3/2}}^2 + \sum_{\ell=0}^{j-1} \left\| \partial_t^\ell u \right\|_{L^\infty H^{2m-2\ell}}^2 + \sum_{\ell=0}^{j-1} \left\| \partial_t^\ell p \right\|_{L^\infty H^{2m-2\ell-1}}^2 ,$$

and

$$(4.79)\quad Y_2 = \sum_{j=0}^{2N} \left\| \partial_t^j F^1 \right\|_{L^2 H^{4N-2j-1}}^2 + \left\| \partial_t^j F^3 \right\|_{L^2 H^{4N-2j-1/2}}^2 + \sum_{\ell=0}^{j-1} \left\| \partial_t^\ell u \right\|_{L^2 H^{2m-2\ell+1}}^2 + \left\| \partial_t^\ell p \right\|_{L^2 H^{2m-2\ell}}^2 .$$

Since $\mathfrak{K}(\eta) \le 1$, we know that $\mathcal{Q}(\mathfrak{E}(\eta))(1+\mathfrak{D}(\eta)) \lesssim (1+\mathfrak{K}(\eta))$, and the bound (4.73) follows immediately from (4.77). □

Next we record an estimates for the difference between $\partial_t v$ and $D_t v$ for a general v. The proof is similar to that of Lemma 4.4, and is thus omitted.

Lemma 4.5. *If $k = 0, \dots, 4N-1$ and v is sufficiently regular, then*

$$(4.80)\quad \|\partial_t v - D_t v\|_{L^2 H^k}^2 \lesssim (1+\mathfrak{K}(\eta)) \|v\|_{L^2 H^k}^2 ,$$

and if $k = 0, \dots, 4N-2$, then

$$(4.81)\quad \|\partial_t v - D_t v\|_{L^\infty H^k}^2 \lesssim (1+\mathfrak{K}(\eta)) \|v\|_{L^\infty H^k}^2 .$$

If $m = 1, \dots, 2N-1$, $j = 1, \dots, m$, and v is sufficiently regular, then

$$(4.82)\quad \left\| \partial_t^j v - D_t^j v \right\|_{L^2 H^{2m-2j+3}}^2 \lesssim (1+\mathfrak{K}(\eta)) \sum_{\ell=0}^{j-1} \left(\left\| \partial_t^\ell v \right\|_{L^2 H^{2m-2\ell+1}}^2 + \left\| \partial_t^\ell v \right\|_{L^\infty H^{2m-2\ell}}^2 \right),$$

$$(4.83)\quad \left\| \partial_t^j v - D_t^j v \right\|_{L^\infty H^{2m-2j+2}}^2 \lesssim (1+\mathfrak{K}(\eta)) \sum_{\ell=0}^{j-1} \left\| \partial_t^\ell v \right\|_{L^\infty H^{2m-2\ell}}^2 ,$$

and

$$(4.84)\quad \left\| \partial_t D_t^m v - \partial_t^{m+1} v \right\|_{L^2 H^1}^2 + \left\| \partial_t^2 D_t^m v - \partial_t^{m+2} v \right\|_{L^2 H^{-1}}^2 \lesssim (1+\mathfrak{K}(\eta)) \sum_{\ell=0}^{m+1} \left(\left\| \partial_t^\ell v \right\|_{L^2 H^{2m-2\ell+1}}^2 + \left\| \partial_t^\ell v \right\|_{L^\infty H^{2m-2\ell}}^2 \right).$$

Also, if $j = 0, \dots, 2N$, and v is sufficiently regular, then

$$(4.85)\quad \left\| \partial_t^j v(0) - D_t^j v(0) \right\|_{4N-2j}^2 \lesssim (1+\mathfrak{E}_0(\eta)) \sum_{\ell=0}^{j-1} \left\| \partial_t^\ell v(0) \right\|_{4N-2\ell}^2 .$$

Now we record an estimate for the terms $\mathfrak{G}^0$ and $\mathfrak{f}^i$ (defined in (4.66) and (4.67), respectively) that will be used in computing initial data.

Lemma 4.6. *Suppose that v, q, G^1, G^3 are evaluated at $t=0$ and are sufficiently regular for the right sides of the following estimates to make sense. For $j = 0, \dots, 2N-1$, we have*

$$(4.86)\quad \left\| \mathfrak{G}^0(G^1, v, q) \right\|_{4N-2j-2}^2 \lesssim (1 + \|\eta(0)\|_{4N}^2 + \|\partial_t \eta(0)\|_{4N-1}^2) \left(\|v\|_{4N-2j}^2 + \|q\|_{4N-2j-1}^2 + \left\|G^1\right\|_{4N-2j-2}^2 \right).$$

If $j = 0, \dots, 2N-2$, *then*

$$\left\|\mathfrak{f}^1(G^1,v)\right\|^2_{4N-2j-3} + \left\|\mathfrak{f}^2(G^3,v)\right\|^2_{4N-2j-3/2} + \left\|\mathfrak{f}^3(G^1,v)\right\|^2_{4N-2j-5/2} \lesssim (1+\|\eta(0)\|^2_{4N}) \left(\left\|G^1\right\|^2_{4N-2j-2} + \left\|G^3\right\|^2_{4N-2j-3/2} + \|v\|^2_{4N-2j} \right). \tag{4.87}$$

For $j = 2N-1$, *if* $\operatorname{div}_{\mathcal{A}(0)} v = 0$ *in* Ω, *then*

$$\left\|\mathfrak{f}^2(G^3,v)\right\|^2_{1/2} + \left\|\mathfrak{f}^3(G^1,v)\right\|^2_{-1/2} \lesssim (1+\|\eta(0)\|^2_{4N}) \left(\left\|G^1\right\|^2_{2} + \left\|G^3\right\|^2_{1/2} + \|v\|^2_{2} \right). \tag{4.88}$$

Proof. The proof of the estimates (4.86) and (4.87) as well as the $\mathfrak{f}^2$ estimate in (4.88) can be carried out as in the proof Lemma 4.4. We omit further details. For the $\mathfrak{f}^3$ estimate of (4.88), we note that $\operatorname{div}_{\mathcal{A}(0)} v = 0$ implies that $\operatorname{div}_{\mathcal{A}(0)} \Delta_{\mathcal{A}(0)} v = 0$, so that Lemmas 3.3 and 2.1 provide the bound $\left\|\Delta_{\mathcal{A}(0)} v \cdot \nu\right\|^2_{H^{-1/2}(\Sigma_b)} \lesssim \left\|\Delta_{\mathcal{A}(0)} v\right\|^2_0$. We may then argue as in Lemma 4.4 to derive the $\mathfrak{f}^3$ bound.

□

Now we assume that $u_0 \in H^{4N}(\Omega)$, $\eta_0 \in H^{4N+1/2}(\Sigma)$, $\mathfrak{F}_0 < \infty$, and that $\|\eta_0\|^2_{4N-1/2} \le \mathfrak{E}_0(\eta) \le 1$ is sufficiently small for the hypothesis of Propositions 3.7 and 3.9 to hold when $k = 4N$. We will iteratively construct the initial data $D_t^j u(0)$ for $j = 0, \dots, 2N$ and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-1$. To do so, we will first construct all but the highest order data, and then we will state some compatibility conditions for the data. These are necessary to construct $D_t^{2N} u(0)$ and $\partial_t^{2N-1} p(0)$, and to construct high-regularity solutions in Theorem 4.7.

We now turn to the construction of $D_t^j u(0)$ for $j = 0, \dots, 2N-1$ and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-2$, which will employ Lemma 4.6 in conjunction with estimate (4.76) of Lemma 4.4 and (4.85) of Lemma 4.5. For $j = 0$ we write $F^{1,0}(0) = F^1(0) \in H^{4N-2}$, $F^{3,0}(0) = F^3(0) \in H^{4N-3/2}$, and $D_t^0 u(0) = u_0 \in H^{4N}$. Suppose now that $F^{1,\ell} \in H^{4N-2\ell-2}$, $F^{3,\ell} \in H^{4N-2\ell-3/2}$, and $D_t^\ell u(0) \in H^{4N-2\ell}$ are given for $0 \le \ell \le j \in [0, 2N-2]$; we will define $\partial_t^j p(0) \in H^{4N-2j-1}$ as well as $D_t^{j+1} u(0) \in H^{4N-2j-2}$, $F^{1,j+1}(0) \in H^{4N-2j-4}$, and $F^{3,j+1}(0) \in H^{4N-2j-7/2}$, which allows us to define all of said data via iteration. By virtue of estimate (4.87), we know that $f^1 = \mathfrak{f}^1(F^{1,j}(0), D_t^j u(0)) \in H^{4N-2j-3}$, $f^2 = \mathfrak{f}^2(F^{3,j}(0), D_t^j u(0)) \in H^{4N-2j-3/2}$, and $f^3 = \mathfrak{f}^3(F^{1,j}(0), D_t^j u(0)) \in H^{4N-2j-5/2}$. This allows us to define $\partial_t^j p(0)$ as the solution to (3.43) with this choice of f^1, f^2, f^3, and then Proposition 3.9 with $k = 4N$ and $r = 4N - 2j - 1 < k$ implies that $\partial_t^j p(0) \in H^{4N-2j-1}$. Now the estimates (4.76), (4.85), and (4.86) allow us to define

$$\begin{aligned} D_t^{j+1} u(0) &:= \mathfrak{G}^0(F^{1,j}(0), D_t^j u(0), \partial_t^j p(0)) \in H^{4N-2j-2}, \\ F^{1,j+1}(0) &:= D_t F^{1,j}(0) + \mathfrak{G}^1(D_t^j u(0), \partial_t^j p(0)) \in H^{4N-2j-4}, \text{ and} \\ F^{3,j+1}(0) &:= \partial_t F^{3,j}(0) + \mathfrak{G}^3(D_t^j u(0), \partial_t^j p(0)) \in H^{4N-2j-7/2}. \end{aligned} \tag{4.89}$$

Using the above analysis, we iteratively construct all of the desired data except for $D_t^{2N} u(0)$ and $\partial_t^{2N-1} p(0)$.

By construction, the initial data $D_t^j u(0)$ and $\partial_t^j p(0)$ are determined in terms of u_0 as well as $\partial_t^\ell F^1(0)$ and $\partial_t^\ell F^3(0)$ for $\ell = 0, \dots, 2N-1$. In order to use these in Theorem 4.3 and to construct $D_t^{2N} u(0)$ and $\partial_t^{2N-1} p(0)$, we must enforce compatibility conditions for $j = 0, \dots, 2N-1$. For such j, we say that the j^{th} compatibility condition is satisfied if

$$\begin{cases} D_t^j u(0) \in \mathcal{X}(0) \cap H^2(\Omega) \\ \Pi_0(F^{3,j}(0) + \mathbb{D}_{\mathcal{A}_0} D_t^j u(0) \mathcal{N}_0) = 0. \end{cases} \tag{4.90}$$

Note that the construction of $D_t^j u(0)$ and $\partial_t^j p(0)$ ensures that $D_t^j u(0) \in H^2(\Omega)$ and that $\operatorname{div}_{\mathcal{A}_0}(D_t^j u(0)) = 0$, so the condition $D_t^j u(0) \in \mathcal{X}(0) \cap H^2(\Omega)$ may be reduced to the condition $D_t^j u(0)|_\Sigma = 0$.

It remains only to define $\partial_t^{2N-1} p(0) \in H^1$ and $D_t^{2N} u(0) \in H^0$. According to the $j = 2N-1$ compatibility condition (4.90), $\operatorname{div}_{\mathcal{A}_0} D_t^{2N-1} u(0) = 0$, which means that we can use estimate (4.88) of Lemma 4.6 to see that $f^2 = \mathfrak{f}^2(F^{3,2N-1}(0), D_t^{2N-1}u(0)) \in H^{1/2}$ and $f^3 = \mathfrak{f}^3(F^{1,2N-1}(0), D_t^{2N-1}u(0)) \in H^{-1/2}$. We also see from (4.90) that we have the equality $g_0 = -\operatorname{div}_{\mathcal{A}_0}(R(0)D_t^{2N-1}u(0))$ so that $g_0 \in H^0$. Then, owing to the fact that $G = -F^{1,2N-1} \in H^0$, we can define $\partial_t^{2N-1}p(0) \in H^1$ as a weak solution to (3.43) in the sense of (3.49) with this choice of f^2, f^3, g_0, and G Then we define

$$D_t^{2N}u(0) = \mathfrak{G}^0(F^{1,2N-1}(0), D_t^{2N-1}u(0), \partial_t^{2N-1}p(0)) \in H^0, \tag{4.91}$$

employing (4.86) for the inclusion in H^0. In fact, the construction of $\partial_t^{2N-1}p(0)$ guarantees that $D_t^{2N}u(0) \in \mathcal{Y}(0)$. In addition to providing the above inclusions, the bounds (4.76), (4.87), (4.86) also imply the estimate

$$\sum_{j=0}^{2N} \left\| D_t^j u(0) \right\|_{4N-2j}^2 + \sum_{j=0}^{2N-1} \left\| \partial_t^j p(0) \right\|_{4N-2j-1}^2 \lesssim (1 + \mathfrak{E}_0(\eta) \left(\|u_0\|_{4N}^2 + \mathfrak{F}_0 \right). \tag{4.92}$$

Note that, owing to estimate (4.85), the bound (4.92) also holds with $\partial_t^j u(0)$ replacing $D_t^j u(0)$ on the left.

Before stating our result on higher regularity for solutions to problem (1.16), we define two quantities associated to u, p. Write

$$\begin{aligned} \mathfrak{D}(u,p) &:= \sum_{j=0}^{2N+1} \left\| \partial_t^j u \right\|_{L^2 H^{4N-2j+1}}^2 + \sum_{j=0}^{2N} \left\| \partial_t^j p \right\|_{L^2 H^{4N-2j}}^2, \\ \mathfrak{E}(u,p) &:= \sum_{j=0}^{2N} \left\| \partial_t^j u \right\|_{L^\infty H^{4N-2j}}^2 + \sum_{j=0}^{2N-1} \left\| \partial_t^j p \right\|_{L^\infty H^{4N-2j-1}}^2, \\ \mathfrak{K}(u,p) &:= \mathfrak{E}(u,p) + \mathfrak{D}(u,p). \end{aligned} \tag{4.93}$$

Note that, again, Lemmas 2.4 and A.3 imply that $\mathfrak{E}(u,p) \le C(T)\mathfrak{D}(u,p)$ for $C(T)$ a constant depending on T. To avoid introducing this constant we will use both $\mathfrak{E}(u,p)$ and $\mathfrak{D}(u,p)$.

Theorem 4.7. *Suppose that $u_0 \in H^{4N}(\Omega)$, $\eta_0 \in H^{4N+1/2}(\Sigma)$, $\mathfrak{F} < \infty$, and that $\mathfrak{K}(\eta) \le 1$ is sufficiently small so that $\mathcal{K}(\eta)$, defined by (4.13), satisfies the hypotheses of Theorem 4.3 and Proposition 3.9. Let $D_t^j u(0) \in H^{4N-2j}(\Omega)$ and $\partial_t^j p(0) \in H^{4N-2j-1}$ for $j = 0, \dots, 2N-1$ along with $D_t^{2N}u(0) \in \mathcal{Y}(0)$ all be determined as above in terms of u_0 and $\partial_t^j F^1(0)$, $\partial_t^j F^3(0)$ for $j = 0, \dots, 2N-1$. Suppose that for $j = 0, \dots, 2N-1$, the initial data satisfy the j^{th} compatibility condition (4.90).*

Then there exists a unique strong solution (u,p) to (1.16) so that

$$\begin{aligned} &\partial_t^j u \in C^0([0,T]; H^{4N-2j}(\Omega)) \cap L^2([0,T]; H^{4N-2j+1}(\Omega)) \text{ for } j = 0, \dots, 2N, \\ &\partial_t^j p \in C^0([0,T]; H^{4N-2j-1}(\Omega)) \cap L^2([0,T]; H^{4N-2j}(\Omega)) \text{ for } j = 0, \dots, 2N-1, \\ &\partial_t^{2N+1} u \in (\mathcal{H}_T^1)^*, \text{ and } \partial_t^{2N} p \in L^2([0,T]; H^0(\Omega)). \end{aligned} \tag{4.94}$$

The pair $(D_t^j u, \partial_t^j p)$ satisfies the PDE

$$\begin{cases} \partial_t(D_t^j u) - \Delta_{\mathcal{A}}(D_t^j u) + \nabla_{\mathcal{A}}(\partial_t^j p) = F^{1,j} & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}}(D_t^j u) = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(\partial_t^j p, D_t^j u)\mathcal{N} = F^{3,j} & \text{on } \Sigma \\ D_t^j u = 0 & \text{on } \Sigma_b \end{cases} \tag{4.95}$$

in the strong sense with initial data $(D_t^j u(0), \partial_t^j p(0))$ for $j = 0, \dots, 2N-1$, and in the weak sense of (4.3) with initial data $D_t^{2N}u(0) \in \mathcal{Y}(0)$ for $j = 2N$. Here the vectors $F^{1,j}$ and $F^{3,j}$ are as defined by (4.68). Moreover, the solution satisfies the estimate

$$\mathfrak{E}(u,p) + \mathfrak{D}(u,p) \lesssim (1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)) \exp\left(C(1 + \mathfrak{E}(\eta))T\right) \left(\|u_0\|_{4N}^2 + \mathfrak{F}_0 + \mathfrak{F} \right) \tag{4.96}$$

for a constant $C > 0$, independent of η.

Proof. For notational convenience, throughout the proof we write

$$\mathcal{Z} := (1 + \mathfrak{E}_0(\eta) + \mathfrak{K}(\eta)) \exp\left(C(1+\mathfrak{E}(\eta))T\right) \left(\|u_0\|_{4N}^2 + \mathfrak{F}_0 + \mathfrak{F} \right). \tag{4.97}$$

Since the 0^{th} order compatibility condition (4.90) is satisfied and $\mathfrak{K}(\eta)$ is small enough for $\mathcal{K}(\eta)$ to satisfy the hypotheses of Theorem 4.3, we may apply Theorem 4.3. It guarantees the existence of (u,p) satisfying the inclusions $\partial_t^j u \in L^2 H^{3-2j}$ for $j = 0,1,2$ and $\partial_t^j p \in L^2 H^{2-2j}$ for $j = 0,1$. The $(D_t^j u, \partial_t^j p)$ are solutions in that (4.95) is satisfied in the strong sense when $j = 0$ and in the weak sense when $j = 1$. Finally, the estimate (4.17) holds, but we may replace its right hand side by $\mathcal{Z}$ since $\mathcal{K}(\eta) \le \mathfrak{E}(\eta) \le \mathfrak{K}(\eta)$.

For an integer $m \ge 0$, let $\mathbb{P}_m$ denote the proposition asserting the following three statements. First, that $(D_t^j u, \partial_t^j p)$ are solutions to (4.95) in the strong sense for $j = 0, \dots, m$ and in the weak sense for $j = m+1$. Second, that $\partial_t^j u \in L^2 H^{2m-2j+3}$ for $j = 0,1,\dots,m+2$, $\partial_t^j u \in L^\infty H^{2m-2j+2}$ for $j = 0,1,\dots,m+1$, $\partial_t^j p \in L^2 H^{2m-2j+2}$ for $j = 0,1,\dots,m+1$, and $\partial_t^j p \in L^\infty H^{2m-2j+1}$ for $j = 0,1,\dots,m$. Third, that the estimate

$$\sum_{j=0}^{m+1} \left\| \partial_t^j u \right\|_{L^\infty H^{2m-2j+2}}^2 + \sum_{j=0}^{m} \left\| \partial_t^j p \right\|_{L^\infty H^{2m-2j+1}}^2 + \sum_{j=0}^{m+2} \left\| \partial_t^j u \right\|_{L^2 H^{2m-2j+3}}^2 + \sum_{j=0}^{m+1} \left\| \partial_t^j p \right\|_{L^2 H^{2m-2j+2}}^2 \lesssim \mathcal{Z} \tag{4.98}$$

holds.

The above analysis implies that $\mathbb{P}_0$ holds. We claim that if $\mathbb{P}_m$ holds for some $m = 0, \dots, 2N-2$, then $\mathbb{P}_{m+1}$ also holds. Once the claim is established, a finite induction implies that $\mathbb{P}_m$ holds for all $m = 0, \dots, 2N-1$, which immediately implies all of the conclusions of the theorem. The rest of the proof is dedicated to the proof of this claim.

Suppose that $\mathbb{P}_m$ holds for some $m = 0, \dots, 2N-2$. We may then combine (4.98) with the estimates (4.73), (4.74), and (4.75) of Lemma 4.4 to see that

$$\begin{aligned} &\sum_{j=1}^{m+1} \left(\|F^{1,j}\|_{L^2 H^{2m-2j+3}}^2 + \|F^{3,j}\|_{L^2 H^{2m-2j+7/2}}^2 + \|F^{1,j}\|_{L^\infty H^{2m-2j+2}}^2 + \|F^{3,j}\|_{L^\infty H^{2m-2j+5/2}}^2 \right) \\ &\quad + \|\partial_t F^{1,m+1}\|_{L^2 H^{-1}}^2 + \|\partial_t F^{3,m+1}\|_{L^2 H^{-1/2}}^2 \lesssim (1+\mathfrak{K}(\eta)) \left(\mathfrak{F} + \sum_{\ell=0}^{m+1} \left\| \partial_t^\ell u \right\|_{L^\infty H^{2m-2\ell+2}}^2 \right. \\ &\quad \left. + \sum_{\ell=0}^{m} \left\| \partial_t^\ell p \right\|_{L^\infty H^{2m-2\ell+1}}^2 + \sum_{\ell=0}^{m+1} \left\| \partial_t^\ell u \right\|_{L^2 H^{2m-2\ell+3}}^2 + \left\| \partial_t^\ell p \right\|_{L^2 H^{2m-2\ell+2}}^2 \right) \\ &\quad \lesssim (1+\mathfrak{K}(\eta)) (\mathfrak{F} + \mathcal{Z}) \lesssim \mathcal{Z}. \end{aligned} \tag{4.99}$$

The last inequality in (4.99) follows from the fact that $\mathfrak{K}(\eta) \le 1$ and the definition of $\mathcal{Z}$.

We now show that the first assertion of $\mathbb{P}_{m+1}$ holds. To this end, we note that the estimate (4.99) implies that $F^{1,m+1} \in L^2 H^1$, $\partial_t F^{1,m+1} \in L^2 H^{-1}$, $F^{3,m+1} \in L^2 H^{3/2}$, and $\partial_t F^{3,m+1} \in L^2 H^{-1/2}$. These inclusions, together with the fact that $D_t^{m+1} u(0)$ satisfies the $(m+1)^{st}$ order compatibility condition (4.90), allow us to apply Theorem 4.3 to solve problem (1.16), with F^1, F^3 replaced by $F^{1,m+1}, F^{3,m+1}$ and with initial data $D_t^{m+1} u(0)$. The resulting strong solution solution must equal $(D_t^{m+1} u, \partial_t^{m+1} p)$, the weak solution to (4.95) provided by $\mathbb{P}_m$, since strong solutions are also weak solutions and Proposition 4.2 guarantees that weak solutions are unique. Furthermore, the theorem implies that $(D_t^{m+2} u, \partial_t^{m+2} p)$ are a weak solution to (4.95). Since $\mathbb{P}_m$ already provided that $(D_t^j u, \partial_t^j p)$ are solutions to (4.95) in the strong sense for $j = 0, \dots, m$, we deduce that the first assertion of $\mathbb{P}_{m+1}$ holds.

It remains to prove the the second and third assertions of $\mathbb{P}_{m+1}$; they are intertwined and will be derived simultaneously. To begin, we note that the previous application of Theorem 4.3

also provides, by way of (4.17), the estimate

(4.100)
$$\begin{aligned}\left\|D_t^{m+1}u\right\|_{L^2H^3}^2 + \left\|\partial_t D_t^{m+1}u\right\|_{L^2H^1}^2 + \left\|\partial_t^2 D_t^{m+1}u\right\|_{L^2H^{-1}}^2 + \left\|\partial_t^{m+1}p\right\|_{L^2H^2}^2 + \left\|\partial_t^{m+2}p\right\|_{L^2H^0}^2 \\ + \left\|D_t^{m+1}u\right\|_{L^\infty H^2}^2 + \left\|\partial_t D_t^{m+1}u\right\|_{L^\infty H^0}^2 + \left\|\partial_t^{m+1}p\right\|_{L^\infty H^1}^2 \\ \lesssim (1+\mathfrak{K}(\eta))\exp\left(C(1+\mathfrak{E}(\eta))T\right)\left(\left\|D_t^{m+1}u(0)\right\|_2^2 + \left\|F^{1,m+1}(0)\right\|_0^2 + \left\|F^{3,m+1}(0)\right\|_{1/2}^2 + \mathfrak{F}\right) \\ \lesssim (1+\mathfrak{E}_0(\eta)+\mathfrak{K}(\eta))\exp\left(C(1+\mathfrak{E}(\eta))T\right)\left(\|u_0\|_{4N}^2 + \mathfrak{F}_0 + \mathfrak{F}\right) \lesssim \mathcal{Z},\end{aligned}$$

where in the second inequality we have employed estimate (4.76) to control the $F^{i,m+1}(0)$ terms and the bound (4.92) to bound the resulting temporal derivatives or u and p at $t = 0$. The estimates of the u terms in (4.100), together with the estimates (4.82)–(4.84) of Lemma 4.5 and the estimate (4.98), imply that

(4.101)
$$\begin{aligned}\left\|\partial_t^{m+1}u\right\|_{L^2H^3}^2 + \left\|\partial_t^{m+2}u\right\|_{L^2H^1}^2 + \left\|\partial_t^{m+3}u\right\|_{L^2H^{-1}}^2 + \left\|\partial_t^{m+1}u\right\|_{L^\infty H^2}^2 + \left\|\partial_t^{m+2}u\right\|_{L^\infty H^0}^2 \\ \lesssim (1+\mathfrak{K}(\eta))\left(\sum_{\ell=0}^{m+2}\left\|\partial_t^\ell u\right\|_{L^2H^{2m-2\ell+3}}^2 + \sum_{\ell=0}^{m+1}\left\|\partial_t^\ell u\right\|_{L^\infty H^{2m-2\ell+2}}^2\right) + \mathcal{Z} \\ \lesssim (1+\mathfrak{K}(\eta))\mathcal{Z} + \mathcal{Z} \lesssim \mathcal{Z}.\end{aligned}$$

Hence

(4.102)
$$\begin{aligned}\sum_{j=m+1}^{m+2}\left\|\partial_t^j u\right\|_{L^\infty H^{2(m+1)-2j+2}}^2 \sum_{j=m+1}^{m+1}\left\|\partial_t^j p\right\|_{L^\infty H^{2(m+1)-2j+1}}^2 \\ + \sum_{j=m+1}^{m+3}\left\|\partial_t^j u\right\|_{L^2H^{2(m+1)-2j+3}}^2 + \sum_{j=m+1}^{m+2}\left\|\partial_t^j p\right\|_{L^2H^{2(m+1)-2j+2}}^2 \lesssim \mathcal{Z},\end{aligned}$$

which means that in order to derive the estimate (4.98) with m replaced by $m+1$, it suffices to prove that

(4.103)
$$\begin{aligned}\sum_{j=0}^{m}\left\|\partial_t^j u\right\|_{L^\infty H^{2(m+1)-2j+2}}^2 + \left\|\partial_t^j p\right\|_{L^\infty H^{2(m+1)-2j+1}}^2 \\ + \sum_{j=0}^{m}\left\|\partial_t^j u\right\|_{L^2H^{2(m+1)-2j+3}}^2 + \left\|\partial_t^j p\right\|_{L^2H^{2(m+1)-2j+2}}^2 \lesssim \mathcal{Z}.\end{aligned}$$

Once (4.103) is established, summing (4.102) and (4.103) implies that (4.98) holds with m replaced by $m+1$, which further implies that the second and third assertions of $\mathbb{P}_{m+1}$ hold, so that then all of $\mathbb{P}_{m+1}$ holds.

In order to prove (4.103) we will use the elliptic regularity of Proposition 3.7 (with $k = 4N$) and an iteration argument. The estimates of $D_t^{m+1}u$ in (4.100), together with (4.98) and the estimates (4.80) and (4.81) of Lemma 4.5, allow us to deduce that

(4.104)
$$\|\partial_t D_t^m u\|_{L^\infty H^2}^2 + \|\partial_t D_t^m u\|_{L^2H^3}^2 \lesssim \mathcal{Z}.$$

Since (4.95) is satisfied in the strong sense for $j = m$, we may rearrange to find that for a.e. $t \in [0,T]$, $(D_t^m, \partial_t^m p)$ solve the elliptic problem (3.13) with F^1 replaced by $F^{1,m} - \partial_t D_t^m u$, $F^2 = 0$, and F^3 replaced by $F^{3,m}$. We may then apply Proposition 3.7 with $r = 5$ to deduce that the estimate (3.27) holds for a.e. $t \in [0,T]$; squaring this estimate and integrating over $[0,T]$ then yields the inequality

(4.105)
$$\begin{aligned}\|D_t^m u\|_{L^2H^5}^2 + \|\partial_t^m p\|_{L^2H^4}^2 \lesssim \left\|F^{1,m} - \partial_t D_t^m u\right\|_{L^2H^3}^2 + \left\|F^{3,m}\right\|_{L^2H^{7/2}}^2 \\ \lesssim \left\|F^{1,m}\right\|_{L^2H^3}^2 + \|\partial_t D_t^m u\|_{L^2H^3}^2 + \left\|F^{3,m}\right\|_{L^2H^{7/2}}^2 \lesssim \mathcal{Z},\end{aligned}$$

where in the last inequality we have used (4.99) and (4.104). Similarly, we may apply Proposition 3.7 with $r = 4$ to deduce

$$\|D_t^m u\|_{L^\infty H^4}^2 + \|\partial_t^m p\|_{L^\infty H^3}^2 \lesssim \left\|F^{1,m} - \partial_t D_t^m u\right\|_{L^\infty H^2}^2 + \left\|F^{3,m}\right\|_{L^\infty H^{5/2}}^2 \lesssim \mathcal{Z}. \tag{4.106}$$

We may argue as before to deduce from (4.105) and (4.106) that

$$\|\partial_t^m u\|_{L^\infty H^4}^2 + \|\partial_t^m u\|_{L^2 H^5}^2 \lesssim \mathcal{Z} \tag{4.107}$$

as well. This argument may be iterated to estimate $\partial_t^j u$, $\partial_t^j p$ for $j = 1, \dots, m$; this yields the estimate

$$\sum_{j=1}^m \left\|\partial_t^j u\right\|_{L^\infty H^{2(m+1)-2j+2}}^2 + \left\|\partial_t^j p\right\|_{L^\infty H^{2(m+1)-2j+1}}^2 + \sum_{j=1}^m \left\|\partial_t^j u\right\|_{L^2 H^{2(m+1)-2j+3}}^2 + \left\|\partial_t^j p\right\|_{L^2 H^{2(m+1)-2j+2}}^2 \lesssim \mathcal{Z}. \tag{4.108}$$

We then apply Proposition 3.7 with $r = 2(m+1) + 2 \le 4N$ to see that

$$\begin{aligned} \|u\|_{L^\infty H^{2(m+1)+1}}^2 + \|p\|_{L^\infty H^{2(m+1)+1}}^2 &\lesssim \left\|F^1 - \partial_t u\right\|_{L^\infty H^{2(m+1)}}^2 + \left\|F^3\right\|_{L^\infty H^{2(m+1)+1/2}}^2 \\ &\lesssim \left\|F^1\right\|_{L^\infty H^{2(m+1)}}^2 + \|\partial_t u\|_{L^\infty H^{2(m+1)}}^2 + \left\|F^3\right\|_{L^\infty H^{2(m+1)+1/2}}^2 \lesssim \mathcal{Z}, \end{aligned} \tag{4.109}$$

and then again with $r = 2(m+1) + 3 \le 4N + 1$ to see that

$$\begin{aligned} &\|u\|_{L^2 H^{2(m+1)+3}}^2 + \|p\|_{L^2 H^{2(m+1)+2}}^2 \lesssim \left\|F^1 - \partial_t u\right\|_{L^2 H^{2(m+1)+1}}^2 + \left\|F^3\right\|_{L^2 H^{2(m+1)+3/2}}^2 \\ &+ \|\eta\|_{L^2 H^{4N+1/2}}^2 \left(\left\|F^1 - \partial_t u\right\|_{L^\infty H^2}^2 + \left\|F^3\right\|_{L^\infty H^{5/2}}^2\right) \lesssim \left\|F^1\right\|_{L^2 H^{2(m+1)+1}}^2 + \|\partial_t u\|_{L^2 H^{2(m+1)+1}}^2 \\ &+ \left\|F^3\right\|_{L^2 H^{2(m+1)+3/2}}^2 + \mathfrak{K}(\eta)(\mathfrak{F} + \mathcal{Z}) \lesssim \mathcal{Z}. \end{aligned} \tag{4.110}$$

Summing (4.108)–(4.110) then gives (4.103), completing the proof. □

5. Preliminaries for the nonlinear problem

5.1. **Forcing estimates.** We want to eventually use our linear theory for the problem (1.16) in order to solve the nonlinear problem (1.10). To do so, we define forcing terms F^1, F^3 to be used in the linear theory that match the terms in (1.10). That is, given u, η, we define

$$\begin{aligned} F^1(u,\eta) &= \partial_t \bar{\eta} \tilde{b} K \partial_3 u - u \cdot \nabla_{\mathcal{A}} u, \text{ and} \\ F^3(u,\eta) &= \eta \mathcal{N} = -\eta D\eta + \eta e_3, \end{aligned} \tag{5.1}$$

where $\mathcal{A}, \mathcal{N}, K$ are determined as usual by η.

We will need to be able to estimate various norms of $F^1(u,\eta)$ and $F^3(u,\eta)$ in terms of the norms of u and η that appear in $\mathfrak{K}(\eta)$, $\mathfrak{E}_0(\eta)$, and $\mathfrak{K}(u,p)$, defined by (4.71), (4.72), and (4.93), respectively. The norms of the F^i terms are contained in $\mathfrak{F}$ and $\mathfrak{F}_0$, as defined by (4.69). We will actually need a slight modification of $\mathfrak{K}(u,p)$, which we define as

$$\mathfrak{K}_{2N}(u) = \sum_{j=0}^{2N} \left\|\partial_t^j u\right\|_{L^2 H^{4N-2j+1}}^2 + \left\|\partial_t^j u\right\|_{L^\infty H^{4N-2j}}^2. \tag{5.2}$$

Our estimates are the content of the following lemma.

Lemma 5.1. *Suppose that $\mathfrak{K}(\eta) \le 1$ and $\mathfrak{K}_{2N}(u) < \infty$. Then*

$$\mathfrak{F}(F^1(u,\eta), F^3(u,\eta)) \lesssim [1 + T + \mathfrak{K}(\eta)]\, \mathfrak{E}(\eta) + \mathfrak{K}(\eta) \left[\mathfrak{K}_{2N}(u) + (\mathfrak{K}_{2N}(u))^2\right] + (\mathfrak{K}_{2N}(u))^2. \tag{5.3}$$

Proof. All terms in the definition of $F^1(u,\eta)$, $F^3(u,\eta)$ are quadratic or higher-order except the term ηe_3 in F^3. As such, we may argue as in the proof of Lemma 4.4 to deduce the bound

$$\mathfrak{F}(F^1(u,\eta),F^3(u,\eta)-\eta e_3) \lesssim \mathfrak{E}(\eta)\mathfrak{K}(\eta)+\mathfrak{K}(\eta)(\mathfrak{K}(\eta)+\mathfrak{K}_{2N}(u)+(\mathfrak{K}_{2N}(u))^2)+(\mathfrak{K}_{2N}(u))^2. \tag{5.4}$$

Here the appearance of the term $\mathfrak{E}(\eta)\mathfrak{K}(\eta)$ is due to the term $\eta D\eta$ in F^3, while the appearance of $\mathfrak{K}_{2N}(u)^2$ is due to the term $u\cdot\nabla u$ that appears when we write $u\cdot\nabla_{\mathcal{A}}u = u\cdot\nabla u + u\cdot\nabla_{\mathcal{A}-I}u$ in F^1.

On the other hand, by definition, we have

$$\begin{aligned}\mathfrak{F}(0,\eta e_3) &= \sum_{j=0}^{2N}\left\|\partial_t^j\eta\right\|^2_{L^2H^{4N-2j-1/2}} + \sum_{j=0}^{2N-1}\left\|\partial_t^j\eta\right\|^2_{L^\infty H^{4N-2j-3/2}} \\ &\lesssim (1+T)\sum_{j=0}^{2N}\left\|\partial_t^j\eta\right\|^2_{L^\infty H^{4N-2j}} = (1+T)\mathfrak{E}(\eta).\end{aligned} \tag{5.5}$$

Then, since $\mathfrak{F}(X,Y+Z)\lesssim\mathfrak{F}(X,Y)+\mathfrak{F}(0,Z)$, we may combine (5.4) with (5.5) to deduce (5.3). □

5.2. **Data estimates.** In the construction of the initial data performed after Lemma 4.6 it was assumed that $\partial_t^j\eta(0)$ for $j=0,\dots,2N$ and $\partial_t^jF^1(0)$, $\partial_t^jF^3(0)$ for $j=0,\dots,2N-1$ were all known. Knowledge of the former allowed us to compute $R(0)$, $\mathcal{A}_0$, $\mathcal{N}_0$, etc along with their temporal derivatives; these quantities then served as coefficients in deriving the initial conditions for u,p and their temporal derivatives. Since for the full nonlinear problem the function η is unknown and its evolution is coupled to that of u and p, we must revise the construction of the data to include this coupling, assuming only that u_0 and η_0 are given. This will also reveal the compatibility conditions that must be satisfied by u_0 and η_0 in order to solve the nonlinear problem (1.10). To this end we first define the quantities

$$\mathcal{E}_0 := \|u_0\|^2_{4N}+\|\eta_0\|^2_{4N}, \text{ and } \mathcal{F}_0 := \|\eta_0\|^2_{4N+1/2}. \tag{5.6}$$

For our estimates we must also introduce the quantity

$$\mathfrak{E}_0(u,p) = \sum_{j=0}^{2N}\left\|\partial_t^j u(0)\right\|^2_{4N-2j} + \sum_{j=0}^{2N-1}\left\|\partial_t^j p(0)\right\|^2_{4N-2j-1}. \tag{5.7}$$

We will also need a more exact enumeration of the terms in $\mathfrak{E}_0(u,p)$, $\mathfrak{E}_0(\eta)$, and $\mathfrak{F}_0$ (as defined in (5.7), (4.72), and (4.69), respectively). For $j=0,\dots,2N-1$ we define

$$\mathfrak{F}_0^j(F^1(u,\eta),F^3(u,\eta)) := \sum_{\ell=0}^{j}\left\|\partial_t^\ell F^1(0)\right\|^2_{4N-2\ell-2} + \left\|\partial_t^\ell F^3(0)\right\|^2_{4N-2\ell-3/2}, \text{ and} \tag{5.8}$$

$$\mathfrak{E}_0^j(\eta) := \|\eta_0\|^2_{4N} + \|\partial_t\eta(0)\|^2_{4N-1} + \sum_{\ell=2}^{j}\left\|\partial_t^\ell\eta(0)\right\|^2_{4N-2\ell+3/2}, \tag{5.9}$$

with the sum in (5.9) only including the first term when $j=0$ and only the first two terms when $j=1$. For $j=0$ we write $\mathfrak{E}_0^0(u,p) := \|u_0\|^2_{4N}$, and for $j=1,\dots,2N$ we write

$$\mathfrak{E}_0^j(u,p) := \sum_{\ell=0}^{j}\left\|\partial_t^\ell u(0)\right\|^2_{4N-2j} + \sum_{\ell=0}^{j-1}\left\|\partial_t^\ell p(0)\right\|^2_{4N-2j-1}. \tag{5.10}$$

The following lemma records more refined versions of the estimates (4.76) and (4.85) as well as some other related estimates that are useful in dealing with the initial data.

Lemma 5.2. *For $F^1(u,\eta)$ and $F^3(u,\eta)$ defined by* (5.1) *and $j=0,\dots,2N-1$, it holds that*

$$\mathfrak{F}_0^j(F^1(u,\eta),F^3(u,\eta)) \le P_j(\mathfrak{E}_0^{j+1}(\eta),\mathfrak{E}_0^j(u,p)) \tag{5.11}$$

for $P_j(\cdot,\cdot)$ a polynomial so that $P_j(0,0)=0$.

For $j = 1, \dots, 2N-1$ let $F^{1,j}(0)$ and $F^{3,j}(0)$ be determined by (4.68) and (5.1), using $\partial_t^\ell \eta(0)$, $\partial_t^\ell u(0)$, and $\partial_t^\ell p(0)$ for appropriate values of ℓ. Then

$$\left\| F^{1,j}(0) \right\|_{4N-2j-2}^2 + \left\| F^{3,j}(0) \right\|_{4N-2j-3/2}^2 \le P_j(\mathfrak{E}_0^{j+1}(\eta), \mathfrak{E}_0^j(u,p)) \tag{5.12}$$

for $P_j(\cdot,\cdot)$ a polynomial so that $P_j(0,0) = 0$.

For $j = 0, \dots, 2N$ it holds that

$$\left\| \partial_t^j u(0) - D_t^j u(0) \right\|_{4N-2j}^2 \le P_j(\mathfrak{E}_0^j(\eta), \mathfrak{E}_0^j(u,p)) \tag{5.13}$$

for $P_j(\cdot,\cdot)$ a polynomial so that $P_j(0,0) = 0$.

For $j = 1, \dots, 2N-1$ it holds that

$$\left\| \sum_{\ell=0}^{j} \binom{j}{\ell} \partial_t^\ell \mathcal{N}(0) \cdot \partial_t^{j-\ell} u(0) \right\|_{H^{4N-2j+3/2}(\Sigma)}^2 \le P_j(\mathfrak{E}_0^j(\eta), \mathfrak{E}_0^j(u,p)) \tag{5.14}$$

for $P_j(\cdot,\cdot)$ a polynomial so that $P_j(0,0) = 0$. Also,

$$\| u_0 \cdot \mathcal{N}_0 \|_{H^{4N-1}(\Sigma)}^2 \lesssim \| u_0 \|_{4N}^2 + \| \eta_0 \|_{4N}^2 . \tag{5.15}$$

Proof. These bounds may be derived by arguing as in the proof of Lemma 4.4. As such, we again omit the details. □

This lemma allows us to modify the construction presented after Lemma 4.6 to construct all of the initial data $\partial_t^j u(0)$, $\partial_t^j \eta(0)$ for $j = 0, \dots, 2N$ and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-1$. Along the way we will also derive estimates of $\mathfrak{E}_0(u,p) + \mathfrak{E}_0(\eta)$ in terms of $\mathcal{E}_0$ and determine the compatibility conditions for u_0, η_0 necessary for existence of solutions to (1.10).

We assume that u_0, η_0 satisfy $\mathcal{F}_0 < \infty$ and that $\|\eta_0\|_{4N-1/2}^2 \le \mathcal{E}_0 \le 1$ is sufficiently small for the hypothesis of Proposition 3.9 to hold when $k = 4N$. As before, we will iteratively construct the initial data, but this time we will use the estimates in Lemma 5.2. Define $\partial_t \eta(0) = u_0 \cdot \mathcal{N}_0$, where $u_0 \in H^{4N-1/2}(\Sigma)$ when traced onto Σ, and $\mathcal{N}_0$ is determined in terms of η_0. Estimate (5.15) implies that $\|\partial_t \eta(0)\|_{4N-1}^2 \lesssim \mathcal{E}_0$, and hence that $\mathfrak{E}_0^0(u,p) + \mathfrak{E}_0^1(\eta) \lesssim \mathcal{E}_0$. We may use this bound in (5.11) with $j = 0$ to find that

$$\mathfrak{F}_0^0(F^1(u,\eta), F^3(u,\eta)) \le P_0(\mathfrak{E}_0^1(\eta), \mathfrak{E}_0^0(u,p)) \le P(\mathcal{E}_0) \tag{5.16}$$

for a polynomial $P(\cdot)$ so that $P(0) = 0$. Note that in this estimate and in the estimates below, we employ a convention with polynomials of $\mathcal{E}_0$ similar to the one we employ with constants: they are allowed to change from line to line, but they always satisfy $P(0) = 0$.

Suppose now that $j \in [0, 2N-2]$ and that $\partial_t^\ell u(0)$ are known for $\ell = 0, \dots, j$, $\partial_t^\ell \eta(0)$ are known for $\ell = 0, \dots, j+1$, and $\partial_t^\ell p(0)$ are known for $\ell = 0, \dots, j-1$ (with the understanding that nothing is known of $p(0)$ when $j = 0$), and that

$$\mathfrak{E}_0^j(u,p) + \mathfrak{E}_0^{j+1}(\eta) + \mathfrak{F}_0^j(F^1(u,\eta), F^3(u,\eta)) \le P(\mathcal{E}_0). \tag{5.17}$$

According to the estimates (5.12) and (5.13), we then know that

$$\left\| F^{1,j}(0) \right\|_{4N-2j-2}^2 + \left\| F^{3,j}(0) \right\|_{4N-2j-3/2}^2 + \left\| D_t^j u(0) \right\|_{4N-2j}^2 \le P(\mathcal{E}_0). \tag{5.18}$$

By virtue of estimates (4.87) and (5.17), we know that

$$\left\| \mathfrak{f}^1(F^{1,j}(0), D_t^j u(0)) \right\|_{4N-2j-3}^2 + \left\| \mathfrak{f}^2(F^{3,j}(0), D_t^j u(0)) \right\|_{4N-2j-3/2}^2 + \left\| \mathfrak{f}^3(F^{1,j}(0), D_t^j u(0)) \right\|_{4N-2j-5/2}^2 \le P(\mathcal{E}_0). \tag{5.19}$$

This allows us to define $\partial_t^j p(0)$ as the solution to (3.43) with f^1, f^2, f^3 given by $\mathfrak{f}^1$, $\mathfrak{f}^2$, $\mathfrak{f}^3$. Then Proposition 3.9 with $k = 4N$ and $r = 4N-2j-1 < k$ implies that

$$\left\| \partial_t^j p(0) \right\|_{4N-2j-1}^2 \le P(\mathcal{E}_0). \tag{5.20}$$

Now the estimates (4.86), (5.17), and (5.18) allow us to define

$$D_t^{j+1}u(0) := \mathfrak{G}^0(F^{1,j}(0), D_t^j u(0), \partial_t^j p(0)) \in H^{4N-2j-2}, \tag{5.21}$$

and owing to (5.13), we have the estimate

$$\left\| \partial_t^{j+1} u(0) \right\|_{4N-2(j+1)}^2 \le P(\mathcal{E}_0). \tag{5.22}$$

Now we define $\partial_t^{j+2}\eta(0) = \sum_{\ell=0}^{j+1} \binom{j}{\ell} \partial_t^\ell \mathcal{N}(0) \cdot \partial_t^{j-\ell} u(0)$. The estimate (5.15), together with (5.17) and (5.22) then imply that

$$\left\| \partial_t^{j+2} \eta(0) \right\|_{4N-2(j+2)+3/2}^2 \le P(\mathcal{E}_0). \tag{5.23}$$

We may combine (5.17) with (5.20)–(5.23) to deduce that

$$\mathfrak{E}_0^{j+1}(u,p) + \mathfrak{E}_0^{j+2}(\eta) \le P(\mathcal{E}_0), \tag{5.24}$$

but then (5.11) implies that $\mathfrak{F}_0^{j+1}(F^1(u,\eta), F^3(u,\eta)) \le P(\mathcal{E}_0)$ as well, and we deduce that the bound (5.17) also holds with j replaced by $j+1$.

Using the above analysis, we may iterate from $j = 0, \dots, 2N-2$ to deduce that

$$\mathfrak{E}_0^{2N-1}(u,p) + \mathfrak{E}_0^{2N}(\eta) + \mathfrak{F}_0^{2N-1}(F^1(u,\eta), F^3(u,\eta)) \le P(\mathcal{E}_0). \tag{5.25}$$

After this iteration, it remains only to define $\partial_t^{2N-1}p(0)$ and $D_t^{2N}u(0)$. In order to do this, we must first impose the compatibility conditions on u_0 and η_0. These are the same as in (4.90), but because now the temporal derivatives of η have been constructed as well, we restate them in a slightly different way. Let $\partial_t^j u(0)$, $F^{1,j}(0)$, $F^{3,j}(0)$ for $j = 0, \dots, 2N-1$, $\partial_t^j \eta(0)$ for $j = 0, \dots, 2N$, and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-2$ be constructed in terms of η_0, u_0 as above. Let Π_0 be the projection defined in terms of η_0 as in (4.14) and D_t be the operator defined by (4.12). We say that u_0, η_0 satisfy the $(2N)^{th}$ order compatibility conditions if

$$\begin{cases} \operatorname{div}_{\mathcal{A}_0}(D_t^j u(0)) = 0 & \text{in } \Omega \\ D_t^j u(0) = 0 & \text{on } \Sigma_b \\ \Pi_0(F^{3,j}(0) + \mathbb{D}_{\mathcal{A}_0} D_t^j u(0) \mathcal{N}_0) = 0 & \text{on } \Sigma \end{cases} \tag{5.26}$$

for $j = 0, \dots, 2N-1$. Note that if u_0, η_0 satisfy (5.26), then the j^{th} order compatibility condition (4.90) is satisfied for $j = 0, \dots, 2N-1$.

Now we define $\partial_t^{2N-1}p(0)$ and $D_t^{2N}u(0)$. We use the compatibility conditions (5.26) and argue as above and in the derivation of (4.88) in Lemma 4.6 to estimate

$$\left\| \mathfrak{f}^2(F^{3,2N-1}(0), D_t^{2N-1}u(0)) \right\|_{1/2}^2 + \left\| \mathfrak{f}^3(F^{1,2N-1}(0), D_t^{2N-1}u(0)) \right\|_{-1/2}^2 \le P(\mathcal{E}_0) \tag{5.27}$$

and

$$\left\| F^{1,2N-1}(0) \right\|_0^2 + \left\| \operatorname{div}_{\mathcal{A}_0}(R(0) D_t^{2N-1} u(0)) \right\|_0^2 \le P(\mathcal{E}_0). \tag{5.28}$$

We then define $\partial_t^{2N-1}p(0) \in H^1$ as a weak solution to (3.43) in the sense of (3.49) with this choice of $f^2 = \mathfrak{f}^2$, $f^3 = \mathfrak{f}^3$, $g_0 = -\operatorname{div}_{\mathcal{A}_0}(R(0)D_t^{2N-1}u(0))$, and $G = -F^{1,2N-1}(0)$. The estimate (3.46), when combined with (5.27)–(5.28), allows us to deduce that

$$\left\| \partial_t^{2N-1} p(0) \right\|_1^2 \le P(\mathcal{E}_0). \tag{5.29}$$

Then we set $D_t^{2N}u(0) = \mathfrak{G}^0(F^{1,2N-1}(0), D_t^{2N-1}u(0), \partial_t^{2N-1}p(0))$, employing (4.86) to see that $D_t^{2N} \in H^0$. In fact, the construction of $\partial_t^{2N-1}p(0)$ guarantees that $D_t^{2N}u(0) \in \mathcal{Y}(0)$. Arguing as before, we also have the estimate

$$\left\| \partial_t^{2N} u(0) \right\|_0^2 \lesssim P(\mathcal{E}_0) \tag{5.30}$$

This completes the construction of the initial data, but we will record a form of the estimates (5.25), (5.29)–(5.30) in the following proposition.

Proposition 5.3. *Suppose that u_0, η_0 satisfy $\mathcal{F}_0 < \infty$ and that $\mathcal{E}_0 \le 1$ is sufficiently small for the hypothesis of Proposition 3.9 to hold when $k = 4N$. Let $\partial_t^j u(0)$, $\partial_t^j \eta(0)$ for $j = 0, \dots, 2N$ and $\partial_t^j p(0)$ for $j = 0, \dots, 2N-1$ be given as above. Then*

$$\mathcal{E}_0 \le \mathfrak{E}_0(u,p) + \mathfrak{E}_0(\eta) \lesssim \mathcal{E}_0 \tag{5.31}$$

Proof. The first inequality in (5.31) is trivial. Summing (5.25) and (5.29)–(5.30) yields the estimate $\mathfrak{E}_0(u,p) + \mathfrak{E}_0(\eta) \le P(\mathcal{E}_0)$ for a polynomial P satisfying $P(0) = 0$. Since $\mathcal{E}_0 \le 1$, we have that $P(\mathcal{E}_0) \lesssim \mathcal{E}_0$, and the last inequality in (5.31) follows directly. □

5.3. **Transport problem.** Thus far we have considered solving for (u,p), given η. Now we discuss how to solve for η, given u (more precisely, its trace on Σ). We do so by considering the transport problem

$$\begin{cases} \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{in } \mathbb{R}^2 \\ \eta(0) = \eta_0. \end{cases} \tag{5.32}$$

We now state a well-posedness theory for (5.32) involving the quantities $\mathcal{E}_0$, $\mathcal{F}_0$, $\mathfrak{K}_{2N}(u)$, $\mathfrak{K}(\eta)$ as defined by (5.6), (5.2), (4.71), respectively. We will also need one more quantity, which we write as

$$\mathcal{F}(\eta) := \|\eta\|^2_{L^\infty H^{4N+1/2}}. \tag{5.33}$$

Theorem 5.4. *Suppose that u_0, η_0 satisfy $\mathcal{F}_0 < \infty$ and that $\mathfrak{E}_0(\eta) \le 1$ is sufficiently small for the hypothesis of Proposition 3.9 to hold when $k = 4N$. Let $\partial_t^j \eta(0), \partial_t^j u(0)$ for $j = 1, \dots, 2N$ be defined in terms of u_0, η_0 as in Section 5.2 and suppose that u satisfies $\mathfrak{K}_{2N}(u) \le 1$ and achieves the initial conditions $\partial_t^j u(0)$ for $j = 0, \dots, 2N$. Then the problem (5.32) admits a unique solution η that satisfies $\mathcal{F}(\eta) + \mathfrak{K}(\eta) < \infty$ and achieves the initial data $\partial_t^j \eta(0)$ for $j = 0, \dots, 2N$. Moreover, there exists a $0 < \bar{T} \le 1$, depending on N, so that if $0 < T \le \bar{T} \min\{1, 1/\mathcal{F}_0\}$, then we have the estimates*

$$\mathcal{F}(\eta) \lesssim \mathcal{F}_0 + T \mathfrak{K}_{2N}(u), \tag{5.34}$$

$$\mathfrak{E}(\eta) \lesssim \mathcal{E}_0 + T \mathfrak{K}_{2N}(u), \tag{5.35}$$

$$\mathfrak{D}(\eta) \lesssim \mathcal{E}_0 + T \mathcal{F}_0 + \mathfrak{K}_{2N}(u). \tag{5.36}$$

Proof. The proof proceeds through three steps. We first establish the solvability of problem (5.32), then we establish the $L^\infty H^k$ estimates needed to bound $\mathfrak{E}(\eta)$ as in (5.35), and then we handle the $L^2 H^k$ estimates for the terms in $\mathfrak{D}(\eta)$ to derive (5.36). Summing the bounds (5.35) and (5.36) shows that $\mathfrak{K}(\eta) < \infty$.

Step 1 – Solving the transport equation

The assumptions on u imply, via trace theory, that $u \in L^2([0,T]; H^{4N+1/2}(\Sigma))$, which allows us to employ the a priori estimates for solutions of the transport equation derived in [13] (more specifically, Proposition 2.1 with $p = p_2 = r = 2$, $\sigma = 4N + 1/2$). Although the well-posedness of (5.32) is not proved in [13], it can be deduced from the a priori estimates in a standard way; full details are provided in Theorem 3.3.1 of the unpublished note [14]. The result is that (5.32) admits a unique solution $\eta \in C^0([0,T]; H^{4N+1/2}(\Sigma))$ with $\eta(0) = \eta_0$ that satisfies the estimate

$$\|\eta\|_{L^\infty H^{4N+1/2}} \le \exp\left(C \int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)} \, dt \right) \left(\sqrt{\mathcal{F}_0} + \int_0^T \|u_3(t)\|_{H^{4N+1/2}(\Sigma)} \, dt \right) \tag{5.37}$$

for $C > 0$. By trace theory, we have $\|u(t)\|_{H^{4N+1/2}(\Sigma)} \lesssim \sqrt{\mathfrak{K}_{2N}(u)}$, so that the Cauchy-Schwarz inequality implies $C \int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)} \, dt \lesssim \sqrt{T} \sqrt{\mathfrak{K}_{2N}(u)} \lesssim \sqrt{T}$, and hence that

$$\exp\left(C \int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)} \, dt \right) \le 2 \tag{5.38}$$

for $T \le \bar{T}$ with $\bar{T} \le 1$ sufficiently small. We deduce from (5.37) and (5.38) that

$$\sqrt{\mathcal{F}(\eta)} \le 2(\sqrt{\mathcal{F}_0} + \sqrt{T \mathfrak{K}_{2N}(u)}), \tag{5.39}$$

from which (5.34) easily follows.

Step 2 – Bounding $\mathfrak{E}(\eta)$

Proposition 2.1 of [13] also implies the a priori estimate

$$\|\eta\|_{L^\infty H^{4N}} \le \exp\left(C\int_0^T \|u(t)\|_{H^{4N+1/2}(\Sigma)}\,dt\right)\left(\|\eta_0\|_{4N} + \int_0^T \|u_3(t)\|_{H^{4N}(\Sigma)}\,dt\right) \lesssim (\sqrt{\mathfrak{E}_0(\eta)} + \sqrt{T\mathfrak{K}_{2N}(u)}), \tag{5.40}$$

where we have used the smallness of $\bar{T}$, trace theory, and Cauchy-Schwarz as above. Since $\partial_t\eta$ satisfies $\partial_t\eta = u_3 - D\eta\cdot u$ and $\mathfrak{K}_{2N}(u) < \infty$, we know that $\partial_t\eta$ is temporally differentiable and satisfies $\partial_t(\partial_t\eta) + u\cdot D(\partial_t\eta) = \partial_t u_3 - \partial_t u\cdot D\eta$ with initial condition $\partial_t\eta(0) = u_0\cdot\mathcal{N}_0$, which matches the initial data constructed in terms of u_0, η_0. We may again apply Proposition 2.1 of [13] and then use (5.40) to find

$$\begin{aligned}\|\partial_t\eta\|_{L^\infty H^{4N-2}} &\le 2\left(\|\partial_t\eta(0)\|_{4N-2} + \int_0^T \|\partial_t u_3\|_{H^{4N-2}(\Sigma)} + \|\partial_t u\cdot D\eta\|_{H^{4N-2}(\Sigma)}\right)\\ &\lesssim \|\partial_t\eta(0)\|_{4N-2} + (1+\|\eta\|_{L^\infty H^{4N-1}})\int_0^T \|\partial_t u\|_{H^{4N-2}(\Sigma)} \lesssim \sqrt{\mathfrak{E}_0(\eta)}\\ &+ \sqrt{T\mathfrak{K}_{2N}(u)}\,(1+\|\eta\|_{L^\infty H^{4N-1}}) \lesssim \sqrt{\mathfrak{E}_0(\eta)} + \sqrt{T\mathfrak{K}_{2N}(u)}\left(1+\sqrt{\mathfrak{E}_0(\eta)}+\sqrt{T\mathfrak{K}_{2N}(u)}\right)\\ &\lesssim P(\sqrt{\mathfrak{E}_0(\eta)}, \sqrt{T\mathfrak{K}_{2N}(u)})\end{aligned} \tag{5.41}$$

for a polynomial $P(\cdot,\cdot)$ with $P(0,0) = 0$. A straightforward modification of this argument allows us to iterate to obtain, for $j = 1,\dots,2N$, the estimate

$$\left\|\partial_t^j\eta\right\|_{L^\infty H^{4N-2j}} \le P(\sqrt{\mathfrak{E}_0(\eta)}, \sqrt{T\mathfrak{K}_{2N}(u)}) \tag{5.42}$$

for $P(\cdot,\cdot)$ a polynomial with $P(0,0) = 0$. We also find that the initial data $\partial_t^j\eta(0)$ is achieved for $j = 0,\dots,2N$. Squaring (5.40) and (5.42) and summing, we then deduce that $\mathfrak{E}(\eta) \le P(\mathfrak{E}_0(\eta), T\mathfrak{K}_{2N}(u))$ for another polynomial with $P(0,0) = 0$. Since $\mathfrak{E}_0(\eta) \le 1$ and $T\mathfrak{K}_{2N}(u) \le \bar{T}\mathfrak{K}_{2N}(u) \le 1$, we then have that

$$\mathfrak{E}(\eta) \lesssim \mathfrak{E}_0(\eta) + T\mathfrak{K}_{2N}(u), \tag{5.43}$$

which yields (5.35) when combined with Proposition 5.3.

Step 3 – Bounding $\mathfrak{D}(\eta)$

Now we control the terms in $\mathfrak{D}(\eta)$. From (5.39), Cauchy-Schwarz, and the fact that $T \le 1$, we see that

$$\|\eta\|_{L^2 H^{4N+1/2}} \le \sqrt{T}\sqrt{\mathcal{F}(\eta)} \le 2(\sqrt{T\mathcal{F}_0} + \sqrt{\mathfrak{K}_{2N}(u)}). \tag{5.44}$$

We may then use the equation (5.32), trace theory, the fact that $H^{4N-1/2}(\Sigma)$ is an algebra, and estimate (5.44) to bound

$$\begin{aligned}\|\partial_t\eta\|_{L^2 H^{4N-1/2}} &\lesssim \|u_3\|_{L^2 H^{4N-1/2}} + \|u\|_{L^\infty H^{4N-1/2}}\|\eta\|_{L^2 H^{4N+1/2}}\\ &\lesssim \sqrt{\mathfrak{K}_{2N}(u)}(1+\sqrt{T\mathcal{F}_0}+\sqrt{\mathfrak{K}_{2N}(u)}) \lesssim P(\sqrt{T\mathcal{F}_0}, \sqrt{\mathfrak{K}_{2N}(u)})\end{aligned} \tag{5.45}$$

for P a polynomial with $P(0,0) = 0$. We argue similarly (employing (5.45) along the way) to find that

$$\begin{aligned}\|\partial_t^2\eta\|_{L^2 H^{4N-3/2}} &\lesssim \|\partial_t u_3\|_{L^2 H^{4N-1/2}} + \|\eta\|_{L^\infty H^{4N-1/2}}\|\partial_t u\|_{L^2 H^{4N-3/2}}\\ &+ \|\partial_t\eta\|_{L^2 H^{4N-1/2}}\|u\|_{L^\infty H^{4N-3/2}} \lesssim \sqrt{\mathfrak{K}_{2N}(u)}(1+\|\eta\|_{L^\infty H^{4N-1/2}} + \|\partial_t\eta\|_{L^2 H^{4N-1/2}})\\ &\lesssim \sqrt{\mathfrak{K}_{2N}(u)}(1+\sqrt{\mathfrak{E}(\eta)} + P(\sqrt{T\mathcal{F}_0}, \sqrt{\mathfrak{K}_{2N}(u)})) \lesssim P(\sqrt{T\mathcal{F}_0}, \sqrt{\mathfrak{K}_{2N}(u)}, \sqrt{\mathfrak{E}(\eta)})\end{aligned} \tag{5.46}$$

for a polynomial P with $P(0,0,0) = 0$. Iterating this argument for $j = 2,\dots,2N+1$ then yields the inequalities

$$\left\|\partial_t^j\eta\right\|_{L^2 H^{4N-2j+5/2}} \le P(\sqrt{T\mathcal{F}_0}, \sqrt{\mathfrak{K}_{2N}(u)}, \sqrt{\mathfrak{E}(\eta)}) \tag{5.47}$$

for a polynomial with $P(0,0,0) = 0$. We may then square and sum (5.44)–(5.47) to find that $\mathfrak{D}(\eta) \le P(T\mathcal{F}_0, \mathfrak{K}_{2N}(u), \mathfrak{E}(\eta))$, but then (5.43) and the bound $T \le 1$ imply that $\mathfrak{D}(\eta) \le P(T\mathcal{F}_0, \mathfrak{K}_{2N}(u), \mathfrak{E}_0(\eta))$ for another P. By assumption, $T\mathcal{F}_0 \le \bar{T} \le 1$, and $\mathfrak{K}_{2N}(u), \mathfrak{E}_0(\eta) \le 1$ as well; hence

$$\mathfrak{D}(\eta) \lesssim T\mathcal{F}_0 + \mathfrak{K}_{2N}(u) + \mathfrak{E}_0(\eta), \tag{5.48}$$

which provides the estimate (5.36) when combined with Proposition 5.3.

□

5.4. **An extension result.** In our nonlinear well-posedness argument we will need to be able to take the initial data $\partial_t^j u(0)$, $j = 0, \dots, 2N$, constructed in Section 5.2 and extend it to a function u satisfying $\mathfrak{K}_{2N}(u) \lesssim \mathfrak{E}_0(u,0)$. This extension is the content of the following lemma.

Lemma 5.5. *Suppose that $\partial_t^j u(0) \in H^{4N-2j}(\Omega)$ for $j = 0, \dots, 2N$. Then there exists an extension u, achieving the initial data, so that*

$$\partial_t^j u \in L^2([0,\infty); H^{4N-2j+1}(\Omega)) \cap L^\infty([0,\infty); H^{4N-2j}(\Omega)) \tag{5.49}$$

for $j = 0, \dots, 2N$. Moreover $\mathfrak{K}_{2N}(u) \lesssim \mathfrak{E}_0(u,0)$, where in the definition of $\mathfrak{K}_{2N}(u)$ we take $T = \infty$.

Proof. Owing to the usual theory of extensions and restrictions in Sobolev spaces, it suffices to prove the result with Ω replaced by $\mathbb{R}^3$ in the non-periodic case and $(L_1\mathbb{T}) \times (L_2\mathbb{T}) \times \mathbb{R}$ in the periodic case. The proof in the periodic case can be derived from the non-periodic proof by trivially changing some integrals over frequencies to sums. As such, we present only the proof in $\mathbb{R}^3$.

Let $f_j \in H^{4N-2j}(\mathbb{R}^3)$ denote the extension of $\partial_t^j u(0) \in H^{4N-2j}(\Omega)$. It suffices to construct $F_j(x,t)$ for $j = 0, \dots, 2N$ so that $\partial_t^k F_j(x,0) = \delta_{j,k} f_j(x)$ ($\delta_{j,k}$ is the Kronecker delta) and

$$\left\|\partial_t^k F_j\right\|^2_{L^2 H^{4N-2k+1}} + \left\|\partial_t^k F_j\right\|^2_{L^\infty H^{4N-2k}} \lesssim \|f_j\|^2_{4n-2j} \tag{5.50}$$

for $k = 0, \dots, 2N$. Indeed, with such F_j in hand, the sum $F = \sum_{j=0}^{2N} F_j$ is the desired extension. Note that in the norms of (5.50) the symbol $L^p H^m$ denotes $L^p([0,\infty); H^m(\mathbb{R}^3))$.

Let $\varphi_j \in C_c^\infty(\mathbb{R})$ be such that $\varphi_j^{(k)}(0) = \delta_{j,k}$ for $k = 0, \dots, 2N$ (here (k) is the number of derivatives). We then define $\hat{F}_j(\xi,t) = \varphi_j(t\langle\xi\rangle^2)\hat{f}_j(\xi)\langle\xi\rangle^{-2j}$, where $\hat{\cdot}$ denotes the Fourier transform and $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. By construction, $\partial_t^k \hat{F}_j(\xi,t) = \varphi_j^{(k)}(t\langle\xi\rangle^2)\hat{f}_j(\xi)\langle\xi\rangle^{2(k-j)}$ so that $\partial_t^k F(\cdot,0) = \delta_{j,k} f_j$. We estimate

$$\begin{aligned} \left\|\partial_t^k F_j(\cdot,t)\right\|^2_{4N-2k} &= \int_{\mathbb{R}^3} \langle\xi\rangle^{2(4N-2k)} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(2k-2j)} d\xi \\ &= \int_{\mathbb{R}^3} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j)} d\xi \le \left\|\varphi_j^{(k)}\right\|^2_{L^\infty} \|f_j\|^2_{4N-2j}, \end{aligned} \tag{5.51}$$

so that $\left\|\partial_t^k F_j\right\|^2_{L^\infty H^{4N-2k}} \lesssim \|f_j\|^2_{4N-2j}$. Similarly,

$$\begin{aligned} \left\|\partial_t^k F_j\right\|^2_{L^2 H^{4N-2k+1}} &= \int_0^\infty \int_{\mathbb{R}^3} \langle\xi\rangle^{2(4N-2k+1)} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(2k-2j)} d\xi dt \\ &= \int_0^\infty \int_{\mathbb{R}^3} \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j+1)} d\xi dt \\ &= \int_{\mathbb{R}^3} \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j+1)} \left(\int_0^\infty \left|\varphi_j^{(k)}(t\langle\xi\rangle^2)\right|^2 dt\right) d\xi \\ &= \int_{\mathbb{R}^3} \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j+1)} \left(\frac{1}{\langle\xi\rangle^2} \int_0^\infty \left|\varphi_j^{(k)}(r)\right|^2 dr\right) d\xi \\ &= \left\|\varphi_j^{(k)}\right\|^2_{L^2} \int_{\mathbb{R}^3} \left|\hat{f}_j(\xi)\right|^2 \langle\xi\rangle^{2(4N-2j)} d\xi = \left\|\varphi_j^{(k)}\right\|^2_{L^2} \|f_j\|^2_{4N-2j} \end{aligned} \tag{5.52}$$

so that $\left\|\partial_t^k F_j\right\|_{L^2 H^{4N-2k+1}}^2 \lesssim \|f_j\|_{4N-2j}^2$. Note that in (5.52), we have used Fubini's theorem to switch the order of integration; this is possible since φ is compactly supported. We then have that F_j satisfies the desired properties, completing the proof.

□

6. Local well-posedness of the nonlinear problem

6.1. **Sequence of approximate solutions.** In order to construct the solution to (1.10), we will pass to the limit in a sequence of approximate solutions. The construction of this sequence is the content of our next result.

Theorem 6.1. *Assume the initial data are given as in Section 5.2 and satisfy the* $(2N)^{th}$ *compatibility conditions* (5.26). *There exist* $0 < \delta < 1$ *and* $0 < \bar{T} < 1$ *so that if* $\mathcal{E}_0 \le \delta$, $\mathcal{F}_0 < \infty$, *and* $0 < T \le T_0 := \bar{T}\min\{1, 1/\mathcal{F}_0\}$, *then there exists an infinite sequence* $\{(u^m, p^m, \eta^m)\}_{m=1}^\infty$ *with the following three properties. First, for* $m \ge 1$ *it holds that*

$$\begin{cases} \partial_t u^{m+1} - \Delta_{\mathcal{A}^m} u^{m+1} + \nabla_{\mathcal{A}^m} p^{m+1} = \partial_t \bar{\eta}^m \tilde{b} K^m \partial_3 u^m - u^m \cdot \nabla_{\mathcal{A}^m} u^m & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}^m} u^{m+1} = 0 & \text{in } \Omega \\ S_{\mathcal{A}^m}(p^{m+1}, u^{m+1}) \mathcal{N}^m = \eta^m \mathcal{N}^m & \text{on } \Sigma \\ u^{m+1} = 0 & \text{on } \Sigma_b \end{cases} \tag{6.1}$$

and

$$\partial_t \eta^{m+1} = u^{m+1} \cdot \mathcal{N}^{m+1} \text{ on } \Sigma, \tag{6.2}$$

where $\mathcal{A}^m, \mathcal{N}^m, K^m$ *are given in terms of* η^m. *Second,* (u^m, p^m, η^m) *achieve the initial data for each* $m \ge 1$, *i.e.* $\partial_t^j u^m(0) = \partial_t^j u(0)$ *and* $\partial_t^j \eta^m(0) = \partial_t^j \eta(0)$ *for* $j = 0, \dots, 2N$, *while* $\partial_t^j p^m(0) = \partial_t^j p(0)$ *for* $j = 0, \dots, 2N-1$. *Third, for each* $m \ge 1$ *we have the estimates*

$$\mathfrak{K}(\eta^m) + \mathfrak{K}(u^m, p^m) \le C(\mathcal{E}_0 + T\mathcal{F}_0), \text{ and } \mathcal{F}(\eta^m) \le C(\mathcal{F}_0 + \mathcal{E}_0 + T\mathcal{F}_0) \tag{6.3}$$

for a universal constant $C > 0$.

Proof. We divide the proof into three steps. First, we construct an initial pair (u^0, η^0) that will be used as a starting point for constructing (u^m, p^m, η^m) for $m \ge 1$. Second, we prove that if (u^m, p^m, η^m) are known and satisfy certain estimates, then we can construct $(u^{m+1}, p^{m+1}, \eta^{m+1})$. Third, we combine the first two steps in an appropriate way to iteratively construct all of the (u^m, p^m, η^m). Throughout the proof we will need to explicitly enumerate the various constants appearing in estimates where previously we have written $\lesssim$. We do so with $C_1, \dots, C_9 > 0$.

Before proceeding to the steps, we define some terms and make some assumptions. Let $\delta_1 > 0$ be such that if $\mathfrak{K}(\eta) \le \delta_1$, then the hypotheses of Theorem 4.7 are satisfied. Similarly, let $\delta_2 > 0$ be the constant such that if $\mathfrak{E}_0(\eta) \le \delta_2$, then the hypotheses of Theorem 5.4 are satisfied. We assume that δ is sufficiently small so that $\mathcal{E}_0 \le \delta$ satisfies the hypotheses of Proposition 5.3, and so that (using the estimate (5.31))

$$\mathfrak{E}_0(\eta) + \mathfrak{E}_0(u, p) \le C_1 \mathcal{E}_0 \le C_1 \delta \le \min\{1, \delta_2\}. \tag{6.4}$$

This allows us to use (5.11) of Lemma 5.2 with $j = 2N-1$ to bound

$$\mathfrak{F}_0(F^1(u, \eta), F^2(u, \eta)) \le C_2 \mathcal{E}_0. \tag{6.5}$$

Step 1 – Seeding the sequence

We begin by extending the initial data $\partial_t^j u(0) \in H^{4N-2j}(\Omega)$ to a time-dependent function u^0 so that $\partial_t^j u^0(0) = \partial_t^j u(0)$. We do so by applying Lemma 5.5. Although this produces a u^0 defined on the time interval $[0, \infty)$, we may restrict to $[0, T]$ without increasing any of the space-time norms in $\mathfrak{K}_{2N}(u^0)$. We may combine the estimate of $\mathfrak{K}_{2N}(u^0)$ provided by Lemma 5.5 with (6.4) to bound

$$\mathfrak{K}_{2N}(u^0) \le C_3 \mathcal{E}_0. \tag{6.6}$$

With u^0 in hand, we define η^0 as the solution to (5.32) with u^0 replacing u. To do so, we apply Theorem 5.4, the hypotheses of which are satisfied by virtue of (6.4) and (6.6) if we further restrict to $C_3\delta \le 1$. Restricting $\bar{T}$ as in the theorem, we find our solution η^0, which satisfies $\partial_t^j \eta^0(0) = \partial_t^j \eta(0)$ as well as the estimates

$$\begin{aligned} \mathcal{F}(\eta^0) &\le C_4(\mathcal{F}_0 + T\mathfrak{K}_{2N}(u^0)) \\ \mathfrak{E}(\eta^0) &\le C_5(\mathcal{E}_0 + T\mathfrak{K}_{2N}(u^0)) \\ \mathfrak{D}(\eta^0) &\le C_6(\mathcal{E}_0 + T\mathcal{F}_0 + \mathfrak{K}_{2N}(u^0)). \end{aligned} \tag{6.7}$$

Step 2 – The iteration argument

We claim that there exist $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ and $0 < \tilde{\delta}, \tilde{T} < 1$ (both depending on the γ_i) so that if $\delta \le \tilde{\delta}$ and $\bar{T} \le \tilde{T}$, then the following property is satisfied. If (u^m, η^m) are known and satisfy the estimates

$$\begin{gathered} \mathfrak{E}(\eta^m) \le \gamma_1(\mathcal{E}_0 + T\mathcal{F}_0),\ \mathfrak{D}(\eta^m) \le \gamma_2(\mathcal{E}_0 + T\mathcal{F}_0), \\ \mathfrak{K}_{2N}(u^m) \le \gamma_3(\mathcal{E}_0 + T\mathcal{F}_0),\ \mathcal{F}(\eta^m) \le C_4\mathcal{F}_0 + \gamma_4(\mathcal{E}_0 + T\mathcal{F}_0), \end{gathered} \tag{6.8}$$

then there exists a unique triple $(u^{m+1}, p^{m+1}, \eta^{m+1})$ that achieves the initial data, satisfies (6.1) and (6.2), and obeys the estimates

$$\begin{gathered} \mathfrak{E}(\eta^{m+1}) \le \gamma_1(\mathcal{E}_0 + T\mathcal{F}_0),\ \mathfrak{D}(\eta^{m+1}) \le \gamma_2(\mathcal{E}_0 + T\mathcal{F}_0), \\ \mathfrak{K}(u^{m+1}, p^{m+1}) \le \gamma_3(\mathcal{E}_0 + T\mathcal{F}_0),\ \mathcal{F}(\eta^{m+1}) \le C_4\mathcal{F}_0 + \gamma_4(\mathcal{E}_0 + T\mathcal{F}_0). \end{gathered} \tag{6.9}$$

To prove the claim, we will first use η^m to solve for (u^{m+1}, p^{m+1}), and then we will use the resulting u^{m+1} to solve for η^{m+1}. Along the way, we will restrict the size of $\tilde{\delta}$ and $\tilde{T}$ in terms of γ_i, $i = 1, 2, 3, 4$. We will define the γ_i in terms of the C_i, so the $\tilde{\delta}$ and $\tilde{T}$ can be thought of as universal constants. Note that the estimates of (6.9) are stronger than those of (6.8) since $\mathfrak{K}_{2N}(u^{m+1}) \le \mathfrak{K}(u^{m+1}, p^{m+1})$. This asymmetry is useful to us since in Step 1 we have not bothered to construct p^0, so only (u^0, η^0) are available to begin the iterative construction of $\{(u^m, p^m, \eta^m)\}_{m=1}^\infty$.

We assume initially that

$$\tilde{\delta}, \tilde{T} \le \frac{1}{2} \min\left\{ \frac{\min\{1, \delta_1\}}{(\gamma_1 + \gamma_2)}, \frac{1}{\gamma_3} \right\}, \tag{6.10}$$

so that (6.8) implies that $\mathfrak{K}_{2N}(u^m) \le 1$ and

$$\mathfrak{K}(\eta^m) = \mathfrak{E}(\eta^m) + \mathfrak{D}(\eta^m) \le (\gamma_1 + \gamma_2)(\mathcal{E}_0 + T_0\mathcal{F}_0) \le \min\{\delta_1, 1\}, \tag{6.11}$$

the latter of which allows us to use Theorem 4.7 to produce a unique pair (u^{m+1}, p^{m+1}) that achieves the desired initial data and satisfies (6.1). Moreover, from (4.96) and (6.4)–(6.5), we have the estimate

$$\begin{aligned} \mathfrak{K}(u^{m+1}, p^{m+1}) \le C_7(1 + \mathcal{E}_0 + \mathfrak{K}(\eta^m)) \exp\left(C_8(1 + \mathfrak{E}(\eta^m))T\right) \times \\ \left[(1 + C_2)\mathcal{E}_0 + \mathfrak{F}(F^1(u^m, \eta^m), F^3(u^m, \eta^m))\right]. \end{aligned} \tag{6.12}$$

Assume that $2\tilde{T}C_8 \le \log 2$; then

$$C_7(1 + \mathcal{E}_0 + \mathfrak{K}(\eta^m)) \exp\left(C_8(1 + \mathfrak{E}(\eta^m))T\right) \le 3C_7 \exp(2C_8\tilde{T}) \le 6C_7. \tag{6.13}$$

On the other hand, we can use our bounds on η^m, u^m in Lemma 5.1 to see that

$$\mathfrak{F}(F^1(u^m, \eta^m), F^3(u^m, \eta^m)) \le C_9\left[3\mathfrak{E}(\eta^m) + 2\mathfrak{K}(\eta^m)\mathfrak{K}_{2N}(u^m) + (\mathfrak{K}_{2N}(u^m))^2\right]. \tag{6.14}$$

Combining (6.12)–(6.14) with (6.8) then shows that

$$\begin{aligned} \mathfrak{K}(u^{m+1}, p^{m+1}) \le 6C_7 [(1 + C_2)\mathcal{E}_0 + 3C_9\gamma_1(\mathcal{E}_0 + T\mathcal{F}_0) \\ +2C_9\gamma_3(\gamma_1 + \gamma_2)(\mathcal{E}_0 + T\mathcal{F}_0)^2 + C_9\gamma_3^2(\mathcal{E}_0 + T\mathcal{F}_0)^2]. \end{aligned} \tag{6.15}$$

We have now enumerated all of the constants C_i, $i = 1, \dots, 9$, that we need to define the γ_i, $i = 1, \dots, 4$. We choose the values of the γ_i according to

$$\begin{aligned} &\gamma_1 := 2C_5, \gamma_3 := 6C_7(3 + C_2 + 3C_9\gamma_1) + C_3, \\ &\gamma_4 := C_4, \gamma_2 := C_6(1 + \gamma_3). \end{aligned} \tag{6.16}$$

Notice that even though we have used γ_1 to define γ_3 and γ_3 to define γ_2, all of the γ_i are determined in terms of the constants C_i.

Now we will use the choice of the γ_i in (6.16) to derive the $\mathfrak{K}(u^{m+1}, p^{m+1})$ estimate of (6.9) from (6.15). To do this, we further restrict

$$\tilde{\delta}, \tilde{T} \le \frac{1}{2} \min \left\{ \frac{1}{2C_9 \gamma_3 (\gamma_1 + \gamma_2)}, \frac{1}{C_9 \gamma_3^2} \right\}. \tag{6.17}$$

Then since $\mathcal{E}_0 + T\mathcal{F}_0 \le \tilde{\delta} + \tilde{T}$, we may use (6.15) to bound

$$\mathfrak{K}(u^{m+1}, p^{m+1}) \le 6C_7(3 + C_2 + 3C_9\gamma_1)(\mathcal{E}_0 + T\mathcal{F}_0) \le \gamma_3(\mathcal{E}_0 + T\mathcal{F}_0). \tag{6.18}$$

Now we construct η^{m+1}. Recall that $\tilde{\delta}, \tilde{T} \le 1/(2\gamma_3)$; this and (6.18) then yield the bound $\mathfrak{K}_{2N}(u^{m+1}) \le 1$. This estimate then allows us to apply Theorem 5.4 to find η^{m+1} that solves (6.2) and achieves the initial data. Estimates (5.34)–(5.36) of the theorem, together with (6.18) and the bound $T_0\gamma_3 \le \tilde{T}\gamma_3 \le 1$, imply that

$$\begin{aligned} \mathcal{F}(\eta^{m+1}) &\le C_4(\mathcal{F}_0 + T_0 \mathfrak{K}_{2N}(u^{m+1})) \le C_4\mathcal{F}_0 + C_4(\mathcal{E}_0 + T\mathcal{F}_0) \\ \mathfrak{E}(\eta^{m+1}) &\le C_5(\mathcal{E}_0 + T_0\mathfrak{K}_{2N}(u^{m+1})) \le 2C_5(\mathcal{E}_0 + T\mathcal{F}_0) \\ \mathfrak{D}(\eta^{m+1}) &\le C_6(\mathcal{E}_0 + T\mathcal{F}_0 + \mathfrak{K}_{2N}(u^{m+1})) \le C_6(1+\gamma_3)(\mathcal{E}_0 + T\mathcal{F}_0). \end{aligned} \tag{6.19}$$

Using the definitions of the γ_i given in (6.16), we see from (6.19) that the η^{m+1} estimates of (6.9) hold. Then, owing to (6.18), all of the estimates in (6.9) hold, which completes the proof of the claim.

Step 3 – Construction of the full sequence

We assume that $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are given by (6.16) and that $\tilde{\delta}$ and $\tilde{T}$ are as small as in Step 2. We assume that $\delta \le \tilde{\delta}$ and $\bar{T} \le \tilde{T}$ in addition to the other restrictions on their size made in Step 1 and before. Returning to (6.6), note that $C_3 \le \gamma_3$, which means that $\mathfrak{K}_{2N}(u^0) \le \gamma_3(\mathcal{E}_0 + T\mathcal{F}_0)$. We can also combine (6.6) and (6.7) and further restrict $\bar{T} \le 1/C_3$ to deduce that

$$\begin{aligned} \mathcal{F}(\eta^0) &\le C_4\mathcal{F}_0 + T_0C_3C_4\mathcal{E}_0 \le C_4\mathcal{F}_0 + \gamma_4(\mathcal{E}_0 + T\mathcal{F}_0) \\ \mathfrak{E}(\eta^0) &\le C_5(1 + T_0C_3)\mathcal{E}_0 \le 2C_5\mathcal{E}_0 \le \gamma_1(\mathcal{E}_0 + T\mathcal{F}_0) \\ \mathfrak{D}(\eta^0) &\le C_6(\mathcal{E}_0 + T\mathcal{F}_0 + C_3\mathcal{E}_0) \le C_6(1+C_3)(\mathcal{E}_0 + T\mathcal{F}_0) \le \gamma_2(\mathcal{E}_0 + T\mathcal{F}_0). \end{aligned} \tag{6.20}$$

Note that in the last inequality we have used the fact that $C_3 \le \gamma_3$ to bound $C_6(1 + C_3) \le C_6(1 + \gamma_3) = \gamma_2$. We are then free to use the pair (u^0, η^0) as the starting point in Step 2, which allows us to construct (u^1, p^1, η^1) satisfying the desired PDE and initial conditions, along with the estimates

$$\begin{gathered} \mathfrak{E}(\eta^1) \le \gamma_1(\mathcal{E}_0 + T\mathcal{F}_0), \ \mathfrak{D}(\eta^1) \le \gamma_2(\mathcal{E}_0 + T\mathcal{F}_0), \\ \mathfrak{K}(u^1, p^1) \le \gamma_3(\mathcal{E}_0 + T\mathcal{F}_0), \ \mathcal{F}(\eta^1) \le C_4\mathcal{F}_0 + \gamma_4(\mathcal{E}_0 + T\mathcal{F}_0). \end{gathered} \tag{6.21}$$

We then iterate from $m = 1, \dots, \infty$, using (u^m, η^m) and Step 2 to produce the next element of the sequence, $(u^{m+1}, p^{m+1}, \eta^{m+1})$, which satisfies (6.9). All of the conclusions of the theorem follow.

□

6.2. **Contraction.** While the estimates (6.3) of Theorem 6.1 will allow us to extract weak limits from the sequence $\{(u^m, p^m, \eta^m)\}_{m=1}^\infty$, weak convergence of a subsequence is not enough to allow us to pass to the limit in (6.1)–(6.2) in order to produce the desired solution to (1.10). We are thus led to study the strong convergence of the sequence, and in particular to consider its contraction in some norm.

We now define the norms in which we will show the sequence contracts. For $T > 0$ we define

$$\begin{aligned} \mathfrak{N}(v, q; T) &= \|v\|^2_{L^\infty H^2} + \|v\|^2_{L^2 H^3} + \|\partial_t v\|^2_{L^\infty H^0} + \|\partial_t v\|^2_{L^2 H^1} + \|q\|^2_{L^\infty H^1} + \|q\|^2_{L^2 H^2}, \\ \mathfrak{M}(\zeta; T) &= \|\zeta\|^2_{L^\infty H^{5/2}} + \|\partial_t \zeta\|^2_{L^\infty H^{3/2}} + \left\|\partial_t^2 \zeta\right\|^2_{L^2 H^{1/2}}, \end{aligned} \tag{6.22}$$

where we write $L^p H^k$ for $L^p([0, T]; H^k(\Omega))$ in $\mathfrak{N}$ and $L^p([0, T]; H^k(\Sigma))$ in $\mathfrak{M}$.

The next result provides a comparison of $\mathfrak{N}$ for pairs of solutions to problems of the form (6.1)–(6.2). We will use it later in Theorem 6.3 to show that the sequence of approximate solutions contracts, but we will also use it to prove the uniqueness of solutions to (1.10). In order to avoid confusion with the sequence $\{(u^m, p^m, \eta^m)\}$, we refer to velocities as v^j, w^j, pressures as q^j, and surface functions as ζ^j for $j = 1, 2$.

Theorem 6.2. *Let w^1, w^2, v^1, v^2, q^1, q^2, and ζ^1, ζ^2 satisfy*

$$\sup\left\{\mathfrak{E}(\zeta^1), \mathfrak{E}(\zeta^2), \mathfrak{E}(v^1, q^1), \mathfrak{E}(v^2, q^2), \mathfrak{E}(w^1, 0), \mathfrak{E}(w^2, 0)\right\} \le \varepsilon, \tag{6.23}$$

where the temporal L^∞ norms in $\mathfrak{E}$ are computed over the interval $[0, T]$ with $0 < T$. Suppose that for $j = 1, 2$,

$$\begin{cases} \partial_t v^j - \Delta_{\mathcal{A}^j} v^j + \nabla_{\mathcal{A}^j} q^j = \partial_t \bar{\zeta}^j \tilde{b} K^j \partial_3 w^j - w^j \cdot \nabla_{\mathcal{A}^j} w^j & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}^j} v^j = 0 & \text{in } \Omega \\ S_{\mathcal{A}^j}(q^j, v^j)\mathcal{N}^j = \zeta^j \mathcal{N}^j & \text{on } \Sigma \\ v^j = 0 & \text{on } \Sigma_b, \\ \partial_t \zeta^j = w^j \cdot \mathcal{N}^j & \text{on } \Sigma, \end{cases} \tag{6.24}$$

where $\mathcal{A}^j, K^j, \mathcal{N}^j$ are determined by ζ^j as usual. Further suppose that $\partial_t^k v^1(0) = \partial_t^k v^2(0)$ for $k = 0, 1$, $\zeta^1(0) = \zeta^2(0)$, and $q^1(0) = q^2(0)$.

Then there exist $\varepsilon_1 > 0$, $T_1 > 0$ so that if $\varepsilon \le \varepsilon_1$ and $0 < T \le T_1$, then

$$\mathfrak{N}(v^1 - v^2, q^1 - q^2; T) \le \frac{1}{2}\mathfrak{N}(w^1 - w^2, 0; T) \tag{6.25}$$

and

$$\mathfrak{M}(\zeta^1 - \zeta^2; T) \le 2\mathfrak{N}(w^1 - w^2, 0; T). \tag{6.26}$$

Proof. The proof proceeds through six steps. First, we define $v = v^1 - v^2$, $w = w^1 - w^2$, $q = q^1 - q^2$, and derive the PDEs satisfied by v, q. We also identify the energy evolution for some norms of $\partial_t v, \partial_t q$. Second, we bound various forcing terms that appear in the energy evolution and on the right side of the PDEs for v, q. Third, we prove some bounds for $\partial_t v, \partial_t q$, using the energy evolution equation. Fourth, we use elliptic estimates to bound norms of v, q. Fifth, we derive estimates for $\zeta^1 - \zeta^2$ in terms of w. Sixth, we close the estimate to derive the contraction estimates (6.25), (6.26).

Step 1 – PDEs and energy evolution for differences

We now derive the PDE satisfied by v, q, which are defined above. We subtract the equations in (6.24) with $j = 2$ from the same equations with $j = 1$. With the help of some simple algebra, we can write the resulting equations in terms of v, q:

$$\begin{cases} \partial_t v - \Delta_{\mathcal{A}^1} v + \nabla_{\mathcal{A}^1} q = \operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2) + H^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}^1} v = H^2 & \text{in } \Omega \\ S_{\mathcal{A}^1}(q, v)\mathcal{N}^1 = \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2 \mathcal{N}^1 + H^3 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b \\ v(t = 0) = 0, \end{cases} \tag{6.27}$$

where H^1, H^2, H^3 are defined by

$$\begin{aligned} H^1 = {} & \operatorname{div}_{(\mathcal{A}^1 - \mathcal{A}^2)}(\mathbb{D}_{\mathcal{A}^2} v^2) - (\mathcal{A}^1 - \mathcal{A}^2)\nabla q^2 \\ & + \partial_t \bar{\zeta}^1 \tilde{b} K^1 (\partial_3 w^1 - \partial_3 w^2) + (\partial_t \bar{\zeta}^1 - \partial_t \bar{\zeta}^2) \tilde{b} K^1 \partial_3 w^2 + \partial_t \bar{\zeta}^1 \tilde{b}(K^1 - K^2)\partial_3 w^2 \\ & - (w^1 - w^2) \cdot \nabla_{\mathcal{A}^1} w^1 - w^2 \cdot \nabla_{\mathcal{A}^1}(w^1 - w^2) - w^2 \cdot \nabla_{(\mathcal{A}^1 - \mathcal{A}^2)} w^2, \end{aligned} \tag{6.28}$$

$$H^2 = -\operatorname{div}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2, \tag{6.29}$$

$$\begin{aligned} H^3 = {} & -q^2(\mathcal{N}^1 - \mathcal{N}^2) + \mathbb{D}_{\mathcal{A}^1} v^2 (\mathcal{N}^1 - \mathcal{N}^2) - \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2 (\mathcal{N}^1 - \mathcal{N}^2) \\ & + (\zeta^1 - \zeta^2)\mathcal{N}^1 + \zeta^2(\mathcal{N}^1 - \mathcal{N}^2). \end{aligned} \tag{6.30}$$

The solutions are sufficiently regular for us to differentiate (6.27) in time, which results in the equations

$$\begin{cases} \partial_t(\partial_t v) - \Delta_{\mathcal{A}^1}(\partial_t v) + \nabla_{\mathcal{A}^1}(\partial_t q) = \operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2) + \tilde{H}^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}^1} \partial_t v = \tilde{H}^2 & \text{in } \Omega \\ S_{\mathcal{A}^1}(\partial_t q, \partial_t v)\mathcal{N}^1 = \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 \mathcal{N}^1 + \tilde{H}^3 & \text{on } \Sigma \\ \partial_t v = 0 & \text{on } \Sigma_b \\ \partial_t v(t=0) = 0, \end{cases} \tag{6.31}$$

where $\tilde{H}^1$, $\tilde{H}^2$, and $\tilde{H}^3$ are given by

$$\begin{aligned} \tilde{H}^1 = \partial_t H^1 + \operatorname{div}_{\partial_t \mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2) + \operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} \partial_t v^2) \\ + \operatorname{div}_{\partial_t \mathcal{A}^1}(\mathbb{D}_{\mathcal{A}^1} v) + \operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{\partial_t \mathcal{A}^1} v) - \nabla_{\partial_t \mathcal{A}^1} q, \end{aligned} \tag{6.32}$$

$$\tilde{H}^2 = \partial_t H^2 - \operatorname{div}_{\partial_t \mathcal{A}^1} v, \tag{6.33}$$

$$\tilde{H}^3 = \partial_t H^3 + \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} \partial_t v^2 \mathcal{N}^1 + \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2 \partial_t \mathcal{N}^1 - S_{\mathcal{A}^1}(q, v)\partial_t \mathcal{N}^1 + \mathbb{D}_{\partial_t \mathcal{A}^1} v \mathcal{N}^1. \tag{6.34}$$

Now we multiply (6.31) by $J^1 \partial_t v$, integrate over Ω, and integrate by parts as in the proof of Theorem 4.3 to deduce the evolution equation

$$\begin{aligned} \partial_t \int_\Omega \frac{|\partial_t v|^2}{2} J^1 + \frac{1}{2}\int_\Omega |\mathbb{D}_{\mathcal{A}^1} \partial_t v|^2 J^1 = \int_\Omega \frac{|\partial_t v|^2}{2}(\partial_t J^1 K^1) J^1 + \int_\Omega J^1 \partial_t q \tilde{H}^2 \\ + \int_\Omega J^1 \left(\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2) + \tilde{H}^1 \right) \cdot \partial_t v \\ - \int_\Sigma \left(\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 \mathcal{N}^1 + \tilde{H}^3 \right) \cdot \partial_t v. \end{aligned} \tag{6.35}$$

Another integration by parts reveals that

$$\begin{aligned} \int_\Omega J^1 \operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2) \cdot \partial_t v = -\frac{1}{2}\int_\Omega J^1 \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 : \mathbb{D}_{\mathcal{A}^1} \partial_t v \\ + \int_\Sigma \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 \mathcal{N}^1 \cdot \partial_t v. \end{aligned} \tag{6.36}$$

We then employ (6.36) to rewrite (6.35), and then we integrate in time from 0 to $t < T$; since $\partial_t v(t=0) = 0$, we arrive at the equation

$$\begin{aligned} \int_\Omega \frac{|\partial_t v|^2}{2} J^1(t) + \frac{1}{2}\int_0^t \int_\Omega |\mathbb{D}_{\mathcal{A}^1} \partial_t v|^2 J^1 = \int_0^t \int_\Omega \frac{|\partial_t v|^2}{2}(\partial_t J^1 K^1) J^1 \\ + \int_0^t \int_\Omega J^1(\tilde{H}^1 \cdot \partial_t v + \tilde{H}^2 \partial_t q) - \frac{1}{2}\int_0^t \int_\Omega J^1 \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 : \mathbb{D}_{\mathcal{A}^1} \partial_t v - \int_0^t \int_\Sigma \tilde{H}^3 \cdot \partial_t v. \end{aligned} \tag{6.37}$$

Step 2 – Estimates of the forcing terms

In order for the equation (6.37) to be useful, we must be able to estimate the terms that appear on its right. To this end, we now derive estimates for $\tilde{H}^1, \tilde{H}^2, \partial_t \tilde{H}^2$ in $H^0(\Omega)$ and $\tilde{H}^3$ in $H^{-1/2}(\Sigma)$. We claim that the following estimates hold; in each we have written $P(\cdot)$ for a polynomial so that $P(0) = 0$.

$$\begin{aligned} \left\| \tilde{H}^1 \right\|_0 \lesssim P(\sqrt{\varepsilon}) \Big[\left\| \zeta^1 - \zeta^2 \right\|_{3/2} + \left\| \partial_t \zeta^1 - \partial_t \zeta^2 \right\|_{1/2} + \left\| \partial_t^2 \zeta^1 - \partial_t^2 \zeta^2 \right\|_0 \\ + \left\| w^1 - w^2 \right\|_1 + \left\| \partial_t w^1 - \partial_t w^2 \right\|_1 + \left\| v \right\|_2 + \left\| q \right\|_1 \Big] \end{aligned} \tag{6.38}$$

$$\left\| \tilde{H}^2 \right\|_0 \lesssim P(\sqrt{\varepsilon}) \left[\left\| \zeta^1 - \zeta^2 \right\|_{1/2} + \left\| \partial_t \zeta^1 - \partial_t \zeta^2 \right\|_{1/2} + \left\| v \right\|_1 \right] \tag{6.39}$$

$$\left\| \partial_t \tilde{H}^2 \right\|_0 \lesssim P(\sqrt{\varepsilon}) \Big[\left\| \zeta^1 - \zeta^2 \right\|_{1/2} + \left\| \partial_t \zeta^1 - \partial_t \zeta^2 \right\|_{1/2} + \left\| \partial_t^2 \zeta^1 - \partial_t^2 \zeta^2 \right\|_{1/2} + \left\| v \right\|_1 + \left\| \partial_t v \right\|_1 \Big] \tag{6.40}$$

$$\left\| \tilde{H}^3 \right\|_{-1/2} \lesssim P(\sqrt{\varepsilon}) \left[\left\| \zeta^1 - \zeta^2 \right\|_{1/2} + \left\| \partial_t \zeta^1 - \partial_t \zeta^2 \right\|_{1/2} + \left\| v \right\|_2 + \left\| q \right\|_1 \right] + \left\| \partial_t \zeta^1 - \partial_t \zeta^2 \right\|_{-1/2} \tag{6.41}$$

According to the definitions (6.32)–(6.34), all of the summands in $\tilde{H}^1$, $\tilde{H}^2$, $\partial_t \tilde{H}^2$ are quadratic, of the form $X \times Y$, where Y is one of v, q, $\partial_t^j \zeta^1 - \partial_t^j \zeta^2$ for $j = 0, 1, 2$, or $\partial_t^j w^1 - \partial_t^j w^2$ for $j = 0, 1$. The bounds (6.38)–(6.40) may be established by estimating the products $X \times Y$ with Lemmas A.1, A.4, A.6, A.7, and A.5 and the usual Sobolev and trace embeddings; the appearance of the terms $P(\sqrt{\varepsilon})$ is due to the X terms, whose appropriate Sobolev norm may be bounded above by a polynomial in

$$\sqrt{\sup \{ \mathfrak{E}(\zeta^1), \mathfrak{E}(\zeta^2), \mathfrak{E}(v^1, q^1), \mathfrak{E}(v^2, q^2), \mathfrak{E}(w^1, 0), \mathfrak{E}(w^2, 0) \}} \leq \sqrt{\varepsilon}. \tag{6.42}$$

The estimate (6.41) follows similarly by using (A.3) of Lemma A.1, except that $\tilde{H}^3$ has a single term, namely $(\partial_t \zeta^1 - \partial_t \zeta^2) e_3$, that is not quadratic and that causes the last term on the right side of (6.41) to not be multiplied by $P(\sqrt{\varepsilon})$. The same sort of argument also allows us to deduce the bound

$$\left\| \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 \right\|_0 \lesssim P(\sqrt{\varepsilon}) \left[\left\| \zeta^1 - \zeta^2 \right\|_{1/2} + \left\| \partial_t \zeta^1 - \partial_t \zeta^2 \right\|_{1/2} \right]. \tag{6.43}$$

We will eventually employ an elliptic estimate with (6.27), so we will also need estimates of H^1, H^2, H^3 and the two other terms appearing on the right side of (6.27). The following estimates hold for $r = 0, 1$ (again P denotes a polynomial with $P(0) = 0$):

$$\left\| H^1 \right\|_r \lesssim P(\sqrt{\varepsilon}) \left[\left\| \zeta^1 - \zeta^2 \right\|_{r+1/2} + \left\| \partial_t \zeta^1 - \partial_t \zeta^2 \right\|_{r-1/2} + \left\| w^1 - w^2 \right\|_{r+1} \right] \tag{6.44}$$

$$\left\| H^2 \right\|_{r+1} \lesssim P(\sqrt{\varepsilon}) \left\| \zeta^1 - \zeta^2 \right\|_{r+3/2} \tag{6.45}$$

$$\left\| H^3 \right\|_{r+1/2} \lesssim P(\sqrt{\varepsilon}) \left\| \zeta^1 - \zeta^2 \right\|_{r+3/2} + \left\| \zeta^1 - \zeta^2 \right\|_{r+1/2} \tag{6.46}$$

$$\left\| \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2) \right\|_r \lesssim P(\sqrt{\varepsilon}) \left\| \zeta^1 - \zeta^2 \right\|_{r+1/2} \tag{6.47}$$

$$\left\| \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2 \mathcal{N}^1 \right\|_{r+1/2} \lesssim P(\sqrt{\varepsilon}) \left\| \zeta^1 - \zeta^2 \right\|_{r+3/2}. \tag{6.48}$$

The proof of (6.44)–(6.48) may be carried out in the same manner we used above to prove (6.38)–(6.41).

Step 3 – Estimates of $\partial_t v$ from (6.37)

Now we employ the estimates of the forcing terms from the previous step in (6.37) in order to deduce estimates for $\partial_t v$. First we note that, owing to (6.42) and Sobolev embeddings, we can bound

$$\left\| J^1 \right\|_{L^\infty} + \left\| K^1 \right\|_{L^\infty} \lesssim 1 + P(\sqrt{\varepsilon}) \text{ and } \left\| \partial_t J^1 \right\|_{L^\infty} \lesssim P(\sqrt{\varepsilon}) \tag{6.49}$$

for P a polynomial with $P(0) = 0$.

Because of the time derivative on q, the most delicate term in (6.37) is the product $J^1 \tilde{H}^2 \partial_t q$. To handle it we integrate by parts in time and use the fact that $q(0) = 0$ to see that

$$\begin{aligned} \int_0^t \int_\Omega J^1 \tilde{H}^2 \partial_t q &= \int_0^t \left[\partial_t \int_\Omega J^1 q \tilde{H}^2 - \int_\Omega \partial_t J^1 q \tilde{H}^2 + J^1 q \partial_t \tilde{H}^2 \right] \\ &= \int_\Omega J^1 q \tilde{H}^2(t) - J^1 q \tilde{H}^2(0) - \int_0^t \int_\Omega \partial_t J^1 q \tilde{H}^2 + J^1 q \partial_t \tilde{H}^2 \\ &= \int_\Omega J^1 q \tilde{H}^2(t) - \int_0^t \int_\Omega \partial_t J^1 q \tilde{H}^2 + J^1 q \partial_t \tilde{H}^2. \end{aligned} \tag{6.50}$$

This, (6.49), and the estimates (6.39) and (6.40) then imply that

$$\int_0^t \int_\Omega J^1 \tilde{H}^2 \partial_t q \lesssim P(\sqrt{\varepsilon}) \left\|q\right\|_{L^\infty H^0} \left[\sum_{j=0}^{1} \left\| \partial_t^j \zeta^1 - \partial_t^j \zeta^2 \right\|_{L^\infty H^{1/2}} + \left\|v\right\|_{L^\infty H^1} \right] + P(\sqrt{\varepsilon}) \int_0^t \left\|q\right\|_0 \left[\sum_{j=0}^{2} \left\| \partial_t^j \zeta^1 - \partial_t^j \zeta^2 \right\|_{1/2} + \left\|v\right\|_1 + \left\|\partial_t v\right\|_1 \right], \tag{6.51}$$

where the L^∞ norms are computed over the temporal interval $[0, T]$.

The other terms on the right of (6.37) are not so delicate and may be estimated directly with (6.38), (6.41), and (6.43). Indeed, these estimates together with trace theory and the Poincaré inequality imply that

$$\begin{aligned} &\int_0^t \int_\Omega J^1 \tilde{H}^1 \cdot \partial_t v - \frac{1}{2} J^1 \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 : \mathbb{D}_{\mathcal{A}^1} \partial_t v - \int_0^t \int_\Sigma \tilde{H}^3 \cdot \partial_t v \\ &\le \int_0^t \left\|J^1\right\|_{L^\infty} \left\|\tilde{H}^1\right\|_0 \left\|\partial_t v\right\|_0 + \frac{1}{2} \left\|J^1\right\|_{L^\infty} \left\|\mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2\right\|_0 \left\|\mathbb{D}_{\mathcal{A}^1} \partial_t v\right\|_0 \\ &\quad + \int_0^t \left\|\tilde{H}^3\right\|_{-1/2} \left\|\partial_t v\right\|_{H^{1/2}(\Sigma)} \lesssim \int_0^t \left\|\partial_t v\right\|_1 \left[P(\sqrt{\varepsilon}) \sqrt{\mathcal{Z}} + \left\|\partial_t \zeta^1 - \partial_t \zeta^2\right\|_{-1/2} \right], \end{aligned} \tag{6.52}$$

where we have written

$$\mathcal{Z} := \left\|\zeta^1 - \zeta^2\right\|_{3/2}^2 + \left\|\partial_t \zeta^1 - \partial_t \zeta^2\right\|_{1/2}^2 + \left\|\partial_t^2 \zeta^1 - \partial_t^2 \zeta^2\right\|_{1/2}^2 + \left\|w^1 - w^2\right\|_1^2 + \left\|\partial_t w^1 - \partial_t w^2\right\|_1^2 + \left\|v\right\|_2^2 + \left\|q\right\|_1^2. \tag{6.53}$$

Also, we may use (6.42) to bound

$$\int_0^t \int_\Omega \frac{\left|\partial_t v\right|^2}{2} (\partial_t J^1 K^1) J^1 \le C\sqrt{\varepsilon} \int_0^t \int_\Omega \frac{\left|\partial_t v\right|^2}{2} J^1 \tag{6.54}$$

for some constant $C > 0$.

We now combine the estimates (6.51), (6.52), and (6.54) with (6.37), employ Lemma 2.1 to bound $\left\|\partial_t v\right\|_1 /2 \le \left\|\sqrt{J^1} \mathbb{D}_{\mathcal{A}^1} \partial_t v\right\|_0$, and utilize Cauchy's inequality to absorb $\int_0^t \left\|\partial_t v\right\|_1^2$ onto the left side of the resulting inequality; this yields the bound

$$\begin{aligned} \frac{1}{2} \int_\Omega \left|\partial_t v\right|^2 J^1(t) + \frac{1}{8} \int_0^t \left\|\partial_t v\right\|_1^2 &\le C\sqrt{\varepsilon} \int_0^t \int_\Omega \frac{\left|\partial_t v\right|^2}{2} J^1 + P(\sqrt{\varepsilon}) \int_0^t \left\|q\right\|_0^2 \\ &\quad + P(\sqrt{\varepsilon}) \left\|q\right\|_{L^\infty H^0} \left[\sum_{j=0}^{1} \left\| \partial_t^j \zeta^1 - \partial_t^j \zeta^2 \right\|_{L^\infty H^{1/2}} + \left\|v\right\|_{L^\infty H^1} \right] \\ &\quad + P(\sqrt{\varepsilon}) \int_0^t \left\|q\right\|_0 \left[\sum_{j=0}^{2} \left\| \partial_t^j \zeta^1 - \partial_t^j \zeta^2 \right\|_{1/2} + \left\|v\right\|_1 \right] \\ &\quad + \int_0^t \left[P(\sqrt{\varepsilon}) \mathcal{Z} + C \left\|\partial_t \zeta^1 - \partial_t \zeta^2\right\|_{-1/2}^2 \right]. \end{aligned} \tag{6.55}$$

This bound can be viewed as a differential inequality of the form

$$x(t) + y(t) \le C\sqrt{\varepsilon} \int_0^t x(s) ds + F(t), \tag{6.56}$$

where $x, y, F \ge 0$, $x(0) = 0$, and $F(t)$ is increasing in t. Gronwall's lemma then implies that

$$x(t) + y(t) \le e^{C\sqrt{\varepsilon} t} F(t). \tag{6.57}$$

We assume that ε_1 and T_1 are sufficiently small for $e^{C\sqrt{\varepsilon}t} \le e^{C\sqrt{\varepsilon_1}T_1} \le 2$. Then from (6.55), (6.57), and Lemma 2.1 we deduce the bound

$$\begin{aligned}(6.58)\quad \|\partial_t v\|_{L^\infty H^0}^2 + \|\partial_t v\|_{L^2 H^1}^2 &\le P(\sqrt{\varepsilon}) \|q\|_{L^2H^0}^2 + C \left\|\partial_t \zeta^1 - \partial_t \zeta^2\right\|_{L^2H^{-1/2}}^2 + \int_0^T P(\sqrt{\varepsilon})\mathcal{Z} \\ &+ P(\sqrt{\varepsilon}) \|q\|_{L^\infty H^0} \left[\sum_{j=0}^1 \left\| \partial_t^j \zeta^1 - \partial_t^j \zeta^2 \right\|_{L^\infty H^{1/2}} + \|v\|_{L^\infty H^1} \right] \\ &+ P(\sqrt{\varepsilon}) \|q\|_{L^2 H^0} \left[\sum_{j=0}^2 \left\| \partial_t^j \zeta^1 - \partial_t^j \zeta^2 \right\|_{L^2 H^{1/2}} + \|v\|_{L^2 H^1} \right],\end{aligned}$$

where again the temporal L^∞ and L^2 norms are computed over $[0,T]$.

Step 4 – Elliptic estimates for v and q

In order to close our estimates, we must be able to estimate v and q. This will be accomplished with an elliptic estimate. We combine Proposition 3.7 with the estimates (6.44)–(6.48) to deduce the bound for $r = 0, 1$,

$$\begin{aligned}(6.59)\quad \|v\|_{r+2}^2 + \|q\|_{r+1}^2 &\lesssim \|\partial_t v\|_r^2 + \left\|H^1\right\|_r^2 + \left\|\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)} v^2)\right\|_r^2 + \left\|H^2\right\|_{r+1}^2 \\ &+ \left\|H^3\right\|_{r+1/2}^2 + \left\|\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)} v^2 \mathcal{N}^1\right\|_{r+1/2}^2 \lesssim \|\partial_t v\|_r^2 + \left\|\zeta^1 - \zeta^2\right\|_{r+1/2}^2 \\ &+ P(\sqrt{\varepsilon}) \left[\left\|\zeta^1 - \zeta^2\right\|_{r+3/2}^2 + \left\|\partial_t \zeta^1 - \partial_t \zeta^2\right\|_{r-1/2}^2 + \left\|w^1 - w^2\right\|_{r+1}^2 \right].\end{aligned}$$

We set $r = 0$ in (6.59) and then take the supremum in time over $[0,T]$ to find

$$\begin{aligned}(6.60)\quad \|v\|_{L^\infty H^2}^2 + \|q\|_{L^\infty H^1}^2 &\lesssim \|\partial_t v\|_{L^\infty H^0}^2 + \left\|\zeta^1 - \zeta^2\right\|_{L^\infty H^{1/2}}^2 \\ &+ P(\sqrt{\varepsilon}) \left[\left\|\zeta^1 - \zeta^2\right\|_{L^\infty H^{3/2}}^2 + \left\|\partial_t \zeta^1 - \partial_t \zeta^2\right\|_{L^\infty H^{-1/2}}^2 + \left\|w^1 - w^2\right\|_{L^\infty H^1}^2 \right].\end{aligned}$$

Then we set $r = 1$ in (6.59) and integrate over $[0,T]$ to find

$$\begin{aligned}(6.61)\quad \|v\|_{L^2 H^3}^2 + \|q\|_{L^2 H^2}^2 &\lesssim \|\partial_t v\|_{L^2 H^1}^2 + \left\|\zeta^1 - \zeta^2\right\|_{L^2 H^{3/2}}^2 \\ &+ P(\sqrt{\varepsilon}) \left[\left\|\zeta^1 - \zeta^2\right\|_{L^2 H^{5/2}}^2 + \left\|\partial_t \zeta^1 - \partial_t \zeta^2\right\|_{L^2 H^{1/2}}^2 + \left\|w^1 - w^2\right\|_{L^2 H^2}^2 \right].\end{aligned}$$

Step 5 – Estimates of $\zeta^1 - \zeta^2$

Now we turn to estimating the difference $\zeta^1 - \zeta^2$ in terms of $w^1 - w^2$. We subtract the equations satisfied by ζ^2 from the one for ζ^1 to find that

$$(6.62)\quad \begin{cases} \partial_t(\zeta^1 - \zeta^2) + w^1 \cdot D(\zeta^1 - \zeta^2) = (w^1 - w^2) \cdot \mathcal{N}^2 & \text{in } \Sigma \\ (\zeta^1 - \zeta^2)(t = 0) = 0. \end{cases}$$

The PDE (6.62) is a transport equation for $\zeta^1 - \zeta^2$, so we can employ Lemma A.8 to estimate

$$\begin{aligned}(6.63)\quad \left\|\zeta^1 - \zeta^2\right\|_{L^\infty H^{5/2}} &\le \exp\left(C \int_0^T \left\|w^1(r)\right\|_{H^{7/2}(\Sigma)} dr \right) \int_0^T \left\|(w^1 - w^2) \cdot \mathcal{N}^2(r)\right\|_{H^{5/2}(\Sigma)} dr \\ &\lesssim e^{C\sqrt{T}\sqrt{\varepsilon}} (1 + P(\sqrt{\varepsilon})) \int_0^T \left\|(w^1 - w^2)(r)\right\|_3 dr \\ &\lesssim e^{C\sqrt{T}\sqrt{\varepsilon}} (1 + P(\sqrt{\varepsilon})) \sqrt{T} \left\|w^1 - w^2\right\|_{L^2 H^3}.\end{aligned}$$

We can further restrict ε_1 and T_1 so that $e^{C\sqrt{T}\sqrt{\varepsilon}} \le 2$ and $1 + P(\sqrt{\varepsilon}) \le 2$; then

$$(6.64)\quad \left\|\zeta^1 - \zeta^2\right\|_{L^\infty H^{5/2}} \lesssim \sqrt{T} \left\|w^1 - w^2\right\|_{L^2 H^3}.$$

Then we use the first equation in (6.62), trace theory, and the estimate (6.64) to see that

$$\begin{aligned}(6.65)\quad \left\|\partial_t\zeta^1-\partial_t\zeta^2\right\|_{L^\infty H^{3/2}} &\le \left\|(w^1-w^2)\cdot\mathcal{N}^2\right\|_{L^\infty H^{3/2}}+\left\|w^1\cdot D(\zeta^1-\zeta^2)\right\|_{L^\infty H^{3/2}}\\ &\lesssim (1+P(\sqrt{\varepsilon}))\left\|w^1-w^2\right\|_{L^\infty H^{3/2}(\Sigma)}+P(\sqrt{\varepsilon})\left\|\zeta^1-\zeta^2\right\|_{L^\infty H^{5/2}}\\ &\lesssim \left\|w^1-w^2\right\|_{L^\infty H^2}+P(\sqrt{\varepsilon})\sqrt{T}\left\|w^1-w^2\right\|_{L^2H^3}.\end{aligned}$$

Similarly, we differentiate (6.62) in time to find that

$$\begin{aligned}(6.66)\quad \left\|\partial_t^2\zeta^1-\partial_t^2\zeta^2\right\|_{L^2H^{1/2}} &\lesssim (1+P(\sqrt{\varepsilon}))\left\|\partial_t w^1-\partial_t w^2\right\|_{L^2H^1}+P(\sqrt{\varepsilon})\Big[\left\|w^1-w^2\right\|_{L^2H^1}\\ &+\left\|\zeta^1-\zeta^2\right\|_{L^2H^{3/2}}+\left\|\partial_t\zeta^1-\partial_t\zeta^2\right\|_{L^2H^{3/2}}\Big]\lesssim \left\|\partial_t w^1-\partial_t w^2\right\|_{L^2H^1}\\ &+P(\sqrt{\varepsilon})\sqrt{T}\Big[\left\|w^1-w^2\right\|_{L^\infty H^1}+\left\|\zeta^1-\zeta^2\right\|_{L^\infty H^{3/2}}+\left\|\partial_t\zeta^1-\partial_t\zeta^2\right\|_{L^\infty H^{3/2}}\Big]\\ &\lesssim \left\|\partial_t w^1-\partial_t w^2\right\|_{L^2H^1}+P(\sqrt{\varepsilon})\sqrt{T}\left\|w^1-w^2\right\|_{L^\infty H^2}+P(\sqrt{\varepsilon})T\left\|w^1-w^2\right\|_{L^2H^3}.\end{aligned}$$

Step 6 – Synthesis: contraction

We now have all of the ingredients to prove our contraction result. We write

$$(6.67)\qquad \begin{aligned}\mathfrak{N}^v(T)&:=\mathfrak{N}(v^1-v^2,q^1-q^2;T)\\ \mathfrak{N}^w(T)&:=\mathfrak{N}(w^1-w^2,0;T)\\ \mathfrak{M}(T)&:=\mathfrak{M}(\zeta^1-\zeta^2;T),\end{aligned}$$

where $\mathfrak{M}$ and $\mathfrak{N}$ are defined by (6.22). We will first rewrite the bounds (6.58), (6.60), and (6.61) in terms of these new quantities.

We begin with the right side of (6.58). According to the definition of $\mathcal{Z}$, (6.53), we may bound

$$(6.68)\qquad \|q\|_{L^2H^0}^2+\int_0^T\mathcal{Z}\lesssim (1+T)\left[\mathfrak{M}(T)+\mathfrak{N}^w(T)\right]+T\mathfrak{N}^v(T)$$

Similarly,

$$\begin{aligned}(6.69)\quad \|q\|_{L^2H^0}&\left[\sum_{j=0}^2\left\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\right\|_{L^2H^{1/2}}+\|v\|_{L^2H^1}\right]\\ &\lesssim \sqrt{T}\sqrt{\mathfrak{N}^v(T)}\left[(1+\sqrt{T})\sqrt{\mathfrak{M}(T)}+\sqrt{T}\sqrt{\mathfrak{N}^v(T)}\right],\end{aligned}$$

$$(6.70)\qquad \left\|\partial_t\zeta^{m+1}-\partial_t\zeta^2\right\|_{L^2H^{-1/2}}^2\le T\mathfrak{M}(T),$$

and

$$\begin{aligned}(6.71)\quad \|q\|_{L^\infty H^0}&\left[\sum_{j=0}^1\left\|\partial_t^j\zeta^1-\partial_t^j\zeta^2\right\|_{L^\infty H^{1/2}}+\|v\|_{L^\infty H^1}\right]\\ &\lesssim \sqrt{\mathfrak{N}^v(T)}\left[\sqrt{\mathfrak{M}(T)}+\sqrt{\mathfrak{N}^v(T)}\right].\end{aligned}$$

Then, using (6.68)–(6.71) and Cauchy's inequality, we may rewrite (6.58) as

$$\begin{aligned}(6.72)\quad \|\partial_t v\|_{L^\infty H^0}^2+\|\partial_t v\|_{L^2H^1}^2 &\lesssim \left[T+P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{M}(T)+\left[P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^w(T)\\ &+\left[P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^v(T).\end{aligned}$$

Now we turn to the elliptic estimates (6.60)–(6.61). The bound (6.60) becomes

$$(6.73)\qquad \|v\|_{L^\infty H^2}^2+\|q\|_{L^\infty H^1}^2\lesssim \|\partial_t v\|_{L^\infty H^0}^2+\left\|\zeta^1-\zeta^2\right\|_{L^\infty H^{1/2}}^2+P(\sqrt{\varepsilon})\left[\mathfrak{M}(T)+\mathfrak{N}^w(T)\right].$$

Note here that we have kept the term with $\zeta^1-\zeta^2$ because it does not yet have a small multiplier in front of it. On the other hand, the bound (6.61) becomes

$$(6.74)\qquad \|v\|_{L^2H^3}^2+\|q\|_{L^2H^2}^2\lesssim \|\partial_t v\|_{L^2H^1}^2+T(1+P(\sqrt{\varepsilon}))\left[\mathfrak{M}(T)+\mathfrak{N}^w(T)\right].$$

We need not retain the $\zeta^1 - \zeta^2$ term in (6.74) since we can control the square of the temporal L^2 norm by the square of the L^∞ norm to pick up a T factor.

Next we reformulate the bounds (6.64)–(6.66) in a similar fashion. The estimate (6.64) becomes

$$\left\|\zeta^1 - \zeta^2\right\|^2_{L^\infty H^{5/2}} \lesssim T\mathfrak{N}^w(T). \tag{6.75}$$

Similarly, we may sum (6.65) and (6.66) to bound

$$\left\|\partial_t\zeta^1 - \partial_t\zeta^2\right\|^2_{L^\infty H^{3/2}} + \left\|\partial_t\zeta^1 - \partial_t\zeta^2\right\|^2_{L^2 H^{1/2}} \lesssim \left[1 + (T+T^2)P(\sqrt{\varepsilon})\right]\mathfrak{N}^w(T). \tag{6.76}$$

Summing (6.75) and (6.76) yields

$$\mathfrak{M}(T) \lesssim \left[1 + (T+T^2)P(\sqrt{\varepsilon})\right]\mathfrak{N}^w(T). \tag{6.77}$$

The estimate (6.26) directly follows from (6.77) and the definitions (6.67) if ε_1 and T_1 are small enough.

We now combine the above to get an estimate for $\mathfrak{N}^v$ from our estimates for v, q. Note that due to (6.75), estimate (6.73) also holds with $\left\|\zeta^1 - \zeta^2\right\|^2_{L^\infty H^{1/2}}$ replaced by $T\mathfrak{N}^w(T)$ on the right. We then add this modified version of (6.73) to (6.74), and then add to this a large constant times (6.72). If the constant is chosen to be sufficiently large, we can absorb the appearances of $\partial_t v$ norms on the right side into the left; doing so, we arrive at the bound

$$\begin{aligned}\mathfrak{N}^v(T) \lesssim \left[T + P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{M}(T) + \left[T + P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^w(T)\\ + \left[P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^v(T).\end{aligned} \tag{6.78}$$

This estimate may be combined with (6.77) to see that

$$\begin{aligned}\mathfrak{N}^v(T) \lesssim \left[1 + (T+T^2)P(\sqrt{\varepsilon})\right]\left[T + P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^w(T)\\ + \left[P(\sqrt{\varepsilon})(1+T)\right]\mathfrak{N}^v(T).\end{aligned} \tag{6.79}$$

By further restricting ε_1 and T_1, we may replace (6.79) by $\mathfrak{N}^v(T) \le \frac{1}{4}\mathfrak{N}^w(T) + \frac{1}{2}\mathfrak{N}^v(T)$, which may be rearranged to see that $\mathfrak{N}^v(T) \le \frac{1}{2}\mathfrak{N}^w(T)$, which gives (6.25) after using the definitions of $\mathfrak{N}^w(T)$, $\mathfrak{N}^v(T)$ given in (6.67). □

6.3. **Local well-posedness: the proof of Theorem 1.1.** Now we combine Theorems 6.1 and 6.2 to produce a solution to problem (1.10). Note that Theorem 1.1 follows directly from the following theorem by changing notation.

Theorem 6.3. *Assume that u_0, η_0 satisfy $\mathcal{E}_0, \mathcal{F}_0 < \infty$ and that the initial data $\partial_t^j u(0)$, etc are as constructed in Section 5.2 and satisfy the $(2N)^{th}$ compatibility conditions (5.26). Then there exist $0 < \delta_0, T_0 < 1$ so that if $\mathcal{E}_0 \le \delta_0$ and $0 < T \le T_0 \min\{1, 1/\mathcal{F}_0\}$, then the following hold. There exists a solution triple (u, p, η) to the problem (1.10) on the time interval $[0, T]$ that achieves the initial data and satisfies*

$$\mathfrak{K}(\eta) + \mathfrak{K}(u,p) \le C(\mathcal{E}_0 + T\mathcal{F}_0) \text{ and } \mathcal{F}(\eta) \le C(\mathcal{F}_0 + \mathcal{E}_0 + T\mathcal{F}_0) \tag{6.80}$$

for a universal constant $C > 0$. The solution is unique among functions that achieve the initial data and satisfy $\mathfrak{E}(\eta) + \mathfrak{E}(u,p) < \infty$. Moroever, η is such that the mapping $\Phi(\cdot, t)$, defined by (1.7), is a C^{4N-2} diffeomorphism for each $t \in [0, T]$.

Proof. We again divide the proof into several steps. First, we use Theorem 6.1 to construct a sequence of approximate solutions. Then we use Theorem 6.2 to show the sequence contracts in the norm $\sqrt{\mathfrak{M}(\eta; T) + \mathfrak{N}(u, p; T)}$, which yields strong convergence of the sequence. Next, we use an interpolation argument to improve the convergence results. These then allow us to pass to the limit in the PDEs to deduce that the limit solves the problem (1.10). Finally, we again use Theorem 6.2 to show that our solution is unique.

We assume throughout the proof that $T_0 \le \min\{T_1, \bar{T}\}$, where $\bar{T}$ is given by Theorem 6.1, and T_1 is given by Theorem 6.2. Let $C > 0$ denote the universal constant in Theorem 6.1. We further assume that $T_0 \le \varepsilon_1/(2C)$, where $\varepsilon_1 > 0$ is the constant from Theorem 6.2.

Step 1 – The sequence of approximate solutions

Suppose that $\delta_0 \le \delta$, where δ is given in Theorem 6.1. The hypotheses then allow us to apply Theorem 6.1 to produce the sequence of triples $\{(u^m, p^m, \eta^m)\}_{m=1}^\infty$, all elements of which achieve the initial data, satisfy the PDEs (6.1), (6.2), and obey the bounds

$$\sup_{m\ge 1} \left(\mathfrak{K}(\eta^m) + \mathfrak{K}(u^m,p^m)\right) \le C(\mathcal{E}_0 + T\mathcal{F}_0) \text{ and } \sup_{m\ge 1} \mathcal{F}(\eta^m) \le C(\mathcal{F}_0 + \mathcal{E}_0 + T\mathcal{F}_0). \tag{6.81}$$

We further assume that δ_0 is small enough for $C\delta_0 \le \varepsilon_1/2$ (with ε_1 again from Theorem 6.2) so that (6.81) implies, in particular, that

$$\sup_{m\ge 1} \max\left\{\mathfrak{E}(\eta^m), \mathfrak{E}(u^m,p^m)\right\} \le C(\mathcal{E}_0 + T\mathcal{F}_0) \le C(\delta_0 + T_0) \le \varepsilon_1. \tag{6.82}$$

The uniform bounds (6.81) allow us to take weak and weak-$*$ limits, up to the extraction of a subsequence:

$$\begin{cases} \partial_t^j u^m \rightharpoonup \partial_t^j u & \text{weakly in } L^2([0,T];H^{4N-2j+1}(\Omega)) \text{ for } j=0,\dots,2N+1 \\ \partial_t^j u^m \overset{*}{\rightharpoonup} \partial_t^j u & \text{weakly-}* \text{ in } L^\infty([0,T];H^{4N-2j}(\Omega)) \text{ for } j=0,\dots,2N \\ \partial_t^j p^m \rightharpoonup \partial_t^j p & \text{weakly in } L^2([0,T];H^{4N-2j}(\Omega)) \text{ for } j=0,\dots,2N \\ \partial_t^j p^m \overset{*}{\rightharpoonup} \partial_t^j p & \text{weakly-}* \text{ in } L^\infty([0,T];H^{4N-2j-1}(\Omega)) \text{ for } j=0,\dots,2N-1 \end{cases} \tag{6.83}$$

and

$$\begin{cases} \eta^m \rightharpoonup \eta & \text{weakly in } L^2([0,T];H^{4N+1/2}(\Sigma)) \\ \partial_t\eta^m \rightharpoonup \partial_t\eta & \text{weakly in } L^2([0,T];H^{4N-1/2}(\Sigma)) \\ \partial_t^j\eta^m \rightharpoonup \partial_t^j\eta & \text{weakly in } L^2([0,T];H^{4N-2j+5/2}(\Sigma)) \text{ for } j=2,\dots,2N+1 \\ \eta^m \overset{*}{\rightharpoonup} \eta & \text{weakly-}* \text{ in } L^\infty([0,T];H^{4N+1/2}(\Sigma)) \\ \partial_t^j\eta^m \overset{*}{\rightharpoonup} \partial_t^j\eta & \text{weakly-}* \text{ in } L^\infty([0,T];H^{4N-2j}(\Sigma)) \text{ for } j=1,\dots,2N. \end{cases} \tag{6.84}$$

Note that in the first convergence result of (6.83) we mean $H^{-1}(\Omega) = ({}_0H^1(\Omega))^*$ when $j = 2N+1$. According to the weak and weak-$*$ lower semicontinuity of the norms in $\mathfrak{K}(\eta^m)$, $\mathfrak{K}(u^m,p^m)$, and $\mathcal{F}(\eta^m)$ we find that the limit (u,p,η) satisfies

$$\mathfrak{K}(\eta) + \mathfrak{K}(u,p) \le C(\mathcal{E}_0 + T\mathcal{F}_0) \text{ and } \mathcal{F}(\eta) \le C(\mathcal{F}_0 + \mathcal{E}_0 + T\mathcal{F}_0). \tag{6.85}$$

The collection of triples (v,q,ζ) that achieve the initial data, i.e. $\partial_t^j v(0) = \partial_t^j u(0)$, $\partial_t^j\zeta(0) = \partial_t^j\eta(0)$, for $j = 0,\dots,2N$ and $\partial_t^j q(0) = \partial_t^j p(0)$ for $j = 0,\dots,2N-1$, is clearly convex; Lemma A.3 implies that it is also closed with respect to the topology generated by the norm $\sqrt{\mathfrak{D}(\zeta) + \mathfrak{D}(v,q)}$. As such, the collection is also closed in the corresponding weak topology. Then, since each (u^m,p^m,η^m) is in this collection, we deduce that the limit (u,p,η) is as well. Hence (u,p,η) achieves the initial data.

Step 2 – Contraction

Now we want to improve the weak convergence results of the previous step to strong convergence in the norm $\sqrt{\mathfrak{M}(\eta;T) + \mathfrak{N}(u,p;T)}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are defined by (6.22). For $m \ge 1$ we set $v^1 = u^{m+2}$, $v^2 = u^{m+1}$, $w^1 = u^{m+1}$, $w^2 = u^m$, $q^1 = p^{m+2}$, $q^2 = p^{m+1}$, $\zeta^1 = \eta^{m+1}$, $\zeta^2 = \eta^m$ in Theorem 6.2. Because of (6.1)–(6.2) we have that (6.24) holds; the initial data of w^j, v^j, q^j, ζ^j match for $j = 1,2$ by construction. Also, (6.82) implies that (6.23) holds, so all of the hypotheses of Theorem 6.2 are satisfied. Then (6.25) and (6.26) imply that

$$\mathfrak{N}(u^{m+2} - u^{m+1}, p^{m+2} - p^{m+1}; T) \le \frac{1}{2}\mathfrak{N}(u^{m+1} - u^m, p^{m+1} - p^m; T) \tag{6.86}$$

and

$$\mathfrak{M}(\eta^{m+1} - \eta^m; T) \le 2\mathfrak{N}(u^{m+1} - u^m, p^{m+1} - p^m; T). \tag{6.87}$$

The bound (6.86) implies that the sequence $\{(u^m,p^m)\}_{m=1}^\infty$ is Cauchy in the norm $\sqrt{\mathfrak{N}(\cdot,\cdot;T)}$, so as $m \to \infty$

$$\begin{cases} u^m \to u & \text{in } L^\infty([0,T];H^2(\Omega)) \cap L^2([0,T],H^3(\Omega)) \\ \partial_t u^m \to \partial_t u & \text{in } L^\infty([0,T];H^0(\Omega)) \cap L^2([0,T],H^1(\Omega)) \\ p^m \to p & \text{in } L^\infty([0,T];H^1(\Omega)) \cap L^2([0,T],H^2(\Omega)). \end{cases} \tag{6.88}$$

Because of (6.87), we further deduce that the sequence $\{\eta^m\}_{m=1}^\infty$ is Cauchy in the norm $\sqrt{\mathfrak{M}(\cdot;T)}$, so that as $m \to \infty$

$$(6.89)\qquad \begin{cases} \eta^m \to \eta & \text{in } L^\infty([0,T];H^{5/2}(\Sigma)) \\ \partial_t \eta^m \to \partial_t \eta & \text{in } L^\infty([0,T];H^{3/2}(\Sigma)) \\ \partial_t^2 \eta^m \to \partial_t^2 \eta & \text{in } L^2([0,T];H^{1/2}(\Sigma)). \end{cases}$$

Step 3 – Interpolation for improved strong convergence

Since (u^m, p^m, η^m) obey the bounds (6.81), we can parlay the convergence results (6.88), (6.89) into convergence in better norms by use of interpolation theory. We first interpolate with L^2H^0 norms of temporal derivatives (such estimates take the form

$$(6.90)\qquad \left\| \partial_t^k f \right\|_{L^2H^0} \le C(T) \left\| f \right\|_{L^2H^0}^\theta \left\| \partial_t^j f \right\|_{L^2H^0}^{1-\theta}$$

for $j > k \ge 0$ and $\theta = \theta(j,k) \in (0,1)$ and $C(T)$ a constant depending on T), which reveals that

$$(6.91)\qquad \begin{cases} \partial_t^j u^m \to \partial_t^j u & \text{in } L^2([0,T];H^0(\Omega)) \text{ for } j = 0,\dots,2N-1 \\ \partial_t^j p^m \to \partial_t^j p & \text{in } L^2([0,T];H^0(\Omega)) \text{ for } j = 0,\dots,2N-1 \\ \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } L^2([0,T];H^0(\Sigma)) \text{ for } j = 0,\dots,2N. \end{cases}$$

Here the range of j is determined by the range of j appearing in $\mathfrak{D}(\eta)$ and $\mathfrak{D}(u,p)$. Then we use spatial interpolation between H^0 and H^k to deduce from (6.91) that

$$(6.92)\qquad \begin{cases} \partial_t^j u^m \to \partial_t^j u & \text{in } L^2([0,T];H^{4N-2j}(\Omega)) \text{ for } j = 0,\dots,2N-1 \\ \partial_t^j p^m \to \partial_t^j p & \text{in } L^2([0,T];H^{4N-2j-1}(\Omega)) \text{ for } j = 0,\dots,2N-1 \\ \eta^m \to \eta & \text{in } L^2([0,T];H^{4N}(\Sigma)) \\ \partial_t \eta^m \to \partial_t \eta & \text{in } L^2([0,T];H^{4N-1}(\Sigma)) \\ \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } L^2([0,T];H^{4N-2j+2}(\Sigma)) \text{ for } j = 2,\dots,2N. \end{cases}$$

Here the Sobolev index is determined by the Sobolev index k in the L^2H^k norms of $\mathfrak{D}(\eta)$ and $\mathfrak{D}(u,p)$. Finally, we use the temporal L^2 convergence of (6.92) to get L^∞ and C^0 convergence by applying Lemma A.3. This yields

$$(6.93)\qquad \begin{cases} \partial_t^j u^m \to \partial_t^j u & \text{in } C^0([0,T];H^{4N-2j-1}(\Omega)) \text{ for } j = 0,\dots,2N-2 \\ \partial_t^j p^m \to \partial_t^j p & \text{in } C^0([0,T];H^{4N-2j-2}(\Omega)) \text{ for } j = 0,\dots,2N-2 \\ \eta^m \to \eta & \text{in } C^0([0,T];H^{4N-1/2}(\Sigma)) \\ \partial_t \eta^m \to \partial_t \eta & \text{in } C^0([0,T];H^{4N-3/2}(\Sigma)) \\ \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } C^0([0,T];H^{4N-2j+1}(\Sigma)) \text{ for } j = 2,\dots,2N-1. \end{cases}$$

Step 4 – Passing to the limit in the PDEs

The strong convergence results of (6.93) are more than sufficient for us to pass to the limit in the equations (6.1), (6.2) for each $t \in [0,T]$. Doing so, we find that the limits (u,p,η) are a strong solution to problem (1.10) on the time interval $t \in [0,T]$.

Step 5 – Uniqueness

We now turn to the question of uniqueness of our solution (u,p,η). Suppose that (v,q,ζ) is another solution to (1.10) on the time interval $[0,T]$ that achieves the same initial data as (u,p,η) and which satisfies $\mathfrak{E}(\zeta) + \mathfrak{E}(v,q) < \infty$. By continuity we may restrict to a temporal subinterval $[0,T_*] \subset [0,T]$ so that $\mathfrak{E}_0(\eta) + \mathfrak{E}_0(u,p) \le \mathfrak{E}(\zeta) + \mathfrak{E}(v,q) \le \varepsilon_1$, where ε_1 is given in Theorem 6.2 and the norms are computed on $[0,T_*]$. We then set $v^1 = w^1 = u$, $v^2 = w^2 = v$, $q^1 = p$, $q^2 = q$, $\zeta^1 = \eta$, and $\zeta^2 = \zeta$ in Theorem 6.2 to deduce that

$$(6.94)\qquad \mathfrak{N}(u-v,p-q;T_*) \le \frac{1}{2}\mathfrak{N}(u-v,p-q;T_*) \text{ and } \mathfrak{M}(\eta-\zeta;T_*) \le 2\mathfrak{N}(u-v,p-q;T_*),$$

which implies that $u = v$, $p = q$, $\eta = \zeta$ on the time interval $[0,T_*]$. This argument can then be iterated in the usual way, repeatedly increasing T_*, to extend the uniqueness to all of the interval $[0,T]$.

Step 6 – Diffeomorphism

It is easy to check that the smallness of $\mathfrak{K}(\eta)$ is sufficient to guarantee that the map Φ, given by (1.7), is a C^1 diffeomorphism for each $t \in [0,T]$. The fact that it is in C^{4N-2} follows easily from Lemma A.7 in the periodic case and Lemma A.5 in the infinite case.

□

Appendix A. Analytic tools

A.1. **Products in Sobolev spaces.** We will need some estimates of the product of functions in Sobolev spaces.

Lemma A.1. *The following hold for sufficiently smooth subsets of* $\mathbb{R}^n$.

(1) *Let* $0 \le r \le s_1 \le s_2$ *be such that* $s_1 > n/2$. *Let* $f \in H^{s_1}$, $g \in H^{s_2}$. *Then* $fg \in H^r$ *and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A.1}$$

(2) *Let* $0 \le r \le s_1 \le s_2$ *be such that* $s_2 > r + n/2$. *Let* $f \in H^{s_1}$, $g \in H^{s_2}$. *Then* $fg \in H^r$ *and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A.2}$$

(3) *Let* $0 \le r \le s_1 \le s_2$ *be such that* $s_2 > r + n/2$. *Let* $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. *Then* $fg \in H^{-s_1}(\Sigma)$ *and*

$$\|fg\|_{-s_1} \lesssim \|f\|_{-r} \|g\|_{s_2}. \tag{A.3}$$

Proof. The proofs of (A.1) and (A.2) are standard; the bounds are first proved in $\mathbb{R}^n$ with the Fourier transform, and then the bounds in sufficiently nice subsets of $\mathbb{R}^n$ are deduced by use of an extension operator. To prove (A.3) we argue by duality. For $\varphi \in H^{s_1}$ we use (A.2)bound

$$\int_\Sigma \varphi f g \lesssim \|\varphi g\|_r \|f\|_{-r} \lesssim \|\varphi\|_{s_1} \|g\|_{s_2} \|f\|_{-r}, \tag{A.4}$$

so that taking the supremum over φ with $\|\varphi\|_{s_1} \le 1$ we get (A.3).

□

We will also need the following variant.

Lemma A.2. *Suppose that* $f \in C^1(\Sigma)$ *and* $g \in H^{1/2}(\Sigma)$. *Then*

$$\|fg\|_{1/2} \lesssim \|f\|_{C^1} \|g\|_{1/2}. \tag{A.5}$$

Proof. Consider the operator $F : H^k \to H^k$ given by $F(g) = fg$ for $k = 0,1$. It is a bounded operator for $k = 0,1$ since

$$\|fg\|_0 \le \|f\|_{C^1} \|g\|_0 \text{ and } \|fg\|_1 \lesssim \|f\|_{C^1} \|g\|_1. \tag{A.6}$$

Then the theory of interpolation of operators implies that F is bounded from $H^{1/2}$ to itself, with operator norm less than a constant times $\sqrt{\|f\|_{C^1}}\sqrt{\|f\|_{C^1}} = \|f\|_{C^1}$, which is the desired result. □

A.2. **Continuity and temporal derivatives.** We will need the following interpolation result, which affords us control of the $L^\infty H^k$ norm of a function f, given that we control f in $L^2 H^{k+m}$ and $\partial_t f$ in $L^2 H^{k-m}$.

Lemma A.3. *Let* Γ *denote either* Σ *or* Ω. *Suppose that* $\zeta \in L^2([0,T]; H^{s_1}(\Gamma))$ *and* $\partial_t \zeta \in L^2([0,T]; H^{s_2}(\Gamma))$ *for* $s_1 \ge s_2 \ge 0$. *Let* $s = (s_1 + s_2)/2$. *Then* $\zeta \in C^0([0,T]; H^s(\Gamma))$ *(after possibly being redefined on a set of measure* 0*), and*

$$\|\zeta\|_{L^\infty H^s} \lesssim \left(1 + \frac{1}{T}\right) \left(\|\zeta\|^2_{L^2 H^{s_1}} + \|\partial_t \zeta\|^2_{L^2 H^{s_2}} \right). \tag{A.7}$$

Proof. According to the usual theory of extensions and restrictions in Sobolev spaces, it suffices to prove the result with $\Gamma = \mathbb{R}^n$ or $\Gamma = (L_1\mathbb{T}) \times (L_2\mathbb{T}) \times \mathbb{R}^m$ for $n = 2, 3$, $m = 0, 1$. We will prove the result assuming that $\Gamma = \mathbb{R}^n$; the proof in the other case may be derived similarly, replacing integrals in Fourier space with sums, etc. Assume for the moment that ζ is smooth. Writing $\hat{}$ for the Fourier transform, we compute

$$\text{(A.8)} \quad \begin{aligned} \partial_t \|\zeta(t)\|_s^2 &= 2\Re\left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s}\hat{\zeta}(\xi,t)\overline{\partial_t\hat{\zeta}(\xi,t)}d\xi\right) \le 2\int_{\mathbb{R}^n}\langle\xi\rangle^{2s}\left|\hat{\zeta}(\xi,t)\right|\left|\partial_t\hat{\zeta}(\xi,t)\right|d\xi \\ &= 2\int_{\mathbb{R}^n}\langle\xi\rangle^{s_1}\left|\hat{\zeta}(\xi,t)\right|\langle\xi\rangle^{s_2}\left|\partial_t\hat{\zeta}(\xi,t)\right|d\xi \le \int_{\mathbb{R}^n}\langle\xi\rangle^{2s_1}\left|\hat{\zeta}(\xi,t)\right|^2 d\xi + \int_{\mathbb{R}^n}\langle\xi\rangle^{2s_2}\left|\partial_t\hat{\zeta}(\xi,t)\right|^2 d\xi \\ &= \|\zeta(t)\|_{s_1}^2 + \|\partial_t\zeta(t)\|_{s_2}^2 . \end{aligned}$$

Hence for $r, t \in [0, T]$, we have that $\|\zeta(t)\|_s^2 \le \|\zeta(r)\|_s^2 + \|\zeta\|_{L^2H^{s_1}}^2 + \|\partial_t\zeta\|_{L^2H^{s_2}}^2$. We can then integrate both sides of this inequality with respect to $r \in [0, T]$ to deduce the bound

$$\text{(A.9)} \quad \sup_{0\le t\le T}\|\zeta(t)\|_s^2 \le \frac{1}{T}\|\zeta\|_{L^2H^s}^2 + \|\zeta\|_{L^2H^{s_1}}^2 + \|\partial_t\zeta\|_{L^2H^{s_2}}^2 \lesssim \left(1+\frac{1}{T}\right)\left(\|\zeta\|_{L^2H^{s_1}}^2 + \|\partial_t\zeta\|_{L^2H^{s_2}}^2\right).$$

If ζ is not smooth, we may employ a standard mollification argument (cf. Section 5.9 of [16]) in conjunction with (A.9) to deduce that $\zeta \in C^0([0, T]; H^s(\mathbb{R}^n))$ and that (A.7) holds. □

A.3. **Poisson integral: non-periodic case.** For a function f, defined on $\Sigma = \mathbb{R}^2$, the Poisson integral in $\mathbb{R}^2 \times (-\infty, 0)$ is defined by

$$\text{(A.10)} \quad \mathcal{P}f(x', x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi)e^{2\pi|\xi|x_3}e^{2\pi i x'\cdot\xi}d\xi.$$

Although $\mathcal{P}f$ is defined in all of $\mathbb{R}^2 \times (-\infty, 0)$, we will only need bounds on its norm in the restricted domain $\Omega = \mathbb{R}^2 \times (-b, 0)$. This yields a couple improvements of the usual estimates of $\mathcal{P}f$ on the set $\mathbb{R}^2 \times (-\infty, 0)$.

Lemma A.4. *Let $\mathcal{P}f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$ (here $\dot{H}^s$ is the usual homogeneous Sobolev space of order s). Then*

$$\text{(A.11)} \quad \|\nabla^q\mathcal{P}f\|_0^2 \lesssim \int_{\mathbb{R}^2}|\xi|^{2q}\left|\hat{f}(\xi)\right|^2\left(\frac{1-e^{-4\pi b|\xi|}}{|\xi|}\right)d\xi,$$

and in particular

$$\text{(A.12)} \quad \|\nabla^q\mathcal{P}f\|_0^2 \lesssim \|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2 \text{ and } \|\nabla^q\mathcal{P}f\|_0^2 \lesssim \|f\|_{\dot{H}^q(\Sigma)}^2 .$$

Proof. Employing Fubini, the horizontal Fourier transform, and Parseval, we may bound

$$\text{(A.13)} \quad \begin{aligned} \|\nabla^q\mathcal{P}f\|_0^2 &\lesssim \int_{\mathbb{R}^2}\int_{-b}^0|\xi|^{2q}\left|\hat{f}(\xi)\right|^2 e^{4\pi|\xi|x_3}dx_3d\xi \le \int_{\mathbb{R}^2}|\xi|^{2q}\left|\hat{f}(\xi)\right|^2\left(\int_{-b}^0 e^{4\pi|\xi|x_3}dx_3\right)d\xi \\ &\lesssim \int_{\mathbb{R}^2}|\xi|^{2q}\left|\hat{f}(\xi)\right|^2\left(\frac{1-e^{-4\pi b|\xi|}}{|\xi|}\right)d\xi. \end{aligned}$$

This is (A.11). To deduce (A.12) from (A.11), we simply note that

$$\text{(A.14)} \quad \frac{1-e^{-4\pi b|\xi|}}{|\xi|} \le \min\left\{4\pi b, \frac{1}{|\xi|}\right\},$$

which means we are free to bound the right hand side of (A.13) by either $\|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2$ or $\|f\|_{\dot{H}^q(\Sigma)}^2$. □

We will also need L^∞ estimates.

Lemma A.5. *Let $\mathcal{P}f$ be the Poisson integral of f, defined on Σ. Let $q \in \mathbb{N}$, $s > 1$. Then*

$$\left\| \nabla^q \mathcal{P} f \right\|_{L^\infty}^2 \lesssim \left\| D^q f \right\|_s^2 . \tag{A.15}$$

Proof. We use the definition of $\mathcal{P}f$ and the trivial estimate $\exp(2\pi |\xi| x_3) \le 1$ in Ω to bound

$$\left\| \nabla^q \mathcal{P} f \right\|_{L^\infty} \lesssim \int_{\mathbb{R}^2} |\xi|^q \left| \hat{f}(\xi) \right| d\xi. \tag{A.16}$$

The estimate (A.15) then follows from this and the easy bound

$$\int_{\mathbb{R}^2} |\xi|^q \left| \hat{f}(\xi) \right| d\xi \lesssim \left\| D^q f \right\|_s \left(\int_{\mathbb{R}^2} \langle \xi \rangle^{-2s} d\xi \right)^{1/2} \lesssim \left\| D^q f \right\|_s , \tag{A.17}$$

which holds when $s > 1$. □

A.4. **Poisson integral: periodic case.** Suppose that $\Sigma = (L_1 \mathbb{T}) \times (L_2 \mathbb{T})$. We define the Poisson integral in $\Omega_- = \Sigma \times (-\infty, 0)$ by

$$\mathcal{P} f(x) = \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} e^{2\pi i n \cdot x'} e^{2\pi |n| x_3} \hat{f}(n), \tag{A.18}$$

where for $n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})$ we have written

$$\hat{f}(n) = \int_\Sigma f(x') \frac{e^{-2\pi i n \cdot x'}}{L_1 L_2} dx'. \tag{A.19}$$

It is well known that $\mathcal{P} : H^s(\Sigma) \to H^{s+1/2}(\Omega_-)$ is a bounded linear operator for $s > 0$. We now show that how derivatives of $\mathcal{P}f$ can be estimated in the smaller domain Ω.

Lemma A.6. *Let $\mathcal{P}f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then*

$$\left\| \nabla^q \mathcal{P} f \right\|_0^2 \lesssim \left\| f \right\|_{\dot{H}^{q-1/2}(\Sigma)}^2 \text{ and } \left\| \nabla^q \mathcal{P} f \right\|_0^2 \lesssim \left\| f \right\|_{\dot{H}^q(\Sigma)}^2 . \tag{A.20}$$

Proof. Since $\mathcal{P}f$ is defined on $\Sigma \times (-\infty, 0)$, it suffices to prove the estimates on $\tilde{\Omega} := \Sigma \times (-b_+, 0)$ since $\Omega \subset \tilde{\Omega}$. By Fubini and Parseval,

$$\begin{aligned} \left\| \nabla^q \mathcal{P} f \right\|_{H^0(\tilde{\Omega})}^2 &\lesssim \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} \int_{-b_+}^0 |n|^{2q} \left| \hat{f}(n) \right|^2 e^{4\pi |n| x_3} dx_3 \\ &\lesssim \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} |n|^{2q} \left| \hat{f}(n) \right|^2 \left(\frac{1 - e^{-4\pi b_+ |n|}}{|n|} \right). \end{aligned} \tag{A.21}$$

However,

$$\frac{1 - e^{-4\pi b_+ |n|}}{|n|} \le \min \left\{ 4\pi b_+, \frac{1}{|n|} \right\}, \tag{A.22}$$

which means we are free to bound the right hand side of (A.21) by either $\left\| f \right\|_{\dot{H}^{q-1/2}(\Sigma)}^2$ or $\left\| f \right\|_{\dot{H}^q(\Sigma)}^2$. □

We will also need L^∞ estimates.

Lemma A.7. *Let $\mathcal{P}f$ be the Poisson integral of a function f that is in $\dot{H}^{q+s}(\Sigma)$ for $q \ge 1$ an integer and $s > 1$. Then*

$$\left\| \nabla^q \mathcal{P} f \right\|_{L^\infty}^2 \lesssim \left\| f \right\|_{\dot{H}^{q+s}}^2 . \tag{A.23}$$

The same estimate holds for $q = 0$ if f satisfies $\int_\Sigma f = 0$.

Proof. We estimate

$$\text{(A.24)}\quad \|\nabla^q \mathcal{P} f\|_{L^\infty} \lesssim \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \left|\hat{f}(n)\right| |n|^q \lesssim \|f\|_{\dot{H}^{q+s}} \left(\sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})\backslash\{0\}} |n|^{-2s} \right)^{1/2} \lesssim \|f\|_{\dot{H}^{q+s}}$$

if $s > 1$. The same estimate works with $q = 0$ if $\hat{f}(0) = 0$. □

A.5. **Transport estimate.** Let Σ be either periodic or non-periodic. Consider the equation

$$\text{(A.25)}\quad \begin{cases} \partial_t \eta + u \cdot D\eta = g & \text{in } \Sigma \times (0,T) \\ \eta(t=0) = \eta_0 \end{cases}$$

with $T \in (0,\infty]$. We have the following estimate of the transport of regularity for solutions to (A.25), which is a particular case of a more general result proved in [13]. Note that the result in [13] is stated for $\Sigma = \mathbb{R}^2$, but the same result holds in the periodic setting $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$, as described in [14].

Lemma A.8 (Proposition 2.1 of [13])**.** *Let η be a solution to* (A.25)*. Then there is a universal constant $C > 0$ so that for any $0 \le s < 2$*

$$\text{(A.26)}\quad \sup_{0\le r\le t} \|\eta(r)\|_{H^s} \le \exp\left(C\int_0^t \|Du(r)\|_{H^{3/2}}\, dr \right) \left(\|\eta_0\|_{H^s} + \int_0^t \|g(r)\|_{H^s}\, dr \right).$$

Proof. Use $p = p_2 = 2$, $N = 2$, and $\sigma = s$ in Proposition 2.1 of [13] along with the embedding $H^{3/2} \hookrightarrow B^1_{2,\infty} \cap L^\infty$. □

A.6. **Poincaré-type inequalities.** Let Σ and Ω be either periodic or non-periodic.

Lemma A.9. *It holds that*

$$\text{(A.27)}\quad \|f\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Sigma)} + \|\partial_3 f\|^2_{L^2(\Omega)}$$

for all $f \in H^1(\Omega)$. Also, if $f \in W^{1,\infty}(\Omega)$, then

$$\text{(A.28)}\quad \|f\|^2_{L^\infty(\Omega)} \lesssim \|f\|^2_{L^\infty(\Sigma)} + \|\partial_3 f\|^2_{L^\infty(\Omega)}.$$

Proof. By density we may assume that f is smooth. Writing $x = (x', x_3)$ for $x' \in \Sigma$ and $x_3 \in (-b(x'), 0)$, we have

$$\text{(A.29)}\quad \left|f(x',x_3)\right|^2 = \left|f(x',0)\right|^2 - 2\int_{x_3}^0 f(x',z)\partial_3 f(x',z) dz \le \left|f(x',0)\right|^2 + 2\int_{-b(x')}^0 \left|f(x',z)\right| \left|\partial_3 f(x',z)\right| dz.$$

We may integrate this with respect to $x_3 \in (-b(x'), 0)$ to get

$$\text{(A.30)}\quad \int_{-b(x')}^0 \left|f(x',x_3)\right|^2 dx_3 \lesssim \left|f(x',0)\right|^2 + 2\int_{-b(x')}^0 \left|f(x',z)\right| \left|\partial_3 f(x',z)\right| dz.$$

Now we integrate over $x' \in \Sigma$ to find

$$\text{(A.31)}\quad \int_\Omega |f(x)|^2\, dx \lesssim \|f\|^2_{L^2(\Sigma)} + 2\int_\Omega |f(x)|\, |\partial_3 f(x)|\, dx \le \|f\|^2_{L^2(\Sigma)} + \varepsilon \|f\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon} \|\partial_3 f\|^2_{L^2(\Omega)}$$

for any $\varepsilon > 0$. Choosing $\varepsilon > 0$ sufficiently small then yields (A.27). The estimate (A.28) follows similarly, taking suprema rather than integrating. □

We will need a version of Korn's inequality, which is proved, for instance, in Lemma 2.7 [6].

Lemma A.10. *It holds that* $\|u\|_1 \lesssim \|\mathbb{D}u\|_0$ *for all* $u \in H^1(\Omega;\mathbb{R}^3)$ *so that* $u = 0$ *on* Σ_b.

We also record the standard Poincaré inequality, which applies for functions taking either vector or scalar values.

Lemma A.11. *It holds that* $\|f\|_0 \lesssim \|f\|_1 \lesssim \|\nabla f\|_0$ *for all* $f \in H^1(\Omega)$ *so that* $f = 0$ *on* Σ_b. *Also,* $\|f\|_{L^\infty(\Omega)} \lesssim \|f\|_{W^{1,\infty}(\Omega)} \lesssim \|\nabla f\|_{L^\infty(\Omega)}$ *for all* $f \in W^{1,\infty}(\Omega)$ *so that* $f = 0$ *on* Σ_b.

A.7. **An elliptic estimate.** The proof of the following estimate may be found in [6] in the non-periodic case. The same proof holds in the periodic case with obvious modification.

Lemma A.12. *Suppose* (u, p) *solve*

$$\begin{cases} -\Delta u + \nabla p = \phi \in H^{r-2}(\Omega) \\ \operatorname{div} u = \psi \in H^{r-1}(\Omega) \\ (pI - \mathbb{D}(u))e_3 = \alpha \in H^{r-3/2}(\Sigma) \\ u|_{\Sigma_b} = 0. \end{cases} \tag{A.32}$$

Then for $r \geq 2$,

$$\|u\|_{H^r}^2 + \|p\|_{H^{r-1}}^2 \lesssim \|\phi\|_{H^{r-2}}^2 + \|\psi\|_{H^{r-1}}^2 + \|\alpha\|_{H^{r-3/2}}^2 . \tag{A.33}$$

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