An Exposition of Shelah's Proof of a Categoricity Theorem for Uncountable Languages

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Introduction

We say that a first-order theory is categorical in a cardinal λ if every model of the theory with cardinality λ is isomorphic. In [Mo 65] Morley proved that a countable complete theory T which is categorical in an uncountable λ is categorical in every uncountable cardinal. Shelah, in [Sh 74], proved the following generalization: for a complete theory T of any infinite cardinality, if it is categorical in some $\lambda > |T|$ then it is categorical in every $\kappa > |T|$.

Briefly, the main idea behind the proof of Morley's categoricity theorem is that a countable theory that is categorical for some uncountable cardinal is totally transcendental and therefore stable in all infinite cardinals. From then, one can prove that there is a strongly minimal formula and a prime model over a strongly minimal set, which can then be proved to be unique and therefore establish categoricity. None of these properties hold for an uncountable language, although for each one there is a corresponding generalization which, while much more difficult to prove, allows one to partially recover this line of attack.

This exposition follows Shelah's originial proof in [Sh 74] for the main theorem of categoricity, and includes all the model-theoretic tools needed to prove it. Shelah actually updated the proof in [Sh 90] as part of constructing the entire spectrum of isomorphic classes for a given theory, and thus introduced many tools in the process. Unfortunately, the updated proof depends heavily on these tools, the exposition of which is not necessary for the original proof and will require a much lengthier exposition. We have thus avoided the unnecessary machinery of [Sh 90] if there is some weaker concept that is sufficient for the current exposition.

Additionally, this exposition was written with the intent of being self-contained with respect to model theory, and the intended audience is only assumed to be familiar with the semantics of first order logic and elementary set theory (in particular ordinal and cardinal arithmetic). Chapter one is thus devoted to an introduction of elementary model theoretical concepts: signatures, models, Skolem functions and types. The reader who is familiar with the Compactness theorem, Löwenheim-Skolem theorems, saturated models, universal models and ultrapowers may skip this chapter entirely. However, readers who have not encountered this subject before will likely find this chapter insufficient in providing familiarity necessary for the results to follow. Readers who have not encountered model theory before are therefore advised to consult other introductory texts (for example, [Ho 97]) in addition to this exposition.

Chapter two introduces most of the machinery which would be needed for the proof of the main theorem: stability and stable formulas, λ-prime models, indiscernibles, definable types, two cardinal theorems and Ehrenfeucht-Mostowski models. The concept of stability is the starting point to stability theory, but as mentioned above we will be avoiding most of the powerful tools of stability theory: this includes ranks, forking and dimensionality, since they are too general for the task at hand and each require a lengthy exposition itself.

Chapter three begins the exposition proper of [Sh 74]: we first fix the notation that we will use for what is to follow (which we cannot do immediately as pedagogically we have not yet introduced any model-theoretic concepts), then follow step-by-step Shelah's original proof. In particular, Shelah's original paper makes references to papers which only proved their respective results for countable languages, and the generalization of those results to uncountable languages is not always straight forward. However, we have structured this exposition so that the relevant generalizations are presented in chapter two.

During a first reading, the various concepts and results proved in chapter one and two may seem unmotivated: the structure of this exposition is such that each section contains most of the results corresponding to the theme of the section, and the application of each result may not seem obvious immediately. Thus it may be helpful to skip over the details of the proofs in chapter one and two and only return to them when the exposition of Shelah's proof in chapter three requires a particular result.

The historical remarks are meant to be a reference to where the results first arose in the literature, and also where the proofs given in this exposition are from. The author has tried to trace these sources to the best of his ability, but it is likely that some of these proofs are based on an idea which pre-dates the cited source of the proof (especially the results in chapter one, which are now considered classical results and found in a large number of introductory texts). Additionally, there are a few lemmas that are unattributed, which are mostly specific details that the original papers did not address explicitly but (in the author's opinion) should be made clearer. In any case, the author does not claim any of these lemmas to be a novel result, as even those which are unattributed are widely known in the field.

One should be aware that the proof given here is probably not the most efficient method of proof, and the lemmas proved along the way are not the most general ones possible. The interested reader should consult [Sh 90] for the details, but there are some details that are worth mentioning: it is possible to prove that a theory which is categorical in some $\lambda > |T|$ is unimodular and superstable, and in particular μ-stable for any $\mu \geq |T|$ (improving on Proposition 3.2.1). This simplifies the proof that every model of T is locally saturated (Theorem 3.2.3), yields a different proof that $D(x = x) < \infty$ (Theorem 3.3.5) and also gives a simpler proof of the existence of a weakly minimal formula (Theorem 3.4.6). Furthermore, using techniques like forking and Morley sequences, one can show that if T is μ-stable then there is a saturated model of T with cardinality μ, giving a stronger result than Proposition 2.1.17 and making Corollary 3.4.9 a trivial claim. Lastly, we proved Theorem 2.5.9 and Theorem 2.6.2 for the simplest cases, but in fact the same results can be proved for a much larger range of cardinalities.

Chapter 1

Model Theoretic Preliminaries

1.1 First Order Languages and Structures

Given a variety of algebraic structures, for examples groups, rings or lattices, we may talk about constants, operations or relations of the structure. We generalize and formalize this using formal languages:

Definition 1.1.1. *A* ***signature*** τ *consists of a set of constant symbols, a set of function symbols and a set of relation symbols, all of which are pairwise disjoint. A* ***term*** *in* τ *is defined inductively by:*

1. *A constant symbol* c *is a term*
2. *A variable* x *is a term*
3. *If* $t_0, \ldots, t_{n-1}$ *are terms and* f *is a function symbol of arity* n*, then* $f(t_0, \ldots, t_{n-1})$ *is a term*
4. *There are no other terms*

A ***closed term*** *is a term which does not contain any variables.*
The ***language*** L_τ *has* $\tau \cup \{(,), \neg, \vee, \wedge, \rightarrow, \leftrightarrow, \forall, \exists\} \cup \{v_i : i < \omega\}$ *as symbols, with* $\{v_i : i < \omega\}$ *as variables, and is defined inductively by the following formation clauses:*

1. *If* t_0, t_1 *are terms, then* $(t_0 = t_1)$ *is a formula in* L_τ *(Read as "*t_0 *equals* t_1*")*
2. *If* $t_0, \ldots, t_{n-1}$ *are terms and* P *is a relation symbol of arity* n*, then* $(P(t_0, \ldots, t_{n-1}))$ *is a formula in* L_τ
3. *If* ϕ *is a formula in* L_τ*, then* $(\neg\phi)$ *is a formula in* L_τ *(Read as "not* ϕ*")*

4. *If ϕ, ψ are formulas in L_τ, then $(\phi \wedge \psi)$ is a formula in L_τ (Read as "ϕ and ψ")*

5. *If ϕ, ψ are formulas in L_τ, then $(\phi \vee \psi)$ is a formula in L_τ (Read as "ϕ or ψ")*

6. *If ϕ, ψ are formulas in L_τ, then $(\phi \rightarrow \psi)$ is a formula in L_τ (Read as "if ϕ then ψ")*

7. *If ϕ, ψ are formulas in L_τ, then $(\phi \leftrightarrow \psi)$ is a formula in L_τ (Read as "ϕ is equivalent to ψ")*

8. *If ϕ is a formula in L_τ and x is a variable, then $(\forall x \phi)$ is a formula in L_τ (Read as "for all x $\phi(x)$")*

9. *If ϕ is a formula in L_τ and x is a variable, then $(\exists x \phi)$ is a formula in L_τ (Read as "there exists x $\phi(x)$")*

10. *No other string of symbols is a formula in L_τ*

*An **atomic formula** is one which is of the forms as in clauses 1 or 2 above.*
*A **literal formula** is either atomic or one formed by clause 3 above from an atomic ϕ i.e. the negation of an atomic formula.*
*A **quantifier-free formula** is one which is formed without using clauses 8 or 9 above i.e. the quantifiers $\forall, \exists$ do not appear in the formula.*
For a formula ϕ, a variable x is free in ϕ if:

1. *ϕ is of the form $(t = s)$ or $(P(t_0, \dots, t_{n-1}))$ and x occurs in ϕ*

2. *ϕ is of the form $(\neg\psi)$ for a formula ψ and x is free in ψ*

3. *ϕ is of the form $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi)$ or $(\varphi \leftrightarrow \psi)$ for formulas φ, ψ, and x is free in either φ or ψ*

4. *ϕ is of the form $(\forall y \psi), (\exists y \psi)$, x is free in ψ and x is not y*

For a formula $\phi \in L$, we write $\phi(x_0, \dots, x_{n-1})$ to emphasize that the variables $x_0, \dots, x_{n-1}$ are either free in ϕ or do not occur in ϕ, though not necessarily including ALL the free variables in ϕ.
*A **closed formula** is a formula with no free variables.*

Remark. We will abbreviate the formula

$$\exists x_0, \dots, x_{n-1}((\bigwedge_{0 \le i < n} \phi(x_i)) \wedge (\bigwedge_{0 \le i < j < n} x_i \neq x_j))$$

by $\exists^{\geq n} x \phi(x)$. Similarly, we abbreviate the formula

$$\exists x_0, \dots, x_{n-1}((\bigwedge_{0 \le i < n} \phi(x_i)) \wedge (\forall y (\bigwedge_{0 \le i < n} x_i \neq y) \rightarrow \neg\phi(y)))$$

by $\exists^{\leq n} x \phi(x)$. We will abbreviate $(\exists^{\geq n} x \phi(x)) \wedge (\exists^{\leq n} x \phi(x))$ by $\exists^{= n} x \phi(x)$.

We will omit parentheses surrounding formulas when there is no ambiguity. We will omit τ and write L when there is no ambiguity, and similarly we will speak of constants symbols of L, function symbols of L, relation symbols of L and terms of L without reference to τ.

Additionally, for any signature τ in this exposition, if C is the set of constant symbols, F the set of function symbols and R the set of relations symbols, we will assume that

$$|C| + |F| \geq |R|$$

While not necessary, this assumption will simplify certain proofs and results about the cardinalities.

Proposition 1.1.2. *For any signature τ, $|L_\tau| = |\tau| + \aleph_0$*

Proof. Note that any formula in L_τ has finite length. The result then follows by induction on formula length. □

For example, the signature of the language of groups (and in fact of monoids) consists of the constant e and the binary operation $\cdot$. The language of partial orders consists only of the binary relation $\leq$, and note that the language of simple graphs also consists only of the binary relation E (denoting the existence of an edge). Since the actual choice of symbols is irrelevant to the language, thus we see that the language itself is insufficient to determining what kind of structures we are interested in. But before we can refine that, we need to first define what a structure for an arbitrary signature is:

Definition 1.1.3. *A **τ-structure** (or **L-structure**, or simply a **structure** when L and τ is clear from context) $\mathscr{M}$ consists of a nonempty set M and an **interpretation** satisfying:*

1. *For every constant symbol c in τ, there is a $c^{\mathscr{M}} \in M$*
2. *For every function symbol f in τ of arity n, there is a function $f^{\mathscr{M}} : M^n \longrightarrow M$*
3. *For every relation symbol P in τ of arity n, there is a $P^{\mathscr{M}} \subseteq M^n$*

*The **domain** of $\mathscr{M}$ is M.*
Let t be a term in τ, $\mathscr{M}$ a τ-structure and $\bar{a} \in M^\omega$. Then $t^{\mathscr{M}}(\bar{a})$ is defined inductively by:

1. *If t is a constant symbol c, then $t^{\mathscr{M}}(\bar{a})$ is $c^{\mathscr{M}}$*
2. *If t is the variable x_i for some $i \in \omega$, then $t^{\mathscr{M}}(\bar{a})$ is $\bar{a}(i)$*
3. *If t is of the form $f(t_0, \dots, t_{n-1})$ for terms $t_0, \dots, t_{n-1}$ and f a function symbol of arity n, then $t^{\mathscr{M}}(\bar{a})$ is $f^{\mathscr{M}}(t_0^{\mathscr{M}}(\bar{a}), \dots, t_{n-1}^{\mathscr{M}}(\bar{a}))$*

For $\bar{a} \in M^\omega$, $i \in \omega$ and $b \in M$, $\bar{a}[i \to b] = (a_0, \dots, a_{i-1}, b, a_{i+1}, \dots)$
For ϕ a formula in L and $\mathscr{M}$ an L-structure, we defined $\phi(\mathscr{M})$ inductively:

1. *If ϕ is of the form $(t = s)$ for terms t, s, then* $\phi(\mathscr{M}) = \{\bar{a} \in M^\omega : t^{\mathscr{M}}(\bar{a}) = s^{\mathscr{M}}(\bar{a})\}$

2. *If ϕ is of the form $(P(t_0, \ldots, t_{n-1}))$ for terms $t_0, \ldots, t_{n-1}$ and P a relation symbol of arity n, then* $\phi(\mathscr{M}) = \{\bar{a} \in M^\omega : (t_0^{\mathscr{M}}(\bar{a}), \ldots, t_{n-1}^{\mathscr{M}}(\bar{a})) \in P^{\mathscr{M}}\}$

3. *If ϕ is of the form $(\neg\psi)$ for a formula ψ, then* $\phi(\mathscr{M}) = M^\omega \backslash \psi(\mathscr{M})$

4. *If ϕ is of the form $(\varphi \wedge \psi)$ for formulas φ, ψ, then* $\phi(\mathscr{M}) = \varphi(\mathscr{M}) \cap \psi(\mathscr{M})$

5. *If ϕ is of the form $(\varphi \vee \psi)$ for formulas φ, ψ, then* $\phi(\mathscr{M}) = \varphi(\mathscr{M}) \cup \psi(\mathscr{M})$

6. *If ϕ is of the form $(\varphi \to \psi)$ for formulas φ, ψ, then* $\phi(\mathscr{M}) = (M^\omega \backslash \varphi(\mathscr{M})) \cup (\varphi(\mathscr{M}) \cap \psi(\mathscr{M}))$

7. *If ϕ is of the form $(\varphi \leftrightarrow \psi)$ for formulas φ, ψ, then* $\phi(\mathscr{M}) = (\varphi(\mathscr{M}) \cap \psi(\mathscr{M})) \cup ((M^\omega \backslash \varphi(\mathscr{M})) \cap (M^\omega \backslash \psi(\mathscr{M})))$

8. *If ϕ is of the form $(\forall v_i \psi)$ for a formula ψ,* $\phi(\mathscr{M}) = \{\bar{a} \in M^\omega : \text{ for every } b \in M, \bar{a}[i \to b] \in \psi(\mathscr{M})\}$

9. *If ϕ is of the form $(\exists v_i \psi)$ for a formula ψ,* $\phi(\mathscr{M}) = \{\bar{a} \in M^\omega : \text{ there is some } b \in M, \bar{a}[i \to b] \in \psi(\mathscr{M})\}$

*We say that $\mathscr{M}$ **satisfies** ϕ or $\mathscr{M}$ is a **model** of ϕ and write $\mathscr{M} \models \phi$ when $\phi(\mathscr{M}) = M^\omega$.*
*For $\Gamma \subseteq L$, $\mathscr{M} \models \Gamma$ if for every formula $\phi \in \Gamma$, $\mathscr{M} \models \phi$. We say Γ **implies** ϕ and also write $\Gamma \models \phi$ if for every L-structure $\mathscr{M}$ such that $\mathscr{M} \models \Gamma$, $\mathscr{M} \models \phi$.*
*We define the **L-theory of $\mathscr{M}$** by $Th_L(\mathscr{M}) = \{\phi \in L : \mathscr{M} \models \phi\}$.*

We will frequently make an abuse of notation and write M in place of $\mathscr{M}$ when there is no ambiguity in interpretation. For brevity, given a (possibly infinite) tuple $\bar{a}$ with elements from a set A we will also write as $\bar{a} \in A$.

Similarly, when the language in consideration is clear we will drop L from subscripts. This also applies to many definitions in the upcoming sections.

Note that it is clear from the definitions that for any formula ϕ with n free variables, $\phi(M)$ is actually well-defined as a subset of M^n, and we will identify $\phi(M)$ as such. In particular, if ϕ has only one free variable then we will regard $\phi(M)$ as a subset of M.

Lastly, it is an elementary fact of propositional logic that any formula using the logical connectives $\{\neg, \wedge, \vee, \to, \leftrightarrow\}$ is equivalent to a formula using only the connectives $\{\neg, \wedge\}$. Additionally:

Proposition 1.1.4. *For any formula ϕ with x a free variable in ϕ and structure M,*

- $M \models \forall x \phi$ *iff* $M \models \neg\exists x \neg\phi$
- $M \models \exists x \phi$ *iff* $M \models \neg\forall x \neg\phi$

- $M \models \phi$ *iff* $M \models \forall x \phi$

Thus when we need to proceed by induction over the structure of formulas, it is sufficient to consider only the formation clauses 1, 2, 3, either one of 4 or 5 and either one of 8 or 9 (as given above in Definition 1.1.1).

Often when discussing a model, we may wish to describe elements in the model even if the language does not explicitly allow us to do so: for example, the language of rings has constant symbols $0, 1$ and two binary functions $+, \cdot$, so that in $\mathbb{Q}$, every element of $\mathbb{Z}$ is the interpretation of a term ($n = \underbrace{1 + 1 + \cdots + 1}_{n}$) whereas any element of $\mathbb{Q} - \mathbb{Z}$ is not an interpretation of any term. However, would still like to make statements such as "$\exists x \frac{1}{2} \cdot x = 1$" about $\mathbb{Q}$. This motivates:

Definition 1.1.5. *If $\mathscr{M}$ is a τ-structure and $A \subseteq M$, we define $\tau_A = \tau \sqcup A$ with A as new constant symbols. We denote the language of τ_A by L_A, and by $\mathscr{M}_A$ we refer to the τ_A-structure formed by interpreting the symbols in A as themselves in M. We refer to the L_A-theory of $\mathscr{M}_A$ by $Th_A(\mathscr{M})$.*
*For any formula $\phi(x_0, \ldots, x_{n-1}) \in L$ and $\bar{a} \in M^n$, $\phi(\bar{a})$ is the formula in $L_{\bar{a}}$ formed by replacing all occurrences of x_i by $a(i)$ for every $0 \leq i < n$. If $\bar{a} \in A$, we say that $\phi(\bar{a})$ is a **formula in L with parameters in A**.*
We write $\mathscr{M} \models \phi(\bar{a})$ if $\mathscr{M}_{\bar{a}} \models \phi(\bar{a})$.

Proposition 1.1.6. *If τ^+ is a signature, $\tau^+ \supsetneq \tau$ and $\mathscr{M}^+$ is a τ^+-structure, then there is a τ-structure $\mathscr{M}$ with the same domain as $\mathscr{M}^+$ and such that:*

1. *For every constant symbol c in τ, $c^{\mathscr{M}} = c^{\mathscr{M}^+}$*
2. *For every function symbol f in τ, $f^{\mathscr{M}} = f^{\mathscr{M}^+}$*
3. *For every relation symbol P in τ, $P^{\mathscr{M}} = P^{\mathscr{M}^+}$*

Definition 1.1.7. *We call $\mathscr{M}$ from above the τ**-reduct** of $\mathscr{M}^+$ and denote such a reduct by $\mathscr{M}^+|_\tau$. If $L = L_\tau$, we may also call it a L**-reduct** and denote it by $\mathscr{M}^+|_L$.*
*Conversely, we call $\mathscr{M}^+$ from above the τ^+**-expansion** or, if $L^+ = L_{\tau^+}$, the L^+**-expansion** of $\mathscr{M}$.*

1.2 Elementary Substructures

One often talks about subgroups, subrings or sublattices. This naturally generalizes to any structure:

Proposition 1.2.1. *If $\mathscr{M}$ is a structure, $N \subseteq M$ and N satisfies:*

1. *For every constant symbol c, $c^{\mathscr{M}} \in N$*
2. *For every function symbol f of arity n and for every $\bar{a} \in N^n$, $f^{\mathscr{M}}(\bar{a}) \in N$*

Then there is a structure $\mathscr{N}$ with domain N such that:

1. *For every constant symbol c, $c^{\mathscr{N}} = c^{\mathscr{M}}$*
2. *For every function symbol f, $f^{\mathscr{N}} = f^{\mathscr{M}}|_N$*
3. *For every relation symbol P of arity n, $P^{\mathscr{N}} = P^{\mathscr{M}} \cap N^n$*

Definition 1.2.2. *For L-structures $\mathscr{M}, \mathscr{N}$, $\mathscr{N}$ is a* ***substructure*** *of $\mathscr{M}$ if it satisfies the conditions above.*
For a $A \subseteq M$, we denote the ***substructure generated by*** *A by $\langle A \rangle_L$, with*

$$\langle A \rangle_L = \bigcap \{N \subseteq M : A \subseteq N, N \text{ a substructure of } M\}$$

In model theory, the cardinality of a model is often an important property, so it will be useful to establish the cardinality of substructures:

Lemma 1.2.3. *For any τ-structure M and $A \subseteq M$, $|\langle A \rangle| = |A| + |\tau| + \aleph_0$*

Proof. We will show this by constructing $\langle A \rangle$ explicitly. Let C, F be the set of constant symbols and function symbols of τ respectively, and let $A_0 = A \cup \{c^M \in M : c \in C\}$. Then inductively, if A_i is defined, let

$$A_{i+1} = A_i \cup \{f^M(\bar{a}) \in M : f \in F, \bar{a} \in A_i\}$$

Then let $A_\omega = \bigcup_{i<\omega} A_i$.
Claim. $A_\omega = \langle A \rangle$

It is clear by induction that for any $N \subseteq M$ a substructure of M with $A \subseteq N$, for every A_i, $A_i \subseteq N$. Thus it suffices to show that A_ω is a substructure. But by constuction, A_ω satisfies the conditions of Proposition 1.2.1, and we are done.

Now, $|A_0| = |A| + |C|$, and for every $i < \omega$, by construction

$$|A_{i+1}| = \sum_{n<\omega} |A_i^n| \times |\{f \in F : f \text{ has arity } n\}|$$

Thus $|A_\omega| = |A| + |C| + |F| + \aleph_0 = |A| + |\tau| + \aleph_0$ ☐

Corollary 1.2.4. *For every $b \in \langle A \rangle$, there is some term $t(\bar{x})$ in L and some $\bar{a} \in A$ such that $\langle A \rangle \models b = t(\bar{a})$.*

Proof. True by induction due to the construction of A_ω above. ☐

In algebra, we often identify an algebraic structure $\mathscr{B}$ with an isomorphic copy which is a substructure of some other structure $\mathscr{A}$ and say that $\mathscr{B}$ is a substructure of $\mathscr{A}$: for example, we may consider $\mathbb{Z}$ as a subring of the ring of functions $\mathbb{R} \longrightarrow \mathbb{R}$. For an arbitrary signature, we can generalize this concept and justify it by the following:

Proposition 1.2.5. *Let M, N be L-structures, and suppose $h : M \longrightarrow N$ satisfies:*

1. *For every constant symbol c, $h(c^M) = c^N$*

2. *For every function symbol f of arity n and $\bar{m} \in M^n$, $h(f^M(\bar{m})) = f^N(h(m_0), \dots, h(m_{n-1}))$*

3. *For every relation symbol P of arity n and $\bar{m} \in M^n$, $\bar{m} \in P^M$ iff $(h(m_0), \dots, h(m_{n-1})) \in P^N$.*

Then $h(M)$ is a substructure of N with the following interpretations:

1. *For constant symbol c, $c^{h(M)} = h(c^M)$*

2. *For a function symbol f of arity n and $\bar{m} \in M^n$, $f^{h(M)}(h(\bar{m})) = h(f^M(\bar{m}))$*

3. *For a relation symbol P, $P^{h(M)} = h(P^M)$*

Proof. Note that $h(M)$ satisfies the conditions of Proposition 1.2.1, and additionally all the interpretations coincide with the ones given in 1.2.1. $\square$

Definition 1.2.6. *An injective $h : M \longrightarrow N$ satisfying the above conditions is called a L-**embedding**. By identifying M with the substructure which is the image of M under h, we also say that M is a substructure of N.*

Corollary 1.2.7. *If M is a substructure of N, then the inclusion map $i : M \longrightarrow N$ is an embedding.*

Proof. It is straightforward to check that the conditions of the above proposition are satisfied by i. $\square$

Proposition 1.2.8. *If $h : M \longrightarrow N$ is an embedding and $\phi(\bar{x}) \in L$ is quantifier-free, then for $\bar{m} \in M$, $M \models \phi(\bar{m})$ iff $N \models \phi(h(\bar{m}))$.*

Proof. We proceed by induction on $\phi \in L$ to show that $M \models \phi(\bar{a})$ iff $N \models \phi(\bar{a})$:

1. If $\phi(\bar{a})$ is $t(\bar{x}, \bar{a}) = s(\bar{x}, \bar{a})$ with all free variables in $\bar{x}$, then for every choice of $\bar{b}$ of the same length as $\bar{x}$ in N, $N \models t(\bar{b}, \bar{a}) = s(\bar{b}, \bar{a})$. Thus for every choice of $\bar{c}$ in M, since $\bar{c} \in N$, $M \models t(\bar{c}, \bar{a}) = s(\bar{c}, \bar{a})$. Conversely, if $N \not\models \phi(\bar{a})$, then $N \models \exists \bar{x} \neg t(\bar{x}, \bar{a}) = s(\bar{x}, \bar{a})$. By induction on the length of $\bar{x}$ and the assumption,there is a $\bar{b} \in M$ such that $M \models \neg t(\bar{x}, \bar{a}) = s(\bar{x}, \bar{a})$.

2. If $\phi(\bar{a})$ is $P(\bar{x}, \bar{a})$ with all free variables in $\bar{x}$ and $N \models \phi(\bar{a})$, then as above for every choice of $\bar{b}$ of the same length as $\bar{x}$ in N, $N \models P(\bar{b}, \bar{a})$. Thus for every choice of $\bar{c}$ in M, since $\bar{c} \in N$, $M \models P(\bar{c}, \bar{a})$. Conversely, if $N \not\models \phi(\bar{a})$, then $N \models \exists \bar{x} \neg P(\bar{x}, \bar{a})$. By induction on the length of $\bar{x}$ and the assumption,there is a $\bar{b} \in M$ such that $M \models \neg P(\bar{b}, \bar{a})$.

3. If ϕ is $\neg\psi$, ψ satisfying the induction hypothesis, then $N \models \phi(\bar{a})$ iff $N \not\models \psi(\bar{a})$ iff $M \not\models \psi(\bar{a})$ iff $M \models \phi(\bar{a})$.

4. If ϕ is $\varphi \wedge \psi$, φ, ψ both satisfying the induction hypothesis, then $N \models \phi(\bar{a})$ iff $N \models \varphi(\bar{a})$ and $N \models \psi(\bar{a})$ iff $M \models \varphi(\bar{a})$ and $M \models \psi(\bar{a})$ iff $M \models \phi(\bar{a})$.

□

Definition 1.2.9. *M, N are **isomorphic as L-structures** if there is a surjective embedding between them and we write $M \cong_L N$. Such an embedding is called an **L-isomorphism**.*

Proposition 1.2.10. *If M, N are isomorphic, then $Th(M) = Th(N)$.*

Proof. We can interpret N as a L_M-structure by $m^N = h(m)$, where $h : M \longrightarrow N$ is an isomorphism. Then proceed by induction on formula complexity for formulas in L. □

On the other hand, we should note that being a substructure does not preserve the L-theory of the structures: for example, $\mathbb{Z}$ is a subring of $\mathbb{Z}[\sqrt{2}]$, but $\mathbb{Z} \models \neg\exists x x \cdot x = 1 + 1$ while $\mathbb{Z}[\sqrt{2}] \models \exists x x \cdot x = 1 + 1$. So if we want to ensure that a substructure has the same L-theory as it's superstructure, we will need a stronger condition:

Definition 1.2.11. *If $h : M \longrightarrow N$ is an embedding such that for every $\bar{m} \in M$ and $\phi \in L$, $M \models \phi(\bar{m})$ iff $N \models \phi(h(\bar{m}))$, then we say h is an **elementary embedding**.*
*If $\mathscr{M}$ is a substructure of $\mathscr{N}$ and the inclusion may $i : M \longrightarrow N$ is an L-embedding, then $\mathscr{M}$ is an **elementary substructure** of $\mathscr{N}$. We also call $\mathscr{N}$ an **elementary extension** of $\mathscr{M}$. We write $\mathscr{M} \preceq \mathscr{N}$, or $\mathscr{M} \preceq_L \mathscr{N}$ when we wish to emphasize M is an elementary substructure as an L-structure.*
*If M, N are L-structures, we say that M and N are **elementarily equivalent** and write $M \equiv N$ if $Th(M) = Th(N)$.*

Proposition 1.2.12. *If $M \preceq N$, then $M \equiv N$*

Proof. Since the inclusion map preserves the truth value of each formula $\phi(\bar{m})$ with parameters in M, in particular it preserves all formulas in L without parameters. That is $Th(M) = Th(N)$. □

Here is a simple example of a proper elementary substructure: the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$ is a proper elementary substructure of $\mathbb{C}$. This can be proven by the method of quantifier elimination, which shows that the recursively axiomatizable theory of algebraically closed fields of characteristic 0 is a complete theory in the language of rings. This is however a special case, and in general we will use the following criteria for determining whether or not a substructure is an elementary substructure.

Theorem 1.2.13 (Tarski-Vaught Test). *For structures M, N with M a substructure of N, $M \preceq N$ iff for every $\phi(x, \bar{y}) \in L$ and $\bar{a} \in M$, if $N \models \exists x \phi(x, \bar{a})$ then there is a $b \in M$ such that $M \models \phi(b, \bar{a})$.*

Proof. The forward direction is straightforward, as $N \models \exists x \phi(x, \bar{a})$ iff $M \models \exists x \phi(x, \bar{a})$ since the inclusion map is an elementary embedding.

For the reverse direction, Corollary 1.2.7 shows that the inclusion map is an embedding. Now, additionally, for any $\bar{a} \in M$ and $\phi \in L$, $N \models \exists(x, \bar{a})$ iff $M \models \exists(x, \bar{a})$, so in fact the induction in Proposition 1.2.8 can be extended to include all formulas in L. This shows that the inclusion map is actually an elementary embedding. $\square$

The following lemma (actually the corollary which follows) that was introduced also by Tarski and Vaught is used in the construction of elementary extensions:

Lemma 1.2.14. *Suppose $(M_i : i \in \omega)$ is a sequence of structures such that for each $i \in \omega$, $M_i \preceq M_{i+1}$. Then $M = \bigcup_{i<\omega} M_i$ is a structure and $M_0 \preceq M$.*

Remark. Note that M has a natural interpretation as an L-structure:

- For constant symbol c, $c^M = c^{M_i}$
- For function symbol f, $f^M = \bigcup_{i<\omega} f^{M_i}$
- For relation symbol P, $P^M = \bigcup_{i<\omega} P^{M_i}$

Proof. It is clear by induction that for every $0 < i < \omega$, $M_0 \preceq M_i$. Now, for $\phi \in L$ and $\bar{a} \in M_0$, assume $M \models \exists x \phi(x, \bar{a})$. Then there is some $b \in M$ such that $M \models \phi(b, \bar{a})$. So there is some $i < \omega$ such that $b \in M_i$ i.e. $M_i \models \exists x \phi(x, \bar{a})$. But $M_0 \preceq M_i$, and so there is some $c \in M_0$ such that $M_0 \models \phi(c, \bar{a})$. Then by the Tarski-Vaught test, $M_0 \preceq M$. $\square$

Corollary 1.2.15. *Suppose α is an limit ordinal and $(M_i : i < \alpha)$ is a sequence of structures such that for each $i < j$, $M_i \preceq M_j$. Then for each $i < \alpha$, $M_i \preceq M_\alpha = \bigcup_{i<\alpha} M_i$.*

Proof. The interpretation of M_α is as in the lemma above with α replacing ω. The proof the Tarski-Vaught test is also as above. $\square$

Definition 1.2.16. *A chain $(M_i : i < \alpha)$ of structures such that*

- *For each successor $\beta + 1$, $M_\beta \preceq M_{\beta+1}$*
- *For each limit δ, $M_\delta = \bigcup_{i<\delta} M_i$*

*is called an **elementary chain**.*

As in the case of substructures, we often wish to identify a structure with an elementary substructure of another structure. Again, this can be easily justified:

Proposition 1.2.17. *If $h : M \longrightarrow N$ is an embedding such that for every $\bar{m} \in M$ and $\phi \in L$, $M \models \phi(\bar{m})$ iff $N \models \phi(h(\bar{m}))$, then $h(M) \preceq N$.*

Proof. Note that $M \cong h(M)$. By the assumption, for $\phi(x, \bar{y}) \in L$ and $\bar{m} \in M$, $N \models \exists x \phi(x, h(\bar{m}))$ iff $M \models \exists x \phi(x, \bar{m})$ iff there is a $b \in M$ such that $M \models \phi(b, \bar{m})$ iff $h(M) \models \phi(h(b), h(\bar{m}))$. Thus $h(M) \preceq N$ by the Tarski-Vaught test. $\square$

In light of this proposition, we extend our notation slightly:

Definition 1.2.18. *If an elementary embedding from M to N exists, then we say M is **elementarily embeddable** in N and also write $M \preceq N$*

1.3 Skolem Functions

While the Tarski-Vaught test allows us to identify whether or not a substructure is an elementary substructure, it does not help us in constructing one. For this task, we need the following process called Skolemization.

Definition 1.3.1. *We say that a $T \subseteq L$ **has Skolem functions** if for every $\phi(\bar{x}, y) \in L$ with $\bar{x}$ nonempty, there is a function symbol f such that $\forall\bar{x}(\exists y\phi(\bar{x}, y) \to \phi(\bar{x}, f(\bar{x}))) \in T$.*

Lemma 1.3.2. *If $T \subseteq L_\tau$, then there is a $\tau' \supseteq \tau$ with a $T' \subseteq L_{\tau'}$, $T' \supseteq T$ such that $|\tau'| = |\tau| + \aleph_0$ and T' has Skolem functions.*

Proof. For every $\phi(\bar{x}, y) \in L_\tau$, let f_ϕ be a new function symbol of arity $|\bar{x}|$ and define $\tau_1 = \tau \cup \{f_\phi : \phi(\bar{x}, y) \in L\}$. Note then that $|\tau_1| = |\tau| + |L_\tau| = |\tau| + \aleph_0$ by Proposition 1.1.2. Then let

$$T_1 = T \cup \{\forall\bar{x}(\exists y\phi(\bar{x}, y) \to \phi(\bar{x}, f_\phi(\bar{x}))) : \phi(\bar{x}, y) \in L_\tau\}$$

We can repeat the construction to get $(\tau_i : i < \omega)$ and $(T_i : i < \omega)$ with $\tau_0 = \tau$ and $T_0 = T$. Thus let $\tau' = \bigcup_{i<\omega} \tau_i$ and $T' = \bigcup_{i<\omega} T_i$. Since any $\phi \in L_{\tau'}$ is contained in some L_{τ_n}, the corresponding Skolem function is included in T_{n+1}. The cardinality requirement is satisfied by induction. $\square$

Definition 1.3.3. *For a $T \subseteq L$, we call T' above the **Skolemization** of T and denote it by T_{Sk}. The language of T_{Sk} is denoted by L_{Sk}.*

Lemma 1.3.4. *If $\mathscr{M} \models T$, then there is an L_{Sk}-expansion $\mathscr{M}_{Sk} \models T_{Sk}$.*

Proof. For a $\phi(\bar{x}, y) \in L$ with f_ϕ the corresponding Skolem function, we define $f_\phi^{\mathscr{M}_{Sk}}$: for $\bar{m} \in M$, if $\mathscr{M} \models \exists y\phi(\bar{m}, y)$, then let $f_\phi^{\mathscr{M}_{Sk}}(\bar{m}) \in \phi(\bar{m}, M)$. Otherwise choose $f_\phi^{\mathscr{M}_{Sk}}(\bar{m})$ arbitrarily in M. Thus $\mathscr{M}_{Sk} \models T_{Sk}$. $\square$

The idea of Skolemization is that we ensure that every substructure in the Skolemized language is in fact an elementary substructure. This is shown in the following:

Definition 1.3.5. *If $M \models T$ and $A \subseteq M$, then the **Skolem Hull** of A is the substructure generated by A under the language L_{Sk}. We denote it by $\langle A \rangle_{Sk}$.*

Theorem 1.3.6 (Downward Löwenheim-Skolem Theorem). *Suppose $T \subseteq L$, $M \models T$ with $|M| \geq |L|$, and $A \subseteq M$. Then there is a $B \supseteq A$ such that $B \preceq M$ and $|B| = |A| + |L|$*

Proof. We will show that the Skolem hull $\langle A \rangle_{Sk}$ is the desired B.

Claim. $\langle A \rangle_{Sk} \preceq_L M$

Suppose $\phi(x, \bar{y}) \in L$, $\bar{a} \in \langle A \rangle_{Sk}$ and $M \models \exists x \phi(x, \bar{a})$. Then let f_ϕ be the corresponding Skolem function in L_{Sk}, and thus $M_{Sk} \models \phi(f_\phi(\bar{a}), \bar{a})$. But by definition $f_\phi^{M_{Sk}}(\bar{a}) \in \langle A \rangle_{Sk}$, and so by the Tarski-Vaught test $\langle A \rangle_{Sk} \preceq_{L_{Sk}} M$. Thus in particular $\langle A \rangle_{Sk} \preceq_L M$.

By Lemma 1.3.2 and Lemma 1.1.2, $|L_{Sk}| = |L|$. Therefore by Lemma 1.2.3, $|\langle A \rangle_{Sk}| = |A| + |L|$ ☐

In addition to constructing elementary substructures, another useful property of Skolemization comes from the following property:

Proposition 1.3.7. *Suppose $T \subseteq L$ has Skolem functions. Then for every formula $\phi \in L$, there is a quantifier-free $\psi \in L$ such that $T \models \phi \leftrightarrow \psi$.*

Proof. By induction, it suffices to show that if $\phi(\bar{y}, x)$ is such that there exists a quantifier-free $\psi(\bar{y}, x) \in L$ with $T \models \phi \leftrightarrow \psi$, then $\exists x \phi$ also satisfies the condition. But since T has Skolem functions, there is some function symbol such that $T \models \exists x \phi(\bar{y}, x) \leftrightarrow \phi(\bar{y}, f(\bar{y})) \leftrightarrow \psi(\bar{y}, f(\bar{y}))$. Then $\psi(\bar{y}, f(\bar{y}))$ is the desired quantifier-free formula. ☐

1.4 The Compactness Theorem

Compactness is one of the most important tools in classical model theory, guaranteeing that models of a theory actually exists. We will divide up the proof using the next few lemmas.

Definition 1.4.1. *Let $T \subseteq L$.*

- *T is **satisfiable** if there is an $M \models T$. We call a satisfiable set of formulas a **theory**.*
- *T is **finitely satisfiable** if for every finite subset $\Delta \subseteq T$, Δ is satisfiable*
- *T has the **witness property** if for every $\phi(x) \in L$ with one free variable, there is a constant symbol c such that $T \models \phi(c) \leftrightarrow \exists x \phi(x)$*
- *T is **complete over** L (or simply **complete** when L is clear from context) if for every $\phi \in L$, either $\phi \in T$ or $\neg\phi \in T$.*

Lemma 1.4.2. *Suppose $T \in L_\tau$ is finitely satisfiable. Then there is a $\tau^* \supseteq \tau$ and a $T^* \subseteq L_{\tau^*}$ such that:*

1. *$T \subseteq T^*$*
2. *T^* is finitely satisfiable*
3. *$|\tau^*| = |\tau| + \aleph_0$*

4. If Γ is such that $T^* \subseteq \Gamma \subseteq L_{\tau^*}$, then Γ has the witness property

Proof. For every $\phi(x) \in L_\tau$ in one free variable, let c_ϕ be a new constant symbol and define $\tau_1 = \tau \sqcup \{c_\phi : \phi \in L_\tau \text{ has one free variable}\}$. Then define $\theta_\phi \in L_{\tau_1}$ to be $(\phi(c_\phi) \leftrightarrow \exists x \phi(x))$, and let

$$T_1 = T \cup \{\theta_\phi \in L_{\tau_1} : \phi \in L_\tau \text{ has one free variable}\}$$

Claim. T_1 is finitely satisfiable.

For any finite $\Delta_1 \subseteq T_1$, if $\Delta_1 = \Delta \cup \{\theta_{\phi_1}, \ldots, \theta_{\phi_n}\}$, $\Delta \subseteq T$, then as T is finitely satisfiable there is an $M \models \Delta$. We will define, for $1 \leq i \leq n$, $c_{\phi_i}^M$: if $M \models \exists v_0 \phi_i(v_0)$, then interpret $c_{\phi_i}^M = b \in \phi_i(M)$. Else if $M \not\models \exists v_0 \phi_i(v_0)$, then interpret arbitrarily $c_{\phi_i}^M = b \in M = M \backslash \phi_i(M)$. Thus $M \models \Delta_1$.

Repeat the above construction to form $(\tau_i : i < \omega)$ and $(T_i : i < \omega)$ (with $\tau_0 = \tau$, $T_0 = T$), and define $\tau^* = \bigcup_{i<\omega} \tau_i$, $T^* = \bigcup_{i<\omega} T_i$. Since every finite $\Delta \subseteq T^*$ is contained in some T_n which is finitely satisfiable by induction, T^* is finitely satisfiable. Further, note that at every stage, $|\tau_{i+1}| = |\tau_i| + |L_{\tau_i}| = |\tau_i| + \aleph_0$ (by Proposition 1.1.2). Therefore $|\tau^*| = |\tau| + \aleph_0$.

Finally, let $\phi \in L_{\tau^*}$ be a formula in one variable. Since ϕ is finite, there is some $i < \omega$ such that $\phi \in L_{\tau_i}$. Then there is a c_ϕ in τ_{i+1} such that $(\phi(c_\phi) \leftrightarrow \exists x \phi(x)) \in T_{i+1} \subseteq T^*$. Thus T^* satisfies the witness property, and further any extension of T^* over τ^* must also satisfy the witness property. $\square$

Lemma 1.4.3. *If T is finitely satisfiable and ϕ is a formula, then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable.*

Proof. Let $\Delta \subseteq T$ be finite, and satisfied by M. Then either $M \models \phi$ or $M \models \neg\phi$. $\square$

Lemma 1.4.4. *If $T \subseteq L$ is finitely satisfiable then there is a finitely satisfiable $T' \supseteq T$ which is complete over L.*

Proof. Consider $\{\Gamma \subseteq L : \Gamma \supseteq T, \Gamma \text{ finitely satisfiable}\}$ under ordering by inclusion. If $\mathcal{C}$ is a chain, then $\bigcup \mathcal{C}$ is finitely satisfiable (as any finite subset is contained in some $\Gamma \in \mathcal{C}$) and contains T. Thus Zorn's lemma applies and there is some maximal element T'.

We claim that T' is complete. For if not then there is a ϕ such that $\phi, \neg\phi \notin T'$. But then by the above lemma T' is not maximal, a contradiction. $\square$

Lemma 1.4.5. *If T is complete and finitely satisfiable, $\Delta \subseteq T$ is finite and $\Delta \models \phi$, then $\phi \in T$.*

Proof. If $\phi \notin T$, then as T is complete $\neg\phi \in T$ and thus $\Delta \cup \{\neg\phi\}$ is finitely satisfiable, contradicting $\Delta \models \phi$. $\square$

Lemma 1.4.6. *If T is finitely satisfiable, complete and has the witness property, then T is satisfiable.*

Proof. Let C be the constants in the language L. Let $\sim$ be the equivalence relation on C defined by $c \sim d$ iff $(c = d) \in T$. We note:

Claim. For a function symbol f of arity n and $\bar{c} \in C^n$, there is a $d \in C$ such that $(f(\bar{c}) = d) \in T$.

Since $T \models \exists x(f(\bar{c}) = x)$ and T has the witness property, there is a d as required.

Claim. For a relation symbol P of arity n and $\bar{c}, \bar{d} \in C^n$ such that $(c_i = d_i) \in T$ for every $0 \leq i < n$, $R(\bar{c}) \in T$ iff $R(\bar{d}) \in T$.

Let $\Delta_0 = \{R(\bar{c}), c_0 = d_0\}$, which is finite and thus satisfiable. But $\Delta_0 \models R(d_0, c_1, \ldots, c_{n-1})$, and so by the above lemma $R(d_0, c_1, \ldots, c_{n-1}) \in T$. Then by induction $R(\bar{d}) \in T$.

Let M be the equivalence classes of C, and we interpret M as an L-structure by the following:

- For a constant symbol c, let $c^M = c_\sim$.
- For a function symbol f, define f^M by $f^M(\bar{c}_\sim) = d_\sim$ where $(f(\bar{c}) = d) \in T$.
- For a relation symbol P of arity n, let $P^M = \{\bar{c}_\sim \in M^n : P(\bar{c}) \in T\}$.

These are all well-defined by the above claims.

Claim. If t is a term with variables in $v_0, \ldots, v_{n-1}$, $c_0, \ldots, c_{n-1}, d \in C$, then $(t(c_0, \ldots, c_{n-1}) = d) \in T$ iff $t^M(c_{0\sim}, \ldots, c_{(n-1)\sim}) = d_\sim$.

We proceed by induction on term complexity:

1. If t is a constant c, then $(c = d) \in T$ iff $c_\sim = d_\sim$ by definition of $\sim$
2. If t is the variable v_i, then $(c_i = d) \in T$ iff $t^M(c_{0\sim}, \ldots, c_{(n-1)\sim}) = c_{i\sim} = d_\sim$
3. If t is of the form $f(t_0, \ldots, t_m)$, and the induction hypothesis holds for each term in $t_0, \ldots, t_m$, then by the witness property of T since $T \models \exists y t_i(c_0, \ldots, c_{n-1}) = y$, there are constants $d_0, \ldots, d_m$ such that $(t_i(c_0, \ldots, c_{n-1}) = d_i) \in T$. But note that

$$f(\bar{t}(c_0, \ldots, c_{n-1})) = d, t_0(c_0, \ldots, c_{n-1}) = d_0, \ldots, t_m(c_0, \ldots, c_{n-1}) = d_m \models f(d_0, \ldots, d_m) = d$$

Thus by the completeness of T, $f(d_0, \ldots, d_m) \in T$. By the induction hypothesis, $t_i^M(c_{0\sim}, \ldots, c_{(n-1)\sim}) = d_{i\sim}$ and by definition $f^M(d_{0\sim}, \ldots, d_{m\sim}) = d_\sim$. Therefore

$$t^M(c_0, \ldots, c_{n-1}) = f^M(\bar{t}^M(c_{0\sim}, \ldots, c_{(n-1)\sim})) = d_\sim$$

Conversely, if $t^M(c_{0\sim}, \ldots, c_{(n-1)\sim}) = d_\sim$, then by completeness of T, $(\exists x t(c_0, \ldots, c_{n-1}) = x) \in T$ and so by the witness property, there is a $e \in C$ such that $(t(c_0, \ldots, c_{n-1}) = e) \in T$. But as shown above this implies that $t^M(c_{0\sim}, \ldots, c_{(n-1)\sim}) = e_\sim$, and so $e_\sim = d_\sim$. Thus by definition of $\sim$ $(e = d) \in T$, and again by completeness $(t_i(c_0, \ldots, c_{n-1}) = d_i) \in T$.

By a further induction on the second term, we see that for any terms t, s and $\bar{c} \in C^\omega$, $(t(\bar{c}) = s(\bar{c})) \in T$ iff $t^M(\bar{c}_\sim) = s^M(\bar{c}_\sim)$

Claim. For any closed formula ϕ, $M \models \phi$ iff $\phi \in T$.

1. If ϕ is $t = s$ for closed terms, then $M \models \phi$ iff $M \models t = s$ iff $t^M = s^M$ iff $(t = s) \in T$ by the above observation.

2. If ϕ is $P(\bar{t})$ with P a relation symbol and $\bar{t}$ a sequence of closed terms, then $M \models \phi$ iff $\bar{t}^M \in P^M$. Now for any $\bar{c} \in C$, since $(\exists x t(\bar{c}) = x) \in T$ as T is complete, by the witness property there are constants $\bar{e} \in C$ such that $(\bar{e} = \bar{t}) \in T$. Thus $\bar{t}^M \in P^M$ iff $\bar{e}^M \in P^M$, which is true iff $(P(\bar{e})) \in T$. Finally, the completeness of T ensures that this is equivalent to $(P(\bar{t})) \in T$.

3. If ϕ is $\neg\psi$, then $M \models \phi$ iff $M \not\models \psi$ iff $\psi \notin T$. As T is complete, this is true iff $\phi \in T$.

4. If ϕ is $\varphi \wedge \psi$, then $M \models \phi$ iff $M \models \varphi$ and $M \models \psi$ iff $\varphi \in T$ and $\psi \in T$. By completeness, this is equivalent to $\phi \in T$.

5. If ϕ is $\exists x \psi(x)$, then $M \models \phi$ iff there is a $c_\sim \in M$ such that $M \models \psi(c_\sim)$. But as $M \models c = c_\sim$, $M \models \psi(c_\sim)$ iff $M \models \psi(c)$ iff $\psi(c) \in T$. This implies $\phi \in T$. Conversely, if $\phi \in T$, then by the witness property there is a constant d such that $\psi(d) \in T$. Thus $M \models \psi(d)$, and so $M \models \phi$.

Therefore $M \models T$ i.e. T is satisfiable. □

Theorem 1.4.7 (Compactness Theorem). *For $T \subseteq L$, T is satisfiable iff T is finitely satisfiable.*

Proof. The forward direction is trivial. For the reverse direction, by Lemma 1.4.2 there is an expansion in the language into $L^* \supseteq L$ and a $T^* \subseteq L^*$, $T^* \supseteq T$ such that T^* is finitely satisfiable and any expansion of T^* in L^* has the witness property. Then by Lemma 1.4.4 there is a finitely satisfiable T' extending T^* which is complete over L^*, and has the witness property. Finally, by Lemma 1.4.6 there is a $M \models T'$, and therefore $M|_L \models T$. □

As mentioned before, the cardinality of models is an important property. One of the most important result of the Compactness theorem is that there are arbitrarily large models.

Definition 1.4.8. *The **elementary diagram** of a L-structure M is defined to be $Th_M(M) = \{\phi \in L_M : M \models \phi\}$.*

Lemma 1.4.9. *If N is an L-structure and there is an L_M interpretation of N such that $N \models Th_M(M)$, then $M \preceq_L N$.*

Proof. Consider the map $f : M \longrightarrow N$ with $f(m) = M^N$. Since for any $m_0, m_1 \in M$, if $m_0 \neq m_1$ then $M \models m_0 \neq m_1$ (with $m_0 \neq m_1$ a formula in L_M), thus $N \models m_0 \neq m_1$. Therefore f is injective, and as $N \models Th_M(M)$,

f is in fact an elementary embedding. Further, since $Th_L(M) = Th_M(M)|_L$, therefore $Th_L(M) \subseteq Th_L(N)$. Since $Th_L(M)$ is complete, therefore $Th_L(N) = Th_L(M)$. $\square$

Theorem 1.4.10 (Upward Löwenheim-Skolem Theorem). *Suppose M is an infinite L-structure, κ an infinite cardinal with $\kappa \geq |M| + |L|$. Then there is a L-structure N such that $|N| = \kappa$ and $M \preceq N$.*

Proof. Let $\tau' = \tau \sqcup \{c_i : i < \kappa\} \sqcup M$ with the new symbols as constant symbols, and consider the $T = Th_M(M) \cup \{c_i \neq c_j : i \neq j, i, j < \kappa\}$. Note that T is finitely satisfiable: given a finite $\Delta \subseteq \{c_i \neq c_j : i \neq j, i, j < \kappa\}$, for every distinct c_i which occurs in Δ, interpret c_i as a distinct member of M. Then $M \models Th_M(M) \cup \Delta$. Therefore by compactness T is satisfiable, say by the model N'. Further, by the above lemma $M \preceq N'$. Finally, by the Downward Löwenheim-Skolem theorem (Theorem 1.3.6), $\langle M \cup \{c_i : i < \kappa\}\rangle_{Sk}$ is the desired N. $\square$

1.5 Types and Saturation

When we wish to identify a particular element of a model or describe some particular property, sometimes a single formula is insufficient. For example, consider $\mathbb{C}$ as a field: every element α of $\bar{\mathbb{Q}}$ can be identified as the solution to it's minimal polynomial $p(x)$ (not necessarily uniquely, but there are at most finitely many solutions). On the other hand, there is no single formula which determines whether or not an element of $\mathbb{C}$ is transcendental. This motivates the following definition:

Definition 1.5.1. *Let M be a L-structure, $A \subseteq M$, and $\Delta \subseteq L$. We define $\Delta_A = \{\phi(\bar{a}) \in L_A : \phi \in \Delta, \bar{a} \in A\}$.*
*For $n < \omega$, a **Δ-n-type of** M **over** A is a set of formulas with n free variables such that:*

- *For each $\phi \in p$, either $\phi \in \Delta_A$ or $\neg\phi \in \Delta_A$, where we identify $\neg\neg\phi$ with ϕ.*
- *$Th_M(M) \cup \{\phi(x_0, \ldots, x_{n-1}) : \phi \in p\}$, where $x_0, \ldots, x_{n-1}$ are new constant symbols, is satisfiable.*

*A Δ-n-type p is **complete** if it is maximal i.e. for every $\phi \in \Delta_A$ with n free variables, either $\phi \in p$ or $\neg\phi \in p$.*
We define $S^M_{\Delta,n}(A) = \{p : p$ is a complete Δ-n-type of M over $A\}$.
*For a $\bar{b} \in M$, the **Δ-type of** $\bar{b}$ **over** A **in** M is defined by $tp^M_\Delta(\bar{b}/A) = \{\phi : M \models \phi(\bar{b}), \phi \in \Delta_A$ or $\neg\phi \in \Delta_A\}$.*
*For any Δ-n-type p, $p(M) = \{\bar{b} \in M^n :$ For every $\phi \in p$, $M \models \phi(\bar{b})\}$. We say that M **realizes** p if $p(M)$ is nonempty i.e. there is some $\bar{m} \in M^n$ such that $M \models p(\bar{m})$. If M does not realize p, then we say that M **omits** p.*
For the above terms, if $\Delta = L$ we will drop Δ from the description e.g. n-types,

$S^M_n(A)$. *Similarly, if* $\Delta = \{\phi\}$*, then we write* ϕ *instead of* $\{\phi\}$ *e.g.*ϕ*-*n*-types,* $S^M_{\phi,n}(A)$*. We may drop the* M *in supscripts if it is clear from context.*

Proposition 1.5.2. *Suppose* p *is a* n*-type of* M *over* A*,* $A \subseteq M$*. Then there is a* $N \succeq M$ *and a* $\bar{n} \in N$ *such that* $N \models p(\bar{n})$*. Conversely, if* $M \preceq N$*,* $A \subseteq M$ *and* $\bar{n} \in N$*, then* $tp^N(\bar{n}/A)$ *is a type of* M *over* A*.*

Proof. By definition of a type, since $Th_M(M) \cup p(\bar{x})$ is satisfiable (where $\bar{x}$ are n new constant symbols) there is a N which models it. Thus $\bar{x}^N$ is the desired $\bar{n}$. Further, by Lemma 1.4.9, $M \preceq N$ as $N \models Th_M(M)$.

The converse is true by definition as N and $\bar{x}^N = \bar{n}$ satisfies $Th_M(M) \cup tp^N(\bar{n}/A)$. □

Using the idea of types, we can generalize the idea of elementary embeddings and relax the requirement that the domain of the function is the entire model.

Definition 1.5.3. *A* $f : A \longrightarrow N$ *is an* ***elementary map*** *if for every* $\bar{a} \in A$*,* $tp^M(\bar{a}/\emptyset) = tp^N(\overline{f(a)}/\emptyset)$*.*

Proposition 1.5.4. $f : M \longrightarrow N$ *is an elementary map iff it is an elementary embedding.*

Proof. f is an elementary map iff $tp^M(\bar{m}/\emptyset) = tp^N(\overline{h(m)}/\emptyset)$ for every $\bar{m} \in M$ iff for every $\bar{m} \in M$ and $\phi \in L$, $M \models \phi(\bar{m}) \Leftrightarrow N \models \phi(h(\bar{m}))$ iff f is an elementary embedding. □

Note that the type $tp^M(\bar{a}/\emptyset)$ is simply a set of formulas in L. In this case, we define:

Definition 1.5.5. *Let* T *be a theory. A* **Δ-*n*-*type of*** T *is a set of formulas* p *with* n *free variables such that:*

- *For each* $\phi \in p$*, either* $\phi \in \Delta$ *or* $\neg\phi \in \Delta$*, where we identify* $\neg\neg\phi$ *with* ϕ*.*
- $T \cup \{\phi(x_0, \dots, x_{n-1}) : \phi \in p\}$*, where* $x_0, \dots, x_{n-1}$ *are new constant symbols, is satisfiable.*

Proposition 1.5.6. *Let* T *be a complete theory, and suppose* $M \models T$*. If* p *is a type of* T*, then* p *is a type of* M *over* $\emptyset$*. Conversely, for any* $\bar{m} \in M$*,* $tp^M(\bar{m}/\emptyset)$ *is a type of* T*.*

Proof. The reverse direction is clear by definition. For the forward direction, let $\Delta_0 \subseteq p$ be finite, and define $\psi = \exists x_0, \dots, x_{n-1} \bigwedge_{\phi \in \Delta_0} \phi(x_0, \dots, x_{n-1})$. Since $T \cup p(\bar{x})$ is satisfiable, $T \models \psi$. Then $M \models T$ and $M \models \psi$ i.e. $Th_M(M) \cup \Delta_0(\bar{x})$ is satisfiable. By compactness, $Th_M(M) \cup p(\bar{x})$ is satisfiable, and so p is a type of M. □

The idea of types is an essential tool to many parts of model theory, and in particular to this exposition. For now, let us define some properties which uses types to differentiate between elementarily equivalent models.

Definition 1.5.7. *Let M be a L-structure, κ an infinite cardinal with $\kappa \leq |M|$. M is κ-**saturated** if for every $A \subseteq M$ with $|A| < \kappa$ and $p \in S_1(A)$, M realizes p.*
*M is κ-**universal** if for every L-structure N with $|N| < \kappa$ such that $N \equiv M$, N is elementarily embeddable in M.*
*If M is $|M|$-saturated or $|M|^+$-universal, then we simply say M is **saturated** or **universal**, respectively.*

To see the difference between saturated and unsaturated models, consider $\bar{\mathbb{Q}}$, the algebraic closure of Q in $\mathbb{C}$, and F, an algebraically closed field of characteristic 0 with transcendence degree $\aleph_0$. Again, since the theory of algebraically closed fields of characteristic 0 is a complete theory, $\bar{\mathbb{Q}} \equiv F$. However, consider the 1-type $\{p(x) \neq 0 : p \in \mathbb{Q}[x]\}$: this is a type omitted by $\bar{\mathbb{Q}}$ but realized by any transcendental element of F. Moreover, F is in fact a saturated model: since F has transcendence degree $\aleph_0$, $|F| = \aleph_0$. Now, for any set $A \subsetneq F$ with $|A| < \aleph_0$, the only 1 types over A are either the type which says that x is a solution to $p(x) \in F(A)[x]$, or the type which says that x is transcendental over $F(A)$. That types of the former kind are realized by F is by virtue of F being algebraically closed, and as A is finite the transcendental degree of F guarantees that the nonalgebraic type is also realized in F.

As in previous cases, this example depends essentially on properties of fields, and in general there may not be saturated models of a theory. However, we can get the following:

Proposition 1.5.8. *Let T be a complete theory, κ an infinite cardinal with $\kappa \geq |T|$. Then there is a $M \models T$ such that $|M| = 2^\kappa$ and M is κ^+-saturated.*

Remark. We recall that for any infinite cardinal κ, κ^+ is a regular cardinal i.e. if $(\alpha_i : i < \kappa)$ is any sequence of ordinals with each $\alpha_i < \kappa^+$, then there is a $\beta < \kappa^+$ such that for every $i < \kappa$, $\alpha_i < \kappa$.

Proof. By the Upward Löwenheim-Skolem theorem, there is a model $M \models T$ with $|M| = 2^\kappa$.

Claim. If $M \models T$ and $|M| = 2^\kappa$, then there is a $N \succeq M$ such that for every $A \subseteq M$ with $|A| = \kappa$ and $p \in S_1(A)$, N realizes p.

Since $|A| + |T| = \kappa$, $|L_A| = \kappa$ and so $|S_1(A) \leq 2^\kappa|$. Now, $|\{A \subseteq M : |A| = \kappa\}| = |M|^\kappa = (2^\kappa)^\kappa = 2^\kappa$, so $P = \bigcup\{S_1(A) : A \subseteq M, |A| = \kappa\}$, then $|P| = 2^\kappa$. Let $(p_i : i < 2^\kappa)$ enumerate P, and let $\{c_i : i < 2^\kappa\}$ be new constant symbols. Since $T \cup \bigcup\{p_i(c_i) : i < 2^\kappa\}$ is satisfiable by compactness, by the Löwenheim-Skolem theorems there is a model N of cardinality 2^κ which satisfies this theory. The reduction of N to the original language gives the desired model.

We construct the desired model by induction on κ^+. Let $M_0 = M$, and for $i < \kappa^+$, if M_i is defined let M_{i+1} be a model of size 2^κ which realizes every 1-type over a $A \subseteq M_i$ with $|A| = \kappa$, as constructed in the claim above. For a limit δ, let $M_\delta = \bigcup_{i<\delta} M_i$, which by Corollary 1.2.15 is an elementary extension of M_i for $i < \delta$. Therefore $(M_i : i < \kappa^+)$ is an elementary chain and so we let $N = M_{\kappa^+} = \bigcup_{i<\kappa^+} M_i$.

We claim that N is κ^+-saturated: for any $A \subseteq N$ with $|A| < \kappa^+$, since κ^+ is regular there must be some $\alpha < \kappa^+$ such that $A \subseteq M_\alpha$. Then any 1-type over A is realized in $M_{\alpha+1}$, and thus in N. $\square$

Proposition 1.5.9. *If M is κ-saturated, then it is κ^+-universal.*

Proof. Suppose $N \equiv M$ and $|N| \leq \kappa$. List the elements of N by $\bar{n} = (n_i : i < \kappa)$, possibly with repetition if $|N| < \kappa$. Let $\bar{c} = (c_i : i < \kappa)$ be new constant symbols, with $c_i^N = n_i$. Then let L_i be the language with the new constants $\{c_j : j < i\}$ and let $T_i = Th_{L_i}(N)$, so that $L_0 = L$ and $T_0 = Th_L(N) = Th_L(M)$.

Claim. If there is a L_i interpretation of M such that $Th_{L_i}(M) = T_i$, then there is an interpretation of c_{i+1}^M such that $Th_{L_{i+1}}(M) = T_{i+1}$.

Let $A_i = \text{dom } \bar{n}|_i$, and $B_i = \{c_j^M \in M : j < i\}$. Consider the 1-type $p_i = tp^N(n_i/A_i)$ as a type over A_i in the language L. For every $\phi \in p_i$, let ϕ' be the L_i formula obtained from ϕ by replacing every occurrence of n_j by c_j, and let $p_i' = \{\phi' \in L_i : \phi \in p\}$. Thus p_i' is a 1-type over $\emptyset$ in N, and by Proposition 1.5.6 it is a 1-type of T_i. By assumption $M \models T_i$, and so again by 1.5.6 p_i' is a 1-type of M.

Now, again for every $\phi' \in p_i'$ let ϕ^* be the formula in L_{B_i} formed by replacing every occurrence of c_j by c_j^M, and let $p_i^* = \{\phi^* \in L_{B_i} : \phi' \in p_i'\}$. Since p_i' is a 1-type in M, p_i^* is a 1-type of M over B_i in the language L. But $|B_i| < \kappa$, and so by the κ-saturation of M, p_i^* is realized by some $m \in M$. Define $c_i^M = m$; that $Th_{L_{i+1}}(M) = T_{i+1}$ is thus true by construction.

Claim. If δ is a limit ordinal $< \kappa$ and there is an interpretation of $\bar{c}|_\delta$ in M such that for each $i < \delta$, $T_i = Th_{L_i}(M)$, then $Th_{L_\delta}(M) = T_\delta$.

This is true by virtue of the fact that any formula is finite, and is thus contained in T_i for some $i < \delta$.

The above claims gives an interpretation of $\bar{c}$ such that $Th_N(N) = Th_{L_\kappa}(N)$ and $M \models Th_{L_\kappa}(N)$, so $N \preceq M$. $\square$

An important consequence of this result is that it allows us to work within what are called **universal models**: Consider a complete theory T and suppose that all the sets which we are interested in are of cardinality $< \bar{\kappa}$. Using Proposition 1.5.8 we can constructed a model $\bar{M}$ of size $2^{\bar{\kappa}}$ which is $\bar{\kappa}^+$-saturated, and thus by Proposition 1.5.9 is $\bar{\kappa}^{++}$-universal. Thus every model of T which we are interested in is elementary embeddable in $\bar{M}$, and therefore we need only to consider elementary submodels of $\bar{M}$. In this case, for this exposition we will loosely refer to submodels and subsets of $\bar{M}$ as being "small" if it has cardinality λ, where $\beth_\mu(\lambda) < \bar{\kappa}$ with $\mu = \beth_{(2^\lambda)^+}(\lambda)$. This is sufficient for the proofs we need for this exposition. For a more general treatment, it is common to define $\bar{\kappa}$ to be an inaccessible cardinal, so that there is no worry of a construction going "out-of-bounds" with respect to $\bar{\kappa}$.

However, for our purposes the most important use of saturation is the following result:

Lemma 1.5.10. *Suppose $N \equiv M$, $|N| = |M|$. If N, M are both infinite and saturated, then $N \cong M$.*

Proof. Suppose $|N| = |M| = \lambda$ and list their elements by $\bar{m} = (m_i : i < \lambda)$ and $\bar{n} = (n_i : i < \lambda)$ respectively. Using a back-and-forth argument, we will construct an isomorphism $h : M \longrightarrow N$. Let $h_0 = \emptyset$, and let $\bar{c} = (c_i : i < \lambda)$ be new constant symbols.

For $i < \lambda$, let $A_i \supseteq \bar{m}|_i$ and suppose $h_i : A \longrightarrow N$ is an elementary map, $|A_i| < \lambda$. Then for each $m_j \in A_i$ we can interpret the constant symbols by $c_j^M = m_j, c_J^N = h_i(m_j)$ for $j < i$. Let L_i be the language L with new constant symbols $\{c_j : m_j \in A_i\}$, and so $M \equiv_{L_i} N$ since h is an elementary map. Therefore, as in the proof of Proposition 1.5.9, the type $tp^M(m_i/A_i)$ is also a type of N over $h(A_i)$ by replacing the occurrences of each $m_j \in A_i$ by $h(m_j)$. But as N is saturated and $|h(A_i)| < \lambda$, this type is satisfied by some $n \in N$. So extend h_i to $h_i' : A_i \cup \{m_i\} \longrightarrow N$ by $h_i'(m_i) = n$; this ensures h_i' is also an elementary map with domain $|A_i \cup \{m_i\}| < \lambda$.

Conversely, suppose $A \subseteq M$, $|A| < \lambda$ and $h_i' : A \longrightarrow N$ is an elementary map. Let $B_i = \text{Rang } h_i'$, and consider the type $tp^N(n_i/B_i)$. By the same reasoning as above, this is a type of M over A by replacing each $n_j \in B_i$ with $h_i'^{-1}(n_j)$. Then the saturation of M guarantees this type is realized by some $m \in M$. If $n_i \notin B_i$, then necessarily $m \notin A$ and so extend h_{i+1}' to $h_{i+1} : A \cup \{m\} \longrightarrow N$ by $h_{i+1}(m) = n_j$. Otherwise simply let $h_{i+1} = h_i'$. This guarantees that h_{i+1} is an elementary map with domain $|A \cup \{m\}| < \lambda$, and $n_i \in \text{Rang } h_{i+1}$.

Therefore, if $h_i : A_i \longrightarrow N$ is an elementary map with $|A_i| < \lambda$, $\bar{m}|_i \subseteq A_i$ and $\bar{n}_i \subseteq \text{Rang } h_i$, there is an elementary map $h_{i+1} : A_{i+1} \longrightarrow N$ which extends h_i with $|A_{i+1}| < \lambda$, $m_i \in A_{i+1}$ and $n_i \in \text{Rang } h_{i+1}$.

So for a limit ordinal $\delta < \lambda$, suppose $(h_i : i < \delta)$ is a sequence of elementary maps into N such that for each $i < \delta$:

- $|\text{dom } h_i| < \lambda$
- If $j < i$ then h_i extends h_j
- $\bar{m}|_i \in \text{dom } h_i$ and $\bar{n}|_i \in \text{Rang } h_i$.

Then clearly $h_\delta = \bigcup_{i<\delta} h_i$ is an elementary map which satisfies the same conditions.

Finally, let $h = \bigcup_{i<\lambda} h_i$. Then $h : M \longrightarrow N$ is an elementary map, and by Proposition 1.5.4 an elementary embedding which is also surjective i.e. an isomorphism. □

1.6 Ultraproducts

Ultraproduct is another widely used tool in model theory, although for this exposition we will only need it for a particular result.

Definition 1.6.1. *Given a partially ordered set $(P, \leq)$, $\mathscr{F} \subseteq P$ is a **filter** if:*

- *$\mathscr{F} \neq \emptyset$*
- *For every $x, y \in \mathscr{F}$, there is a $z \in \mathscr{F}$ such that $z \leq x, z \leq y$*
- *If $x \in \mathscr{F}$ and $x \leq y$, then $y \in \mathscr{F}$*

A filter $\mathscr{F}$ is ***proper*** *if $\mathscr{F} \neq P$.*

For our purposes, we are only interested in filters which are subfamilies of a powerset. This motivates the following definition:

Definition 1.6.2. *Given any set X, a filter $\mathscr{F} \subseteq \mathscr{P}(X)$ is an* ***ultrafilter*** *if for every $A \subseteq X$, either $A \in \mathscr{F}$ or $X - A \in \mathscr{F}$.*

Lemma 1.6.3. *Given $\mathscr{S} \subseteq \mathscr{P}(X)$, if for every finite choice of $A_0, \dots, A_n \in \mathscr{S}$, $\bigcap_{i \leq n} A_i \neq \emptyset$, then there is a proper filter $\mathscr{F} \subseteq \mathscr{P}(X)$ with $\mathscr{F} \supseteq \mathscr{S}$.*

Proof. Defining $\mathscr{F} = \{A \in \mathscr{P}(X) : \exists B_0, \dots, B_n \in \mathscr{S}, \bigcap_{i \leq n} B_i \subseteq A\}$ fulfils the requirements. □

Lemma 1.6.4. *For a filter $\mathscr{F} \subseteq \mathscr{P}(X)$ for some set X, $\mathscr{F}$ is improper iff $\emptyset \in \mathscr{F}$*

Proof. Trivial. □

Proposition 1.6.5 (Ultrafilter lemma). *Given any set X and a filter $\mathscr{F} \subseteq \mathscr{P}(X)$, there is an ultrafilter $\mathscr{U} \supseteq \mathscr{F}$*

Proof. Note that if $\mathscr{C}$ is a chain of proper filters containing $\mathscr{F}$ and ordered by inclusion, then $\bigcup \mathscr{C}$ is also a proper filter containing $\mathscr{F}$. Thus by Zorn's lemma, a maximal element $\mathscr{U}$ exists in the lattice of filters containing $\mathscr{F}$. Now, if there is an $A \subseteq X$ such that $A, X - A \notin \mathscr{U}$, then $\mathscr{U} \cup \{A\}$ satisfies the conditions of the above lemma, so there is a proper filter $\mathscr{U}' \supsetneq \mathscr{U}$, contradicting the maximality of $\mathscr{U}$. □

Remark. Although the above proof uses Zorn's lemma, it is known that the Ultrafilter lemma itself if independent of ZF but strictly weaker than the Axiom of Choice.

Definition 1.6.6. *Suppose I is a set, and M_i an L-structure for each $i \in I$. Given an ultrafilter $\mathscr{F} \subseteq \mathscr{P}(I)$, we define the equivalence relation $\sim_{\mathscr{F}}$ on $\prod_{i \in I} M_i$: $(a_i : i \in I) \sim_{\mathscr{F}} (b_i : i \in I)$ if $\{i \in I : M_i \models a_i = b_i\} \in \mathscr{F}$*
Abbreviating $\sim_{\mathscr{F}}$ with $\sim$, we define the ***ultraproduct*** *$\prod_{i \in I} M_i / \mathscr{F}$ to be an L-structure with:*

- *The domain is $\prod_{i \in I} M_i / \sim$*
- *For a constant symbol c, $c^{\prod_{i \in I} M_i / \mathscr{F}} = (c^{M_i} : i \in I) / \sim$*
- *For a function symbol f with arity n, $f^{\prod_{i \in I} M_i / \mathscr{F}}(\bar{a}_0 / \sim, \dots, \bar{a}_{n-1} / \sim) = (f^{M_i}(a_{0,i}, \dots, a_{n-1,i}) : i \in I) / \sim$*

- *For a relation symbol P with arity n, $(\bar{a}_0/\sim, \dots, \bar{a}_{n-1}/\sim) \in P^{\prod_{i\in I} M_i/\mathscr{F}}$ iff $\{i \in I : M_i \models P(a_{0,i}, \dots, a_{n-1,i})\} \in \mathscr{F}$*

*If $M_i = M$ for each $i \in I$, then we call $\prod_{i\in I} M/\mathscr{F}$ an **ultrapower** and denote it by $M^I/\mathscr{F}$*

An important use of ultrapowers is in nonstandard analysis: let $\mathscr{F} \subseteq \mathscr{P}(\omega)$ be an ultrafilter containing the cofinite subsets of ω, and let $\mathscr{R} = \mathbb{R}^\omega/\mathscr{F}$ where $\mathbb{R}$ is the real numbers as an ordered field. To see why we can use $\mathscr{R}$ for nonstandard analysis, we need the following results:

Lemma 1.6.7. *Given an ultraproduct $\prod_{i\in I} M_i/\mathscr{F}$, for any two terms s, t of L and $\bar{a}_0, \dots, \bar{a}_n \in \prod_{i\in I} M_i/\mathscr{F}$, $\prod_{i\in I} M_i/\mathscr{F} \models s(\bar{a}_0, \dots, \bar{a}_n) = t(\bar{a}_0, \dots, \bar{a}_n)$ iff $\{i \in I : M_i \models s(a_{0,i}, \dots, a_{n-1,i}) = t(a_{0,i}, \dots, a_{n-1,i})\} \in \mathscr{F}$*

Proof. This is trivial by induction on complexity of terms. □

Theorem 1.6.8 (Łoś' Theorem). *For any L-formula ϕ, and $\bar{a}_0, \dots, \bar{a}_n \in \prod_{i\in I} M_i/\mathscr{F}$, $\prod_{i\in I} M_i/\mathscr{F} \models \phi(\bar{a}_0, \dots, \bar{a}_n)$ iff $\{i \in I : M_i \models \phi(a_{0,i}, \dots, a_{n-1,i})\} \in \mathscr{F}$*

Proof. For any formula ϕ, we define $||\phi(\bar{a}_0, \dots, \bar{a}_n)|| = \{i \in I : M_i \models \phi(\bar{a}_0, \dots, \bar{a}_n)\}$. We then proceed by induction on formula complexity:

- If ϕ is $s(x_0, \dots, x_n) = t(x_0, \dots, x_n)$ for some term s, t, then this is just the case as in the above lemma.

- If ϕ is $P(t_0(x_0, \dots, x_n), \dots, t_m(x_0, \dots, x_n))$, then let $\bar{b}_0, \dots, \bar{b}_m \in \prod_{i\in I} M_i/\mathscr{F}$ be such that $\prod_{i\in I} M_i/\mathscr{F} \models t_j(\bar{a}_0, \dots, \bar{a}_n) = \bar{b}_j$. Then

$$\prod_{i\in I} M_i/\mathscr{F} \models P(t_0(\bar{a}_0, \dots, \bar{a}_n), \dots, t_m(\bar{a}_0, \dots, \bar{a}_n))$$
$$\Leftrightarrow \prod_{i\in I} M_i/\mathscr{F} \models P(\bar{b}_0, \dots, \bar{b}_m) \Leftrightarrow ||P(\bar{b}_0, \dots, \bar{b}_m)|| = D' \in \mathscr{F}$$

  For $j \leq m$, let $D_j = ||t_i(\bar{a}_0, \dots, \bar{a}_n) = \bar{b}_j|| \in \mathscr{F}$. By definition of a filter, $D = D' \cap \bigcap_{j\leq m} D_m \in \mathscr{F}$, and so

$$\{i \in I : M_i \models P(b_{0,i}, \dots, b_{m,i}) \wedge \bigwedge_{j\leq m} t_j(a_{0,i}, \dots, a_{n,i}) = b_{j,i}\} = D \in \mathscr{F}$$

  This is equivalent to

$$\{i \in I : M_i \models P(t_0(a_{0,i}, \dots, a_{n,i}), \dots, t_m(a_{0,i}, \dots, a_{n,i})\} \in \mathscr{F}$$

  Thus proving the claim.

- If ϕ is $\neg\psi$, then

$$\prod_{i\in I} M_i/\mathscr{F} \models \neg\psi(\bar{a}_0,\dots,\bar{a}_n) \Leftrightarrow \prod_{i\in I} M_i/\mathscr{F} \not\models \psi(\bar{a}_0,\dots,\bar{a}_n)$$
$$\Leftrightarrow \{i \in I : M_i \models \psi(a_{0,i},\dots,a_{n,i})\} = D \notin \mathscr{F}$$

  As $\mathscr{F}$ is an ultrafilter, $D \notin \mathscr{F}$ iff $I - D \in \mathscr{F}$, and this is true iff

$$\{i \in I : M_i \not\models \psi(a_{0,i},\dots,a_{n,i})\} = I - D \in \mathscr{F}$$
$$\Leftrightarrow \{i \in I : M_i \models \neg\psi(a_{0,i},\dots,a_{n,i})\} \in \mathscr{F}$$

- If ϕ is $\psi \wedge \varphi$, then

$$\prod_{i\in I} M_i/\mathscr{F} \models \psi(\bar{a}_0,\dots,\bar{a}_n) \wedge \varphi(\bar{a}_0,\dots,\bar{a}_n) \Leftrightarrow$$
$$\prod_{i\in I} M_i/\mathscr{F} \models \psi(\bar{a}_0,\dots,\bar{a}_n), \prod_{i\in I} M_i/\mathscr{F} \models \varphi(\bar{a}_0,\dots,\bar{a}_n)$$
$$\Leftrightarrow ||\psi(\bar{a}_0,\dots,\bar{a}_n)|| \in \mathscr{F}, ||\varphi(\bar{a}_0,\dots,\bar{a}_n)|| \in \mathscr{F}$$

  Since $\mathscr{F}$ is a filter, this is true iff $||\psi(\bar{a}_0,\dots,\bar{a}_n)|| \cap ||\varphi(\bar{a}_0,\dots,\bar{a}_n)|| \in \mathscr{F}$, and we see that $||\psi(\bar{a}_0,\dots,\bar{a}_n) \wedge \varphi(\bar{a}_0,\dots,\bar{a}_n)|| = ||\psi(\bar{a}_0,\dots,\bar{a}_n)|| \cap ||\varphi(\bar{a}_0,\dots,\bar{a}_n)||$, proving the claim.

- If ϕ is $\exists x\psi$, then $\prod_{i\in I} M_i/\mathscr{F} \models \exists x\psi(x,\bar{a}_0,\dots,\bar{a}_n)$ iff there is a $\bar{b} \in \prod_{i\in I} M_i/\mathscr{F}$ such that $\prod_{i\in I} M_i/\mathscr{F} \models \psi(\bar{b},\bar{a}_0,\dots,\bar{a}_n)$ which is true iff there are $(b_i \in M_i : i \in I)$ such that $\{i \in I : M_i \models \psi(b_i, a_{0,i},\dots,a_{n,i})\} \in \mathscr{F}$ iff $||\exists x\psi(x,\bar{a}_0,\dots,\bar{a}_n)|| \in \mathscr{F}$

This completes the proof. □

Corollary 1.6.9. *For any set I, a ultrafilter $\mathscr{F} \subseteq \mathscr{P}(I)$ and M an L-structure, the diagonal map $M \longrightarrow M^I/\mathscr{F}, m \mapsto (m : i \in I)/\sim$ is an elementary embedding.*

Proof. This is trivial by the above theorem. □

To finish our remark regarding nonstandard analysis, let $\bar{a} = (a_n : n < \omega)$ be a sequence of positive reals such that $\lim_{n\to\infty} a_n = 0$. Then for every positive real $r \in \mathbb{R}$, if we consider $\mathbb{R}$ as a elementary substructure of $\mathscr{R}$ by identifying r with $(r : n < \omega)_\sim$, $\mathscr{R} \models 0 < (\bar{a}_\sim) < r$. Thus $\bar{a}$ is an infinitesimal element of $\mathscr{R}$.

Chapter 2

Stability Results

Note. From this point on, by a theory we will always mean a complete theory with infinite models. For every theory, we will work within some universal model, and unless otherwise stated by a set we will always refer to a subset of the universal model. In particular, for a given set, $S_n(A)$ will simply refer to the set of n-types over A relative to the universal model, and similarly $tp(b/A)$. Moreover, we will simply write $\models \phi(c)$ as an abbreviation of $\bar{M} \models \phi(c)$, where $\bar{M}$ is the universal model. Similarly, Also:

Definition 2.0.1. *Let X be a linearly ordered set. Then for any ordinal α, $[X]^\alpha$ denotes the set of increasing α-sequences from X, and $[X]^{<\alpha} = \bigcup_{0<\beta<\alpha}[X]^\beta$*

2.1 Stability

The concept of stability is central to many recent developments in model theory, and is indispensable for this exposition.

Definition 2.1.1. *For an infinite cardinal λ, a complete theory T is **λ-stable** if for every $M \models T$ and $A \subseteq M$ with $|A| \leq \lambda$, $|S_1(A)| \leq \lambda$.*
*T is **stable** if it is λ-stable for some infinite λ.*

As an example, the theory of algebraically closed fields with characteristic zero is $\aleph_0$-stable, whereas the theory of dense linear orders with endpoints is unstable, although it will take quite a lot of work for us to prove these claims.

Stable theories are in many sense "well-behaved", and in particular in the sense that there cannot be too many nonisomorphic models of a stable theory. Making these claims precise is beyond the scope of this exposition, but let us begin by some basic properties.

Lemma 2.1.2. *If T is λ-stable, then for every $n < \omega$, $M \models T$ and $A \subseteq M$ with $|A| \leq \lambda$, $|S_n^M(A)| \leq \lambda$*

Proof. By induction on n: if $n = 1$, the claim is trivial. Inductively, assume for a contradiction that $|S_{n+1}^M(A)| \geq \lambda^+$, $|S_n^M(A)| \leq \lambda$. Thus in a λ^+-saturated elementary extension $N \succeq M$ (see Proposition 1.5.8), there is $(\bar{a}_i : i < \lambda^+)$ such that each $\bar{a}_i$ realizes a distinct type over A. Partitioning $(\bar{a}_i : i < \lambda^+)$ by $tp^N(\bar{a}_i|_n/A)$, since $|S^M(A)| \leq \lambda$ this partitions $(\bar{a}_i : i < \lambda^+)$ into at most λ equivalence classes, and as λ^+ is regular there is some $I \subseteq \lambda^+$, $|I| = \lambda^+$ such that for $i \in I$, $tp^N(\bar{a}_i|_n/A)$ is constant over I. So let $\bar{a} \in N$ be such that for any $i \in I$, $tp^N(\bar{a}/A) = tp^N(\bar{a}_i|_n/A)$. As each $\bar{a}_i$ realizes a distinct type over A, there exists $(b_i : i < \lambda^+)$ such that each $tp^N(\bar{a} \frown b_i/A)$ is distinct. This implies each $tp^N(b_i/A \cup \{\bar{a}\})$ is distinct, with $|A \cup \{\bar{a}\}| \leq \lambda$. Thus $|S^N(A \cup \{\bar{a}\})| > \lambda$, contradicting that T is λ-stable. □

In the literature, different authors often use different but equivalent definitions for the term "stable", often referring to the existence of some formula in T which satisfies some properties. Following [Sh 90], we will first prove a few lemmas, then proceed to demonstrate the equivalence of stability with some other properties.

Proposition 2.1.3. *Given a theory T in L, for a $\phi(x, \bar{y}) \in L$ the following are equivalent:*

1. *For every infinite λ, there is a $M \models T$ with a $A \subseteq M$ such that $|A| \leq \lambda < |S^M_{\phi,1}(A)|$*
2. *There is some infinite λ with a $M \models T$ and a $A \subseteq M$ such that $|A| \leq \lambda < |S^M_{\phi,1}(A)|$*
3. *There is a $M \models T$ with sequences $(c_i : i < \omega), (\bar{a}_j : j < \omega) \in M$ such that either $M \models \phi(c_i, \bar{a}_j)$ iff $i < j$ or $M \models \neg\phi(c_i, \bar{a}_j)$ iff $i < j$*
4. *ϕ has the order property: there is a sequence $(\bar{a}_i : i < \omega)$ from some model M such that for every $n < \omega$, $\{\phi(x, \bar{a}_i) : i < n\} \cup \{\neg\phi(x, \bar{a}_i) : i \geq n\}$ is a type of M.*

We will prove these equivalences in the following lemmas.

Definition 2.1.4. *For any type p over A and $B \subseteq A$, we define $p|_B = \{\phi(\bar{x}, \bar{b}) \in p : \bar{b} \in B\}$.*
For any Δ_0-type p and $\Delta_1 \subseteq \Delta_0$, we define $p|_{\Delta_1} = \{\phi(\bar{x}, \bar{a}) \in p : \phi \in \Delta_1\}$.
*We say that a type p (Δ_0, Δ_1)-**splits over** A if there are $\bar{b}, \bar{c}$ such that $tp_{\Delta_0}(\bar{b}/A) = tp_{\Delta_0}(\bar{c}/A)$ but there is a $\phi(\bar{x}, \bar{y}) \in \Delta_1$ such that $\phi(\bar{x}, \bar{b}), \neg\phi(\bar{x}, \bar{c}) \in p$.*
*If $\Delta_0 = \Delta_1 = L$, then we simply say that p **splits over** A.*

Lemma 2.1.5. (1) ⇒ (2)

Proof. This is trivially true. □

Lemma 2.1.6. (2) ⇒ (3)

Proof. WLOG, we can assume that M realizes enough types such that there is a sequence $(c_i : i < |A|^+) \in M$ with each $tp^M_\phi(c_i/A)$ distinct. Let $l = |\bar{y}|$, define $\psi(\bar{y}, x) = \phi(x, \bar{y})$, and for $j \leq \omega$, define inductively A_j:

1. $A_0 = A$

2. If A_j is defined, let A_{j+1} be such that for every $p \in S^M_{\phi,1}(A_j) \cup S^M_{\psi,l}(A_j)$ and $B \subseteq A_j$ with $|B| < \aleph_0$, $p|_B$ is realized by some tuple in A_{j+1}. Note that we can achieve this while ensuring $|A_{j+1}| = |A_j|$ as B is finite, and so there are only $|A|$ choices of B

3. $A_\omega = \bigcup_{j<\omega} A_j$

Claim. There is an $i < |A|^+$ such that for every $n < \omega$ and $B \subseteq A_n$ with $|B| < \aleph_0$, $tp^M_\phi(c_i/A_{n+1})$ (ψ, ϕ)-splits over B.

Suppose not, so for every $i < |A|^+$ there are $n_i < \omega$ and $B_i \subseteq A_{n_i}$ such that there are no tuples $\bar{a}_i, \bar{b}_i \in A^l_{n_i+1}$ with $tp^M_\psi(\bar{a}_i/B_i) = tp^M_\psi(\bar{b}_i/B_i)$ and $M \models \phi(c_i, \bar{a}_i) \wedge \neg\phi(c_i, \bar{b}_i)$. Note that since $|A|^+ > \aleph_0$, there is some $C' \subseteq |A|^+$ with $|C'| = |A|^+$ and some $n' < \omega$ such that for every $i \in C'$, $n_i = n'$ i.e. the indices $i < |A|^+$ such that $n_i = n'$ is cofinal in $|A|^+$. Now, there are $|A_{n_i}| = |A|$ many choices of $B \subseteq A_{n_i}$ with $|B| < \aleph_0$, so again by restricting to a cofinal $C'' \subseteq C'$ (with $|C''| = |C'|$) there is a finite $B'' \subseteq A_{n_i}$ such that for every $i \in C''$, $B_i = B''$. By construction of $A_{n'+1}$, there is a $D \subseteq A_{n'+1}$ with $B'' \subseteq D$ and $|D| \leq (l+1)2^{|B''|}$ such that every $p \in S^M_{\psi,l}(B'')$ is realized by some tuple in D. We note that $|S^M_{\phi,1}(D)| \leq 2^{|D|} < |A|^+$, so once again we can restrict to a cofinal $C^{(3)} \subseteq C''$, $|C^{(3)}| = |C''|$ with a type p such that for all $i \in C^{(3)}$, $tp^M_\phi(c_i/D) = p$.

Assume WLOG that $0, 1 \in C^{(3)}$ and recall that $tp^M_\phi(c_0/A) \neq tp^M_\phi(c_1/A)$. Thus there is some $\bar{a} \in A^l$ with $M \models \phi(c_0, \bar{a}) \leftrightarrow \neg\phi(c_1, \bar{a})$. By definition of D above, we can find a $\bar{a}' \in D^l$ such that $tp^M_\psi(\bar{a}/B'') = tp^M_\psi(\bar{a}'/B'')$. By assumption, $tp_\phi(c_0/A_{n'+1}), tp_\phi(c_1/A_{n'+1})$ does not (ψ, ϕ)-split over B'', so $M \models \phi(c_0, \bar{a}) \leftrightarrow \phi(c_0, \bar{a}')$ and $M \models \phi(c_1, \bar{a}) \leftrightarrow \phi(c_1, \bar{a}')$. This implies that $M \models \phi(c_0, \bar{a}') \leftrightarrow \neg\phi(c_1, \bar{a}')$, thus contradicting that $tp^M_\phi(c_0, D) = p = tp^M_\phi(c_1/D)$. This proves the above claim.

Let $i_0 < |A|^+$ be such that it satisfies the above claim. For $j < \omega$, define $\bar{a_j}, \bar{b_j}, c_j$ inductively: If $\bar{a_k}, \bar{b_k}, c_k$ is defined for $k < j$, let $B_j = \bigcup_{k<j} \bar{a_k} \cup \bar{b_k} \cup \{c_k\}$. By the above claim, $tp^M_\phi(c_{i_0}/A_{2j+1})$ (ψ, ϕ)-splits over B_j, so there is $\bar{a_j}, \bar{b_j} \in A^l_{2j+1}$ such that $tp^M_\psi(\bar{a_j}/B_j) = tp^M_\psi(\bar{b_j}/B_j)$ and $M \models \phi(c_{i_0}, \bar{a_j}) \wedge \neg\phi(c_{i_0}, \bar{b_j})$. Then let $c_j \in A_{2j+2}$ be such that $tp^M_\phi(c_j/B_j \cup \bar{a_j} \cup \bar{b_j}) = tp^M_\phi(c_{i_0}/B_j \cup \bar{a_j} \cup \bar{b_j})$ (such a c_j exists by definition of A_{2j+2}).

Note that by construction, for any $i \leq j < \omega$, $M \models \phi(c_{i_0}, \bar{a_i}) \wedge \neg\phi(c_{i_0}, \bar{b_j})$. However, since $tp^M_\phi(c_j/B_j \cup \bar{a_j} \cup \bar{b_j}) = tp^M_\phi(c_{i_0}/B_j \cup \bar{a_j} \cup \bar{b_j})$ and $\bar{a_i}, \bar{b_i} \in B_j \cup \bar{a_j} \cup \bar{b_j}$, this implies that $M \models \phi(c_{i_0}, \bar{a_i}) \leftrightarrow \phi(c_j, \bar{a_i})$ and $M \models \phi(c_{i_0}, \bar{b_i}) \leftrightarrow \phi(c_j, \bar{b_i})$. Therefore $M \models \phi(c_j, \bar{a_i}) \wedge \neg\phi(c_j, \bar{b_j})$. On the other hand, if $j < i < \omega$, since $tp^M_\psi(\bar{a_i}/B_i) = tp^M_\psi(\bar{b_i}/B_i)$ and $c_j \in B_i$, $M \models \psi(\bar{a_i}, c_j) \leftrightarrow \psi(\bar{b_i}, c_j)$ and thus $M \models \phi(c_j, \bar{a_i}) \leftrightarrow \phi(c_j, \bar{b_i})$.

Let us define the function $f : [\omega]^2 \longrightarrow 2$ by $f(i, j)$ if $M \models \phi(c_i, \bar{a_j})$, and $f(i, j) = 1$ otherwise. By Ramsey's theorem (see Appendix A, Theorem A.0.1) there is a $W \subseteq \omega$ such that f is constant on $[W]^2$ with $|W| = \aleph_0$, and by renaming the elements we can WLOG identify W with ω. Now, if $M \models \neg\phi(c_0, \bar{a_1})$, then $f(0,1) = 1$, and so for $i < j < \omega$, $M \models \neg\phi(c_i, \bar{a_j})$. But as shown above for $j \leq i < \omega$, $M \models \phi(c_i, \bar{a_j})$, and so $M \models \phi(c_i, \bar{a_j})$ iff $j \leq i$. Conversely, if $M \models \phi(c_0, \bar{a_1})$, then $f(0,1) = 0$ and so for $i < j\omega$, $M \models \phi(c_i, \bar{a_j})$. But from above, this implies that $M \models \phi(c_i, \bar{b_j})$. On the other hand, if $j \leq i < \omega$, then $M \models \neg\phi(c_i, \bar{b_j})$, and so $M \models \phi(c_i, \bar{b_j})$ iff $i < j$. This proves (3). $\square$

Lemma 2.1.7. (3) $\Rightarrow$ (4)

Proof. Suppose $(c_i : i < \omega), (\bar{a_j} : j < \omega) \in M$ are such that either $M \models \phi(c_i, \bar{a_j}) \Leftrightarrow i < j$ or $M \models \neg\phi(c_i, \bar{a_j}) \Leftrightarrow i < j$. For $n < \omega$, let $p_n(x) = \{\phi(x, \bar{a}_j) : j < n\} \cup \{\neg\phi(x, \bar{a}_j) : j \geq n\}$. If $M \models \neg\phi(c_i, \bar{a_j}) \Leftrightarrow i < j$, then $M \models p_n(c_n)$, and thus $p_n(x)$ is a type of M i.e. ϕ has the order property.

On the other hand, if $M \models \phi(c_i, \bar{a_j}) \Leftrightarrow i < j$, then we proceed by the compactness theorem: let $(\bar{d_j} : j < \omega)$ be new constants, and for $n < \omega$, consider $Th_M(M) \cup \{\phi(x, \bar{d_j}) : j < n\} \cup \{\neg\phi(x, \bar{d_j}) : j \geq n\}$. Then for any $k > n$, by interpreting $x^M = c_{k-n+1}$, $\bar{d}_j^M = a_{k-n+1+j}$ for $j < n$ and $\bar{d}_j^M = a_{j-n}$ for $n \leq j < k$, $M \models Th_M(M) \cup \{\phi(x, \bar{d_j}) : j < n\} \cup \{\neg\phi(x, \bar{d_j}) : n \leq j < k\}$. Therefore by compactness there is some $N \succeq M$ with $\bar{d}_j^N \in N$ such that $q_n(x) = \{\phi(x, \bar{d_j}) : j < n\} \cup \{\neg\phi(x, \bar{d_j}) : j \geq n\}$ is a type of N i.e. ϕ has the order property. $\square$

Definition 2.1.8. *For a formula ϕ, we use ϕ^0 to denote ϕ and ϕ^1 to denote $\neg\phi$.*

Lemma 2.1.9. (4) $\Rightarrow$ (1)

Proof. Suppose ϕ has the order property. For an ordinal α, let us define

$$\Gamma_\phi(\alpha) = \{\phi(x_\eta, \bar{y}_{\eta|_\beta})^{\eta(\beta)} : \eta \in 2^\alpha, \beta < \alpha\}$$

where for each η and β, $x_\eta, \bar{y}_{\eta|_\beta}$ are new constants.

Claim. $\Gamma_\phi(n)$ is satisfiable for each $n < \omega$.

Let us define the order $<$ on $2^{<\omega}$ by:

- If $\eta|_k = \nu|_k$, $\eta(k) = 0$ and $\nu(k) = 1$, then $\eta < \nu$
- If $l(\eta) = k$, $\eta = \nu|_k$ and $\nu(k) = 1$, then $\eta < \nu$
- If $l(\eta) = k$, $\eta = \nu|_k$ and $\nu(k) = 0$, then $\nu < \eta$

Obviously $<$ is a linear order. Now, as ϕ has the order property, by interpreting the new constants suitably we see that for every $n < \omega$,

$$T \cup \{\phi(x_\eta, \bar{y}_\nu) : l(\nu) < l(\eta) = n, \eta < \nu\} \cup \{\neg\phi(x_\eta, \bar{y}_\nu) : l(\nu) < l(\eta) = n, \nu \leq \eta\}$$

is satisfiable, and thus by compactness $\Gamma_\phi(n)$ is satisfiable.

Note that for any ordinal α, every finite subset of $\Gamma_\phi(\alpha)$ is a finite subset of $\Gamma_\phi(n)$ for some large enough $n < \omega$ (after renaming the new constants), and thus by compactness every $\Gamma_\phi(\alpha)$ is satisfiable.

To show (1), let λ be any infinite cardinal. Let λ_0 be the least ordinal such that $2^{\lambda_0} > \lambda$ (so $\lambda_0 \leq \lambda$), and let $M \models \Gamma_\phi(\lambda_0)$ with $\bar{a}_\nu = \bar{y}_\nu^M$ and $c_\eta = x_\eta^M$ for $\nu \in 2^{<\lambda_0}$ and $\eta \in 2^{\lambda_0}$. Let $A = \bigcup\{\bar{a}_\nu : \nu \in 2^{<\lambda_0}\}$, so

$$|A| \leq \aleph_0 (\sum_{\alpha<\lambda_0} |2^\alpha|) \leq \aleph_0 \cdot \lambda \cdot \lambda_0 \leq \lambda$$

Now, for $\eta \in 2^{\lambda_0}$, let $p_\eta = tp_\phi^M(c_\eta/A)$. Note for $\eta, \zeta \in 2^{\lambda_0}$, if $\eta \neq \zeta$ and $\beta < \lambda_0$ is the least ordinal such that $\eta(\beta) \neq \zeta(\beta)$, then $\phi(x, \bar{a}_{\eta|_\beta})^{\eta(\beta)} \in p_\eta$ and $\phi(x, \bar{a}_{\zeta|_\beta})^{\zeta(\beta)} = \neg\phi(x, \bar{a}_{\eta|_\beta})^{\eta(\beta)} \in p_\zeta$ i.e. $\eta \neq \zeta$ implies $p_\eta \neq p_\zeta$. Therefore:

$$|S_{\phi,1}^M(A)| \geq |\{p_\eta : \eta \in 2^{\lambda_0}\}| = |2^{\lambda_0}| > \lambda \geq |A|$$

which proves (1). This completes the proof of Proposition 2.1.3. □

Definition 2.1.10. *An* ***unstable formula relative to*** *T is a formula satisfying the conditions of Proposition 2.1.3.*

Note. Using Lemma 2.1.2, we see that we can state and prove Proposition 2.1.3 for $\phi(\bar{x}, \bar{y})$, where $l(\bar{x})$ is not necessarily 1. In this case, we also consider $\phi(\bar{x}, \bar{y})$ to be an unstable formula relative to T.

Proposition 2.1.11. *The following are equivalent:*

1. *T is unstable.*
2. *There is some infinite λ such that $\lambda = \lambda^{|T|}$ and T is not λ-stable.*
3. *Some formula $\phi(x, \bar{y})$ is unstable relative to T.*
4. *Some formula $\phi(\bar{x}, \bar{y})$ is unstable relative to T.*

Proof. (1) $\Rightarrow$ (2) be definition. Assuming (2), then there is an $M \models T$ with $A \subseteq M$ such that $|A| \leq \lambda = \lambda^{|T|} < |S_1^M(A)|$. So consider the mapping:

$$g : S_1^M(A) \longrightarrow \prod_{\phi(x,\bar{y})\in L} S_{\phi,1}^M(A), g(p) \mapsto (p|_\phi)_{\phi\in L}$$

Note that this map is injective. Thus $\lambda < |S_1^M(A)| \leq |\prod_{\phi\in L} S_{\phi,1}^M(A)|$. Let $\lambda_\phi = |S_{\phi,1}^M|$, and note that if $\lambda_\phi \leq \lambda$ for all $\phi \in L$ then $|\prod_{\phi\in L} \lambda_\phi| = \lambda^{|T|} = \lambda$, a contradiction. Thus there is some $\phi(x, \bar{y}) \in L$ such that $|S_{\phi,1}^M(A)| = \lambda_\phi > \lambda$. By Proposition 2.1.3, this implies that ϕ is unstable and thus (3).

Finally, assuming (3), note that for any formula $\phi(x, \bar{y})$, distinct types in $S_{\phi,1}^M(A)$ extend to distinct types in $S_1^M(A)$, and thus $|S_1^M(A)| \geq |S_{\phi,1}^M(A)|$. Therefore if there is some formula ϕ which is unstable relative to T, then by Proposition 2.1.3 there is some infinite λ with a $M \models T$ and $A \subseteq M$, $|A| = \lambda$

such that $|S^M_{\phi,1}(A)| > |A|$. Thus $|S^M_1(A)| > |A|$, and so T is unstable in λ. This implies (1).

For (4), by the previous note we see that Proposition 2.1.3 also holds for $\phi(\bar{x}, \bar{y})$, and so using Lemma 2.1.2, the proof for the equivalence between (1), (2), (4) follows in exactly the same manner as for (3). $\square$

Corollary 2.1.12. *If there is a formula $\phi(\bar{x}, \bar{y})$ and a sequence $(\bar{a}_n : n < \omega)$ such that for every $W \subseteq \omega$, $\{\phi(\bar{x}, \bar{a}_n) : n \in W\} \cup \{\neg\phi(\bar{x}, \bar{a}_n) : n \notin W\}$ is satisfiable, then T is unstable.*

Proof. If such a ϕ and $(\bar{a}_n : n < \omega)$ exists, then in particular the assumption is true for any $W = k < \omega$, and so ϕ has the order property and is therefore an unstable formula. $\square$

A frequently given rationale for the unstable formula is that it is a kind of "generalized ordering". We can formalize this by:

Lemma 2.1.13. *The following are equivalent:*

1. *There is an unstable formula relative to T.*
2. *There is some model $M \models T$, $(\bar{c}_i : i < \omega) \subseteq M^l$ and a formula $\psi(\bar{x}, \bar{y})$ such that for $m, n < \omega$, $M \models \psi(\bar{c}_m, \bar{c}_n)$ iff $m < n$.*

Proof. For the forward direction, suppose that $\phi(x, \bar{y})$ is an unstable formula. By Proposition 2.1.3, this implies that ϕ has the order property, and so there is a $(\bar{a}_i : i < \omega) \subseteq M$ such that for every $n < \omega$

$$p_n(x) = \{\phi(x, \bar{a}_i) : i < n\} \cup \{\neg\phi(x, \bar{a}_i) : i \geq n\}$$

is a satisfiable 1-type of M. Let b_n realize $p_n(x)$, b_n an element of some sufficiently saturated elementary extension N of M, so that $N \models \phi(b_m, \bar{a}_n)$ iff $m \leq n$. Define $\bar{c}_n = b_n \frown \bar{a}_n$, and let $\psi(x_0, \bar{y}_1, x_1, \bar{y}_0) = \phi(x_0, \bar{y}_1) \wedge x_0 = x_1$. Therefore $N \models \psi(\bar{c}_m, \bar{c}_n)$ iff $m < n$.

For the reverse direction, define $\bar{a}_n = \bar{c}_{2n+1}$. Then $\bar{c}_{2n}$ satisfies $\{\psi(\bar{x}, \bar{a}_i) : i < n\} \cup \{\neg\psi(\bar{x}, \bar{a}_i) : i \geq n\}$, and so $\psi(\bar{x}, \bar{y})$ has the order property i.e. ψ is unstable. $\square$

The definition of λ-stability concerns the number of types over a fixed set, which is sometimes insufficient for our purposes. The following lemma shows how the number of types over different sets can still be bounded:

Lemma 2.1.14. *For an infinite λ, let λ_0 be the least cardinal such that $2^{\lambda_0} > \lambda$. Suppose for every $\nu \in 2^{<\lambda_0}$ there exists a set D_ν and a type $p_\nu \in S_1(D_\nu)$ satisfying:*

1. *If σ is an initial segment of ν, then $D_\sigma \subseteq D_\nu$ and $p_\sigma \subseteq p_\nu$*
2. *If $l(\nu)$ is a limit ordinal, then $D_\nu = \bigcup_{\alpha < l(\nu)} D_{\nu|_\alpha}$*

3. *For every ν, $D_{\nu^\frown 0} = D_{\nu^\frown 1}$ and $p_{\nu^\frown 0} \neq p_{\nu^\frown 1}$ i.e. $p_{\nu^\frown 0}, p_{\nu^\frown 1}$ are distinct extensions of p_ν over the same set*

Then T is not λ-stable.

Proof. Note that for $\nu \in 2^{<\lambda_0}$, there is a formula $\phi_\nu(x, \bar{a}_\nu)$ with parameters in $D_{\nu+1}$ such that $\phi_\nu(x, \bar{a}_\nu)^i \in p_{\nu^\frown i}$. So for $\alpha < \lambda_0$, let $A_\alpha = \{\bar{a}_\nu : \nu \in 2^\alpha\}$ and let $B_\alpha = \bigcup_{\beta<\alpha} A_\alpha$. If $B = B_{\lambda_0}$, then for each $\eta \in 2^{\lambda_0}$, $p_\eta|_B = \bigcup_{\alpha<\lambda_0} p_{\eta|_\alpha}|_{B_\alpha}$ is a 1-type over B. Moreover:

- Since for each $\alpha < \lambda_0$, $|2^\alpha| \leq \lambda$ by definition of λ_0, $|B| = |\bigcup_{\alpha<\lambda_0} A_\alpha| \leq \sum_{\alpha<\lambda_0} \aleph_0 \cdot |2^\alpha| \leq \lambda_0 \cdot \lambda = \lambda$
- For $\eta \neq \zeta$, $\eta, \zeta \in 2^{\lambda_0}$, by definition of B we have $p_\eta|_B \neq p_\zeta|_B$

This implies that $|S_1(B)| \geq 2^{\lambda_0} > \lambda \geq |B|$, and therefore T is not λ-stable. □

This result motivates the following definition:

Definition 2.1.15. *For a stable theory T, $\mu(T)$ is defined to be the least cardinal such that there does not exist D_ν, p_ν satisfying the conditions of the above lemma for all $\nu \in 2^{<\mu(T)}$.*

Proposition 2.1.16. *For any stable, complete theory T, $\mu(T) \leq |T|^+$*

Proof. By 2.1.11, if T is stable then it is $2^{|T|}$-stable, and as $|T|^+$ is the least cardinal λ satisfying $2^\lambda > 2^{|T|}$, thus by the above lemma it does not have D_ν, p_ν satisfying the conditions of the above lemma for all $\nu \in 2^{<|T|+}$. Therefore $\mu(T) \leq |T|^+$. □

Before introducing more tools that can be applied to stable theories, let us give a justification of how a stable theory does not have "too many models".

Proposition 2.1.17. *Suppose T is λ-stable for some $\lambda \geq |T|$, and $M \models T$ with $|M| \leq \lambda$.*

1. *T has a saturated model N of size λ^+ with $M \preceq N$.*
2. *If λ is regular, then T has a saturated model N of size λ with $M \preceq N$*

Proof. For (1), by the Löwenheim-Skolem theorems, let M_0 be a model of T of size λ with $M \preceq M_0$. Then $|S_1(M_0)| \leq \lambda$, so by compactness and using new constant symbols there is a model $M_1 \succeq M_0$ of size λ which realizes every 1-type of M_0 over M_0. Repeating this process, let $(M_i : i < \lambda^+)$ be an elementary chain where for each $i < \lambda^+$, $|M_i| = \lambda$ and M_{i+1} realizes every 1-type over M_i. If $N = \bigcup_{i<\lambda^+} M_i$, then as λ^+ is regular, any $A \subseteq N$ with $|A| < \lambda^+$ is such that $A \subseteq M_i$ for some $i < \lambda^+$ i.e. N is a saturated model of size λ^+.

For (2), the prove is exactly the same except that if λ is regular, then M_λ is already saturated: any $A \subseteq M_\lambda$ with $|A| < \lambda$ is contained in some M_i by the regularity of λ, and thus any type over A is realized in M_{i+1}. Therefore taking $N = M_\lambda$ gives the desired saturated model. □

Remark. The above result can be considerably strengthened: in fact, for any stable theory T, if T is λ-stable then there is a saturated model of size λ. To prove this, however, will require several techniques which are not required for this exposition, and we will thus avoid doing so. The interested reader can consult any textbook on stability theory, for example [Bu 96].

2.2 Prime Models

The concept of prime models stems from algebra, and in particular the idea of the characteristic ring of a ring: it is a structure that is embeddable in any other member of some class of structures. In particular, for a complete theory a prime model is a model which is elementarily embeddable in any model of the theory. This idea is an essential tool in proving Morley's categoricity theorem for countable languages, but for uncountable languages a prime model may not exist. We are however interested in a weaker property:

Definition 2.2.1. *For a complete theory T, $\lambda \geq |T|$, a model M and a set $C \subseteq M$, M is λ-**prime over** C if for every λ-saturated model N and an elementary map $f : C \longrightarrow N$, f extends to an elementary embedding $\tilde{f} : M \longrightarrow N$.*

The construction of a λ-prime model is not trivial, and one way of doing so is by defining a construction sequence:

Definition 2.2.2. *Given a set C, a type $p \in S_n(C)$ is λ-**isolated** if there is a $\Gamma(\bar{x}) \subseteq p$, $|\Gamma(\bar{x})| < \lambda$ such that p is the only type in $S_n(C)$ extending $\Gamma(\bar{x})$.*
*A set $D \supsetneq C$ is λ-**constructible over** C if there is an enumeration (possibly with repetition) $\{d_\alpha : \alpha < \gamma\} = D - C$ such that for every α, $tp(d_\alpha/C \cup \{d_\beta : \beta < \alpha\})$ is λ-isolated. In this case, we call $\{d_\alpha : \alpha < \gamma\}$ a λ-**construction of** D **over** C.*

Proposition 2.2.3. *If D is λ-constructible over C and E is λ-constructible over D, then E is λ-constructible over C. If $(D_i : i < \delta)$ is an increasing sequence such that each D_{i+1} is λ-constructible over D_i, then $\bigcup_{i<\delta} D_i$ is λ-constructible over C.*

Proof. Let $D - C = \{d_\alpha : \alpha < \kappa\}, E - D\{e_\beta : \beta < \mu\}$ be λ-constructions over C, D respectively. Then trivially $(d_\alpha : \alpha < \kappa) \frown (e_\beta : \beta < \mu)$ is a λ-construction over C. This also holds for the limit case. □

Lemma 2.2.4. *If M is a λ-saturated model, then every (possible partial) type q over M with $|q| < \lambda$ is realized in M. Further, every λ-isolated type over M is realized in M.*

Proof. If q is a type over M with $|q| < \lambda$, then let $C \subseteq M$ be the set of parameters which appears in the formulas of q, so necessarily $|C| < \lambda$. So q has a completion over C which is realized in M as M is λ-saturated.

If p is a λ-isolated type over M, let $q \subseteq p$ be such that $|q| < \lambda$ and p is the unique completion of q over M. Thus q is realized in M, say by $m \in M$. But as p is the unique completion of q over $M \supseteq C$, thus m realizes p. □

Proposition 2.2.5. *If M is a λ-saturated model which is λ-constructible over C, then it is λ-prime over C.*

Proof. Let $\{d_\alpha : \alpha < \gamma\}$ be a λ-construction of M over C. For any λ-saturated N and an elementary map $f : C \longrightarrow N$, we will define inductively elementary maps $f_\alpha : C \cup \{d_\beta : \beta < \alpha\} \longrightarrow N$:

- $f_0 = f$
- For limit $\delta < \gamma$, let $f_\delta = \bigcup_{\alpha<\delta} f_\alpha$
- If f_α is defined, note that as $p_\alpha = tp^M(d_\alpha / C \cup \{d_\beta : \beta < \alpha\})$ is λ-isolated by assumption, there is a $q_\alpha \subseteq p_\alpha$ such that $|q_\alpha| < \lambda$ and for every $\phi(x, \bar{d}) \in p_\alpha$, $q_\alpha \models \phi(x, \bar{d})$. So let $\tilde{q_\alpha} = \{\phi(x, f_\alpha(\bar{d})) : \phi(x, \bar{d}) \in q_\alpha\}$. $\tilde{q_\alpha}$ is a type over N as f_α is an elementary map, and as N is λ-saturated, by the above lemma $\tilde{q_\alpha}$ is realized by some $n \in N$. Then for every $\phi(x, \bar{d}) \in p_\alpha$, $N \models \phi(n, f_\alpha(\bar{d}))$ and so extending f_α by $f_{\alpha+1}(d_\alpha) = n$ ensures that $f_{\alpha+1}$ is an elementary map.

Taking $\tilde{f} = \bigcup_{\alpha<\gamma} f_\alpha$ then gives the desired elementary embedding from M into N. $\square$

Lemma 2.2.6. *If T is stable, $\lambda \geq |T|^+$, $M \models T$ and $C \subseteq M$ then every (possibly partial) 1-type $\Sigma(x)$ over C with $|\Sigma(x)| < \lambda$ is contained in a λ-isolated $p \in S_1^M(C)$.*

Proof. Suppose for a contradiction $\Sigma(x)$ is a 1-type over C which is not contained in any λ-isolated type over C with $|\Sigma(x)| < \lambda$. Let $p_{<>} = \Sigma(x)$, and let $D_{<>}$ be the set of elements in C which appear as parameters in $\Sigma(x)$. Then for $\nu \in 2^{<\lambda}$, we will define D_ν and a 1-type p_ν over D_ν with $|p_\nu| < \lambda$:

- If D_ν, p_ν are defined, as $p_\nu \supseteq \Sigma(x)$, p_ν is not contained in any λ-isolated type, and in particular p_ν does not isolate a type over C. Thus there is $\phi_\nu(x, \bar{y}) \in L$ and a $\bar{c} \in C$ such that both $p_\nu \cup \{\phi(x, \bar{c})\}, p_\nu \cup \{\neg\phi(x, \bar{c})\}$ are satisfiable. So let $D_{\nu\frown 0} = D_{\nu\frown 1} = D_\nu \cup \{\bar{c}\}$, $p_{\nu\frown 0} = p_\nu \cup \{\phi(x, \bar{c})\}$ and $p_{\nu\frown 1} = p_\nu \cup \{\neg\phi(x, \bar{c})\}$. So if $|p_\nu| < \lambda$, then $|p_{\nu\frown i}| < \lambda$.
- If ν is of limit length, let $D_\nu = \bigcup_{\alpha<l(\nu)} D_{\nu|_\alpha}$ and $p_\nu = \bigcup_{\alpha<l(\nu)} p_{\nu|_\alpha}$. Note as $p_\nu = p_{<>} \cup \bigcup_{\alpha<l(\nu)} p_{\nu|_{\alpha+1}} - p_{\nu|_\alpha}$, and $|p_{\nu|_{\alpha+1}} - p_{\nu|_\alpha}| = 1$, this implies $|p_\nu| \leq |\Sigma(x)| + \aleph_0 |l(\nu)| < \lambda$.

Then D_ν, p_ν for $\nu \in 2^{<\lambda}$ satisfies the conditions for Lemma 2.1.14, contradicting that T is stable (since by Proposition 2.1.16, $\mu(T) \leq |T|^+$). $\square$

Theorem 2.2.7. *If T is stable, $\lambda \geq |T|^+$ and C is a set (in the universal model), then there is a λ-prime model over C.*

Proof. The idea is to use a long λ-construction over C which realizes enough types to be λ-saturated, which is then λ-prime by Proposition 2.2.5. Note that if D is a set such that every type p over D with $|p| < \lambda$ is realized in D, then D is a model of T by the Tarski-Vaught test and in fact a λ-saturated model.

So let $(p_i : i < \kappa)$ enumerate $\{p \subseteq L_C : p \in S_1(A), A \subseteq C, |A| < \lambda\}$. We will define c_i inductively to realize p_i and so that $(c_i : i < \kappa)$ is a λ-construction over C: suppose c_j has been defined for $j < i$. Since p_i is a type over some A with $|A| < \lambda$, $|p_i| < \lambda$, by the above lemma there is a $p' \in S_1(C \cup \{c_j : j < i\})$ which contains p_i and is λ-isolated. So define c_i to realize p'.

This construction gives a $C_1 = C_0 \cup \{c_i : i < \kappa\}$ which is λ-constructible over $C_0 = C$. Repeating this process, we can define C_α for $\alpha \leq \lambda^+$: $C_{\alpha+1}$ is λ-constructible over C_α, and for a limit $\delta < \lambda^+$, $C_\delta = \bigcup_{\alpha<\delta} C_\alpha$. By Proposition 2.2.3, C_{λ^+} is thus λ-constructible over C. Moreover, if p is a type over C_{λ^+} with $|p| < \lambda$, then p is a type over some C_i and so is realized in C_{i+1}. Therefore C_{λ^+} is a λ-saturated λ-constructible model over C, as desired. $\square$

Corollary 2.2.8. *Moreover, if λ is regular, T is λ-stable and $|C| = \lambda$, then there is a λ-prime model over C with cardinality λ.*

Proof. The construction is almost the same as above, except that given C, we can let $(p_i : i < \lambda)$ enumerate $\{p \subseteq L_C : p \in S_1(C), p \text{ is } \lambda\text{-isolated}\}$. λ-stability of T guarantees that there is at most λ such types. Then if c_j has been defined to satisfy p_j, let $q_i \subseteq p_i$ be such that $|q_i| < \lambda$ and q isolates p. Then again by the above lemma there is a $p' \in S_1(C \cup \{c_j : i < j\})$ which is λ-isolated and contains q_i, and as q_i isolates p_i, $p'|_C = p_i$. So defining c_i to satisfy p' satisfies the requirements of the construction.

So given C with $|C| = \lambda$, we can construct a $C_1 \supseteq C$ with $|C| < \lambda$ which is λ-constructible over C. Again, we repeat this process inductively for $\alpha < \lambda$; the same justification shows that C_λ is then λ-constructible over C. Moreover, for any type p over C_λ with $|p| < \lambda$, as λ is regular there is some C_i such that p is a type over C_i, and therefore by the previous lemma contained in some λ-isolated type in $S_1(C)$. Thus p is realized in C_{i+1}, and therefore again C_λ is a λ-saturated λ-constructible model over C. $\square$

2.3 Indiscernibles

Definition 2.3.1. *For a L-structure M, $A \subseteq M$ a linearly ordered set and $\phi(x_0, \ldots, x_{n-1}) \in L$, A is a **ϕ-indiscernible sequence in** M if for every $\bar{a}, \bar{b} \in [A]^n$, $M \models \phi(\bar{a}) \leftrightarrow \phi(\bar{b})$.*
*For a $\Delta \subset L$, A is a **Δ-indiscernible sequence in** M if for every $\phi \in \Delta$, A is ϕ-indiscernible. If $\Delta = L$, we simply say A is an **indiscernible sequence in** M.*
*Given a set $C \subseteq M$, A is a **Δ-indiscernible sequence over** C **in** M if it is Δ_C-indiscernible (see Definition 1.5.1) i.e. for every $\phi(\bar{x}, \bar{y}) \in \Delta$, $\bar{c} \in C$ and $\bar{a}, \bar{b} \in [A]^n$, $M \models \phi(\bar{a}, \bar{c}) \leftrightarrow \phi(\bar{b}, \bar{c})$.*
*For a set Δ of formulas, A is a **Δ-n-indiscernible sequence in** M **over** C*

if it is Δ_n-indiscernible in M over C, where $\Delta_n = \{\phi(x_0, \ldots, x_{n-1}, \bar{y}) : \phi \in \Delta\}$ i.e. we consider only the formulas in Δ which (besides parameters) has n free variables.
Note that by definition, A is Δ-indiscernible iff it is Δ-n-indiscernible for every $n < \omega$.
*If A is a set which is a ϕ-indiscernible sequence under ANY linear ordering on A, then we say that A is a **ϕ-indiscernible set in M** i.e. for any finite $\bar{a} \in A$ of distinct elements, $M \models \phi(\bar{a})$. **Δ-indiscernible sets over C in M** are defined similarly.*
If $A = \{\bar{a}_i : i \in \eta, \bar{a}_i \in M^k\}$ for some linear order η and some $n \in \omega$ i.e. A is a ordered set of finite tuples from M, then the definition for ϕ-indiscernibility is given correspondingly for $\phi(\bar{x_0}, \ldots, \bar{x}_{n-1})$ with $|\bar{x_i}| = k$.
We omit "in M" when the model is clear from context.

Remark. Unless otherwise specified, we will adopt the convention where if A comes with some linear order then by "A is indiscernible" we mean that "A is an indiscernible sequence" under that order. Conversely, if A does not come with a linear order then we mean that "A is an indiscernible set".

The existence of indiscernible sequences is easily proved by:

Lemma 2.3.2. *Let M be a model, $I \subseteq M^n$ infinite but $A \subsetneq M$, $n < \omega$ and $\Delta \subsetneq L$ are all finite. Then there is an infinite and linearly ordered $(\bar{a}_i : i < \omega) \subseteq I$ which is a Δ-n-indiscernible sequence over A. In particular, for any formula ϕ there is an infinite ϕ-indiscernible sequence over A in I.*

Proof. Arbitrarily linearly order I, and assume $|I| = \aleph_0$ (say by taking only a countably infinite subset). Note by assumption, $\Delta_{A,n}$ (see the above definition) is finite and thus $2^{\Delta_{A,n}}$ is finite. Defining $f : [I]^n \longrightarrow 2^{\Delta_{A,n}}$ by $M \models \phi(\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{a})$ iff $\phi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{a}) \in f(\bar{a}_0, \ldots, \bar{a}_{n-1})$, Ramsey's theorem (Theorem A.0.1) guarantees that there is an infinite subset of $J \subseteq I$ such that f is constant on J. J is then the desired Δ-n-indiscernible sequence over A. □

The following property, first exposited by Ehrenfeucht in [Eh 57], is useful for studying stable theories, and makes heavy use of indiscernibility:

Definition 2.3.3. *Let $I \subseteq M^l$ be an infinite set of l-tuples of M. $\phi(\bar{x}_0, \ldots, \bar{x}_{n-1})$ is **connected over** I if for every choice of $\bar{a}_0, \ldots, \ldots a_{n-1} \in I$ of distinct tuples, there is a $\sigma \in S_n$ (where S_n is the permutation group of n elements) such that $M \models \phi(\bar{a}_{\sigma(0)}, \ldots, \bar{a}_{\sigma(n-1)})$.*
*$\phi(\bar{x}_0, \ldots, \bar{x}_{n-1})$ is **asymmetric over** I if for every choice of $\bar{a}_0, \ldots, \ldots a_{n-1} \in I$ of distinct tuples, there is a $\sigma \in S_n$ (where S_n is the permutation group of n elements) such that $M \models \neg\phi(\bar{a}_{\sigma(0)}, \ldots, \bar{a}_{\sigma(n-1)})$.*

Lemma 2.3.4. *If $\phi(\bar{x}_0, \ldots, \bar{x}_{n-1})$ is connected and asymmetric over I, then there is a $\sigma, \tau \in S_n$ and an infinite $J = (\bar{b}_i : i < \omega) \subseteq I$ such that for every $\zeta \in [\omega]^n$,*

$$M \models \phi(\bar{b}_{\zeta(\sigma(0))}, \ldots, \bar{b}_{\zeta(\sigma(n-1))})$$

$$M \models \neg\phi(\bar{b}_{\zeta(\tau(0))}, \ldots, \bar{b}_{\zeta(\tau(n-1))})$$

In this case, we say that ϕ is connected by σ over I and asymmetric by τ over I.

Proof. Let $I = (\bar{a}_i : i < \omega)$, and consider the map $f : [\omega]^n \longrightarrow S_n \times S_n$ with $f(\zeta) = (\sigma_\zeta, \tau_\zeta)$ such that

$$M \models \phi(\bar{a}_{\zeta(\sigma_\zeta(0))}, \ldots, \bar{a}_{\zeta(\sigma_\zeta(n-1))})$$

$$M \models \neg\phi(\bar{a}_{\zeta(\tau_\zeta(0))}, \ldots, \bar{a}_{\zeta(\tau_\zeta(n-1))})$$

Note σ_ζ, τ_ζ exists as ϕ is connected and asymmetric over I. Since $S_n \times S_n$ is finite, by Ramsey's theorem (Theorem A.0.1), there is an infinite $W \subseteq \omega$ such that f is constant on $[W]^n$. Letting $J = (\bar{a}_i : i \in W)$ and $\sigma = \sigma_\zeta, \tau = \tau_\zeta$ for $\zeta \in [W]^n$ gives the desired result. ☐

Lemma 2.3.5. *If there is a $M \models T$ with an infinite $I \subseteq M^l$, a $\bar{m} \in M$ and a formula $\phi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{m})$ which is asymmetric and connected over I, then there is an infinite indiscernible sequence J over $\bar{m}$ and a $\sigma, \tau \in S_n$ such that ϕ is connected by σ and asymmetric by τ over J.*

Proof. We will in fact take M to be an universal model here, so that any type is realized in M. For simplicity, we may assume WLOG that $I = (\bar{a}_i : i < \omega)$, and by the above lemma we may assume that there is $\sigma, \tau \in S_n$ such that for every $\zeta \in [\omega]^n$, I is connected by σ and I is asymmetric by τ.

List L by $(\psi_i : i < |L|)$, and define $\Delta_i = \{\psi_j : j < i\}$. We will construct inductively sequences I_i for $i \leq |L|$, which satisfies:

1. I_i is infinite
2. I_i is Δ_i-indiscernible over $\bar{m}$
3. ψ is connected by σ and asymmetric by τ over I_i
4. For $j < i$, if $I_j = (\bar{a}_k : k < \omega)$ and $I_i = (\bar{b}_k : k < \omega)$, for every $l < j$, $M \models \psi_l(\bar{a}_{\zeta(0)}, \ldots, \bar{a}_{\zeta(n-1)}, \bar{m})$ for every $\zeta \in [\omega]^n$ iff $M \models \psi_l(\bar{b}_{\zeta(0)}, \ldots, \bar{b}_{\zeta(n-1)}, \bar{m})$

For the base case, as $\Delta_0 = \emptyset$, let $I_0 = I$; for the successor case, if I_j satisfies the inductive hypothesis, by Lemma 2.3.2 let $I_{j+1} \subseteq I_j$ be an infinite ψ_{j+1}-indiscernible subsequence over $\bar{m}$. Note then ϕ is naturally connected by σ and asymmetric by τ over I_{j+1}, and (4) is satisfied as $I_{j+1} \subseteq I_j$, and I_j is ψ_l-indiscernible over $\bar{m}$ for $l < j$.

For the limit case $\delta < |L|$, let $(\bar{c}_i : i < \omega)$ be new constants and consider the set

$$\begin{aligned}\{\psi_j(\bar{c}_{\zeta(0)}, \ldots, \bar{c}_{\zeta(h(j)-1)}, \bar{m})^i : j < \delta, \zeta \in [\omega]^{h(j)}, M \models \psi_j(\bar{a}_0, \ldots, \bar{a}_{h(j)-1}, \bar{m})^i \\ \text{where } I_{j+1} = (\bar{a}_k : k < \omega)\} \\ \cup \{\phi(\bar{c}_{\zeta(\sigma(0))}, \ldots, \bar{c}_{\zeta(\sigma(n-1))}, \bar{m}) : \zeta \in [\omega]^n\} \\ \cup \{\neg\phi(\bar{c}_{\zeta(\tau(0))}, \ldots, \bar{c}_{\zeta(\tau(n-1))}, \bar{m}) : \zeta \in [\omega]^n\}\end{aligned}$$

Any finite subset is satisfiable by interpreting $(\bar{c}_i : i < \omega)$ as a suitable I_j for some $j < \delta$, and so by compactness this set of formulae is satisfiable in M by a $I_\delta = (\bar{c}_i^M : i < \omega)$. By construction, this satisfies all the conditions (1) – (4).

Since $\Delta_{|L|} = L$, taking $J = I_{|L|}$ then gives the desired indiscernible sequence. □

Proposition 2.3.6. *If there is a $M \models T$ with an infinite $I \subseteq M^l$, a $\bar{m} \in M$ and a formula $\phi(\bar{x}_0, \ldots, \bar{x}_{n-1}, \bar{m})$ which is asymmetric and connected over I, then T is not stable.*

Proof. Again, we will assume M is the universal model so that all types are realized in M. Using the above lemma, we may assume WLOG that $I = (\bar{a}_i : i < \omega)$ is countable, indiscernible over $\bar{m}$ and there is $\sigma, \tau \in S_n$ such that ϕ is connected by σ and asymmetric by τ over I.

So assume that T is stable. Let $\lambda = 2^{|T|}$, let λ_0 be the least such that $2^{\lambda_0} > \lambda$ and consider the set $J_1 = 2^{\lambda_0} - \{\mathbf{0}, \mathbf{1}\}$ (where $\mathbf{0}$ is the constant string of 0's and $\mathbf{1}$ the constant string of 1's) with the following ordering: if $\alpha < \lambda_0$ is the least such that $\nu(\alpha) \neq \zeta(\alpha), \nu(\alpha) = 0$, then $\nu < \zeta$. Note that as $\lambda_0 \leq \lambda$ is the least such that $2^{\lambda_0} > \lambda$, then $2^{<\lambda_0} \leq \lambda \cdot \lambda_0 \leq \lambda$. Thus, defining J to be the strings which are eventually constant, we see that:

1. $|J_1| > \lambda \geq |J|$
2. Neither J_1 nor J has endpoints
3. For every $\sigma, \tau \in J_1 - J$, there exists $\nu, \zeta, \xi \in J$ such that $\nu < \sigma < \zeta < \tau < \xi$

Since I is an infinite indiscernible sequence over $\bar{m}$, by compactness we can define $(\bar{b}_s : s \in J_1)$ such that for every $(s_1, \ldots, s_n) \in [J_1]^n$, $\bar{b}_{s_1} \frown \cdots \frown \bar{b}_{s_n}$ realizes the same type over $\bar{m}$ as $\bar{a}_0 \frown \cdots \frown \bar{a}_{n-1}$.

Define $B = \bar{m} \cup \bigcup\{\bar{b}_s : s \in J\}$, and for $s \in J_1 - J$ let $p_s = tp(\bar{b}_s/B)$. Note as $|B| \leq \lambda$, the stability of T (by Proposition 2.1.11, as $\lambda^{|T|} = \lambda$) implies that T is λ-stable, and so as $|J_1 - J| > \lambda$ there exists $s < t \in J_1 - J$ such that $p_s = p_t$.

WLOG, assume that $M \models \phi(\bar{a}_0, \ldots, \bar{a}_{n-1}, \bar{m})$ (otherwise replace ϕ by $\neg\phi$). As ϕ is asymmetric (or connected, if $\neg\phi$ is used) over I, there is a $\theta \in S_n$ such that $M \models \neg\phi(\bar{a}_{\theta(0)}, \ldots, \bar{a}_{\theta(n-1)}, \bar{m})$. So let θ be such that $r(\theta) = \min\{k : \theta(k) \neq k\}$ is maximal with $M \models \neg\phi(\bar{a}_{\theta(0)}, \ldots, \bar{a}_{\theta(n-1)}, \bar{m})$, which implies that if $r = r(\theta)$ then

$$M \models \neg\phi(\bar{a}_0, \ldots, \bar{a}_{r-1}, \bar{a}_{\theta(r)}, \ldots, \bar{a}_{\theta(n-1)}, \bar{m})$$

By property (3) above of J_1 and J, we can find $s_0 < \cdots < s_{n-1} \in J$ such that $s_{r-1} < s < s_r$ and $s_{\theta(r)-1} < t < s_{\theta(r)}$, and as $\bar{b}_{s_0} \frown \cdots \frown \bar{b}_{s_{n-1}}$ realize the same type over $\bar{m}$ as $\bar{a}_0 \frown \cdots \frown \bar{a}_{n-1}$, we see that

$$M \models \neg\phi(\bar{b}_{s_0}, \ldots, \bar{b}_{s_{r-1}}, \bar{b}_{s_{\theta(r)}}, \ldots, \bar{a}_{s_{\theta(n-1)}}, \bar{m})$$

So by the indiscernibility of $(\bar{b}_s : s \in J_1)$,

$$M \models \neg\phi(\bar{b}_{s_0}, \ldots, \bar{b}_{s_{r-1}}, \bar{b}_t, \bar{b}_{s_{\theta(r+1)}}, \ldots, \bar{a}_{s_{\theta(n-1)}}, \bar{m})$$

But as $p_s = p_t$,

$$M \models \neg\phi(\bar{b}_{s_0}, \dots, \bar{b}_{s_{r-1}}, \bar{b}_s, \bar{b}_{s_{\theta(r+1)}}, \dots, \bar{a}_{s_{\theta(n-1)}}, \bar{m})$$

So again by indiscernibility

$$M \models \neg\phi(\bar{b}_{s_0}, \dots, \bar{b}_{s_{r-1}}, \bar{b}_{s_r}, \bar{b}_{s_{\theta(r+1)}}, \dots, \bar{a}_{s_{\theta(n-1)}}, \bar{m})$$

Which contradicts that $r = r(\theta)$ is maximal. Therefore T is not stable. □

In fact, one can show that for a complete theory, having a formula which is asymmetric and connected over an infinite set is equivalent to having an unstable formula, hence T is unstable iff there is an infinite set and a formula which is asymmetric and connected over it (see [Sh 90] for details).

For our purposes, one of the main goals of introducing the notion of asymmetric and connected is the following result:

Proposition 2.3.7. *If T has a model M with an infinite $I \subseteq M$ that is an indiscernible sequence over some $A \subseteq M$ but not an indiscernible set over A, then T is unstable.*

Proof. Since I is not an indiscernible set over A, there is a formula $\phi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y})$, a $\bar{a} \in A$, $\bar{a}_0 < \dots < \bar{a}_{n-1} \in I$ and a $\sigma \in S_n$ such that

$$M \models \phi(\bar{a}_0, \dots, \bar{a}_{n-1}, \bar{a}) \wedge \neg\phi(\bar{a}_{\sigma(0)}, \dots, \bar{a}_{\sigma(n-1)}, \bar{a})$$

So let

$$\psi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{a}) = \phi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{a}) \wedge \neg\phi(\bar{x}_{\sigma(0)}, \dots, \bar{x}_{\sigma(n-1)}, \bar{a})$$

But as I is an indiscernible sequence over A, $M \models \psi(\zeta)$ for any $\zeta \in [I]^n$. Therefore ϕ is connected and asymmetric over I. By the above proposition, T is thus unstable. □

Corollary 2.3.8. *We can weaken the above proposition by requiring only that I is a Δ-indiscernible sequence over A but not a Δ-indiscernible set over A, where Δ is closed under permutation of variables i.e. for every $\phi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y}) \in \Delta$ and $\sigma \in S_n$, $\phi(\bar{x}_{\sigma(0)}, \dots, \bar{x}_{\sigma(n-1)}, \bar{y}) \in \Delta$.*

Proof. Again, if the assumption is true then there is a is a formula $\phi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y})$, a $\bar{a} \in A$, $\bar{a}_0 < \dots < \bar{a}_{n-1} \in I$ and a $\sigma \in S_n$ such that

$$M \models \phi(\bar{a}_0, \dots, \bar{a}_{n-1}, \bar{a}) \wedge \neg\phi(\bar{a}_{\sigma(0)}, \dots, \bar{a}_{\sigma(n-1)}, \bar{a})$$

Since Δ is closed under permutation of variables, the formula

$$\psi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y}) = \phi(\bar{a}_{\sigma(0)}, \dots, \bar{a}_{\sigma(n-1)}, \bar{y})$$

is also in Δ, and therefore as I is a Δ-indiscernible sequence, $M \models \phi(\zeta), \psi(\zeta)$ for any $\zeta \in [I]^n$. Therefore ϕ is connected an asymmetric over I, as in the proof above. □

This proposition gives a taste of how indiscernible sequences are used when T is stable. An application is:

Proposition 2.3.9. *For a complete theory T and a model $M \models T$, suppose $C \subseteq M$ is a set and $D = \{d_\alpha : \alpha < |D|\} \subseteq M$ a sequence. Let $p_\alpha = tp(d_\alpha/C \cup \{d_\beta : \beta < \alpha\})$ and suppose for $\beta < \alpha$, $p_\beta \subseteq p_\alpha$. Moreover suppose p_α does not split over C. Then $\{d_\alpha : \alpha < \kappa\}$ is an indiscernible sequence over C.*

Proof. By induction on the length of the tuple $\bar{d} \in [D]^{<\omega}$: for the base case, suppose $tp(d_\alpha/C) \neq tp(d_\beta/C)$, $\beta < \alpha$. So there is a $\phi(x) \in L_C$ such that $\models \phi(d_\alpha) \wedge \phi(d_\beta)$. But $p_\beta \subseteq p_\alpha$ implies $p_\beta|_C = p_\alpha|_C$, a contradiction.

For the inductive case, suppose every $\bar{d} \in [D]^n$ has the same type over C. Assume for a contradiction that there are $(\bar{d}_0, d'_0), (\bar{d}_1, d'_1) \in [D]^{n+1}$ such that $tp(\bar{d}_0 \frown d'_0/C) \neq tp(\bar{d}_1 \frown d'_1/C)$. Supposing WLOG that $d'_1 = d_\alpha$ and $d'_0 = d_\beta$, $\beta < \alpha$, thus there is a $\phi(y_0, \ldots, y_{n-1}, x) \in L_C$ such that $M \models \phi(\bar{d}_0, d'_0) \wedge \neg\phi(\bar{d}_1, d'_1)$. Now by the inductive hypothesis $tp(\bar{d}_0/C) = tp(\bar{d}_1/C)$, so as p_α does not split over C, $\neg\phi(\bar{d}_1, x) \in p_\alpha$ implies $\neg\phi(\bar{d}_0, x) \in p_\alpha$. But as $p_\beta \subseteq p_\alpha$, therefore $M \models \neg\phi(\bar{d}_0, d'_0)$, a contradiction. Thus every $\bar{d} \in [D]^{n+1}$ realizes the same type over C, completing the induction. Thus D is an indiscernible sequence over C. □

Corollary 2.3.10. *If T is stable, then D is in fact an indiscernible set over C*

Proof. Follows directly from Proposition 2.3.7. □

The following technical lemma will be used later to prove a useful result, but since we have all the concepts needed to prove the lemma we will do so here.

Lemma 2.3.11. *For a n, ω, let $\Delta_1, \ldots, \Delta_{n-1}$ be sets of formulas, each of which are closed under permutation of variables (i.e. if $\phi(x_0, \ldots, x_{n-1}) \in \Delta_k$, then for every $\sigma \in S_n$, $\phi(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}) \in \Delta$). Given a model $M \models T$, if $I = (\bar{a}_i : i < \alpha) \subseteq M$ is a sequence of tuples from M, $A \subseteq M$ a set and defining $A_i = A \cup \bigcup_{j<i} A_j$, assume that for every $i < \alpha$, $2 \leq k+1 < n$, $p_i = tp(\bar{a}_i/A_i)$ does not (Δ_k, Δ_{k+1})-split over A. If in addition $p_j|_{\Delta_k} \subseteq p_i|_{\Delta_k}$ for $j < i < \alpha$ and $1 \leq k < n$, then for every $1 \leq k < n$, I is a Δ_k-k-indiscernible sequence over A.*

Proof. Suppose $\zeta, \xi \in [\alpha]^k$, $\phi(\bar{x}_0, \ldots, \bar{x}_{k-1}, \bar{y}) \in \Delta_k$ and $\bar{c} \in A$, so that it suffices to show

Claim. $M \models \phi(\bar{a}_{\zeta(0)}, \ldots, \bar{a}_{\zeta(k-1)}, \bar{c})$ iff $M \models \phi(\bar{a}_{\xi(0)}, \ldots, \bar{a}_{\xi(k-1)}, \bar{c})$

We will proceed by induction on k: For $k = 1$, since $p_0|_{\Delta_1} = p_{\zeta(0)}|_{A,\Delta_1} = p_{\xi(0)}|_{A,\Delta_1}$, thus the claim is true. Inductively, if the claim is true for k with $k + 1 < n$, let $\beta = \max(\zeta(k), \xi(k))$ where $\zeta, \xi \in [\alpha]^{n+1}$. By the inductive hypothesis $tp_{\Delta_k}(\bar{a}_{\zeta(0)} \frown \cdots \frown \bar{a}_{\zeta(k-1)}/A) = tp_{\Delta_k}(\bar{a}_{\xi(0)} \frown \cdots \frown \bar{a}_{\xi(k-1)}/A)$, and in addition p_β does not (Δ_k, Δ_{k+1})-split over A. So for $\phi(\bar{x}_0, \ldots, \bar{x}_{k-1}, \bar{x}, \bar{y}) \in \Delta_{k+1}$, since

$\bar{c} \in A$, $\phi(\bar{a}_{\zeta(0)}, \ldots, \bar{a}_{\zeta(k-1)}, \bar{x}, \bar{c}) \in p_\beta$ iff $\phi(\bar{a}_{\xi(0)}, \ldots, \bar{a}_{\xi(k-1)}, \bar{x}, \bar{c}) \in p_\beta$. Finally, as $p_{\zeta(k)}|_{\Delta_{k+1}}, p_{\xi(k)}|_{\Delta_{k+1}} \subseteq p_\beta|_{\Delta_{k+1}}$

$$\begin{aligned}
M \models \phi(\bar{a}_{\zeta(0)}, \ldots, \bar{a}_{\zeta(k)}, \bar{c}) &\Leftrightarrow \phi(\bar{a}_{\zeta(0)}, \ldots, \bar{a}_{\zeta(k-1)}, \bar{x}, \bar{c}) \in p_{\zeta(k)}|_{\Delta_{k+1}} \\
&\Leftrightarrow \phi(\bar{a}_{\zeta(0)}, \ldots, \bar{a}_{\zeta(k-1)}, \bar{x}, \bar{c}) \in p_\beta|_{\Delta_{k+1}} \\
&\Leftrightarrow \phi(\bar{a}_{\xi(0)}, \ldots, \bar{a}_{\xi(k-1)}, \bar{x}, \bar{c}) \in p_\beta|_{\Delta_{k+1}} \\
&\Leftrightarrow \phi(\bar{a}_{\xi(0)}, \ldots, \bar{a}_{\xi(k-1)}, \bar{x}, \bar{c}) \in p_{\xi(k)}|_{\Delta_{k+1}} \\
&\Leftrightarrow M \models \phi(\bar{a}_{\xi(0)}, \ldots, \bar{a}_{\xi(k-1)}, \bar{a}_{\xi(k)}, \bar{c})
\end{aligned}$$

This completes the inductive proof. □

Indiscernible sets are a versatile tool, as illustrated by the following lemmas:

Lemma 2.3.12. *Suppose T is λ-stable, $M \models T$, $C \subseteq M$ and $X \subseteq M$ is an indiscernible set over C in M. If λ_0 is the least such that $2^{\lambda_0} > \lambda$, then for any $m \in M$, there exists $X' \subsetneq X$, $|X'| < \lambda_0$ such that $X - X'$ is an indiscernible set over $C \cup \{m\}$ in M.*

Proof. WLOG we may replace M by a κ-saturated elementary extension for some large κ so that all types over the sets we are interested in are realized in M (e.g. take M to be the universal model). Suppose there exists C, X, m contradicting the claim. We will construct a tree of types satisfying the assumptions of Lemma 2.1.14, thus contradicting that T is λ-stable.

For $\nu \in 2^{<\lambda_0}$, we will construct an elementary map f_ν with domain $D'_\nu \subseteq C \cup X$, $D'_\nu \supseteq C$ inductively by the length of ν, requiring that $|D'_\nu \cap X| < \lambda_0$, f_ν has range in M and that if σ is an initial segment of ν, then $D'_\sigma \subseteq D'_\nu$ and $f_\sigma = f_\nu|_{D'_\sigma}$. Let $D'_{<>} = C$, $f_{<>} = \mathrm{id}_C$, and note that as X is an indiscernible set over C, unless $|X| = 1$ then $C \cap X = \emptyset$, and so in any case $|C \cap X| < \lambda_0$, satisfying the inductive hypothesis. Also, for any ν with length a limit ordinal δ, let $D'_\nu = \bigcup_{\alpha<\delta} D'_{\nu|_\alpha}$ and $f_\nu = \bigcup_{\alpha<\delta} f_{\nu|_\alpha}$.

So assume that D'_ν, f_ν has been constructed. Since we assume that C, X, m contradicts the claim of the lemma, as $|D'_\nu \cap X| < \lambda_0$, $X - D'_\nu$ is not indiscernible over $C \cup \{m\}$ and so there exists tuples $\bar{d}_0, \bar{d}_1 \in X - D'_\nu$ and a $\phi(x, \bar{y}) \in L_C$ such that $M \models \phi(m, \bar{d}_0) \wedge \neg\phi(m, \bar{d}_1)$. Let $D'_{\nu\frown i} = D_\nu \cup \bar{d}_i$, and we note:

Claim. $tp^M(\bar{d}_0/D'_\nu) = tp^M(\bar{d}_1/D'_\nu)$

For every $\phi(\bar{x}, \bar{y}, \bar{z}) \in L$, $\bar{c} \in C$ and $\bar{d}' \in D'_\nu - C$, note as X is indiscernible over C and $\bar{d}', \bar{d}_0, \bar{d}_1 \in X$, $tp^M(\bar{d}_0 \frown \bar{d}'/C) = tp^M(\bar{d}_1 \frown \bar{d}'/C)$ and therefore $M \models \phi(\bar{d}_0, \bar{d}', \bar{c})$ iff $M \models \phi(\bar{d}_1, \bar{d}', \bar{c})$ i.e. $tp^M(\bar{d}_0/D'_\nu) = tp^M(\bar{d}_1/D'_\nu)$

Since f_ν has range in M, which we assume to be sufficiently saturated, the following type:

$$q_i = \{\phi(\bar{x}, f_\nu(\bar{d}')) : \phi \in L, \bar{d} \in D'_\nu, M \models \phi(\bar{d}_i, \bar{d}')\}$$

is realized in M by some tuple $\bar{d}$, and we define $f_{\nu\frown i}$ to extend f_ν by having $f_{\nu\frown i}(\bar{d}_i) = \bar{d}$. Moreover, as $tp^M(\bar{d}_0/D'_\nu) = tp^M(\bar{d}_1/D'_\nu)$ we see that $q_0 = q_1$, which implies that we can define $f_{\nu\frown 0}(\bar{d}_0) = f_{\nu\frown 1}(\bar{d}_1) = \bar{d}$. Since $\bar{d}_0, \bar{d}_1$ are finite, if $|D'_\nu \cap X| < \lambda_0$ then $|D'_{\nu^\frown i} \cap X| < \lambda_0$. Moreover, since $D'_{\nu^\frown i} - D_\nu$ is always finite, this guarantees that $|D'_\nu| \leq \aleph_0 \cdot l(\nu)$ and so the cardinality condition is satisfied even at limit stages. This completes the inductive construction.

So define $D_\nu = f_\nu(D'_\nu)$, and $p_\nu = \{\phi(x, f_\nu(\bar{d})) : \bar{d} \in D'_\nu, M \models \phi(m, \bar{d})\}$. That f_ν is an elementary map guarantees that $p_\nu \in S_1^M(D_\nu)$, and for every ν by construction there is $\bar{d}_0 \in D'_{\nu^\frown 0}$, $\bar{d}_1 \in D'_{\nu^\frown 1}$ and a $\phi \in L_C$ such that $M \models \phi(m, \bar{d}_0) \wedge \neg\phi(m, \bar{d}_1)$. Therefore $tp^M(m/D_{\nu^\frown 0}) \neq tp^M(m/D_{\nu^\frown 1})$, and so $p_{\nu^\frown 0} \neq p_{\nu^\frown 1}$. Finally, that $f_{\nu^\frown 0}(\bar{d}_0) = f_{\nu^\frown 1}(\bar{d}_1)$ guarantees by induction that $D_{\nu^\frown 0} = D_{\nu^\frown 1}$, and thus D_ν, p_ν for $\nu \in 2^{<\lambda_0}$ satisfies the conditions of Lemma 2.1.14, contradicting that T is λ-stable. □

Corollary 2.3.13. *In fact, we can have $|X'| < \mu(T)$.*

Proof. The above proof is valid by replacing λ_0 with $\mu(T)$, by definition of $\mu(T)$ (see Definition 2.1.15). □

Corollary 2.3.14. *If the assumptions of the above lemma hold, $\kappa > \mu(T)$ and $D \subseteq M$ is such that $|D| < \kappa$, then there is a $X' \subsetneq X$ with $|X'| < \kappa$ such that $X - X'$ is an indiscernible set over $C \cup D$. If $\mu(T)$ is regular, then this applies even for the case of $\kappa = \mu(T)$.*

Proof. List $D = \{d_\alpha : \alpha < |D|\}$, and for $\alpha < |D|$ define inductively X_α with $|X_\alpha| < \kappa$ such that $X - X_\alpha$ is indiscernible over $C \cup \{d_\beta : \beta < \alpha\}$:

- Let $X_0 = \emptyset$
- If X_α is defined, let $Y_{\alpha+1} \subsetneq X - X_\alpha$ be such that $|Y_{\alpha+1}| < \mu(T)$ and $(X - X_\alpha) - Y_{\alpha+1}$ is indiscernible over $C \cup \{d_\beta : \beta < \alpha + 1\}$ by the above lemma. Then let $X_{\alpha+1} = X_\alpha \cup Y_{\alpha+1}$, which guarantees that $X_\alpha \subseteq X_{\alpha+1}$, $|X_{\alpha+1}| < |X_\alpha| + |Y_{\alpha+1}| < \kappa$ and $X - X_{\alpha+1}$ is indiscernible over $C \cup \{d_\beta : \beta < \alpha + 1\}$.
- For a limit ordinal $\delta < |D|$, let $X_\delta = \bigcup_{\alpha<\delta} X_\alpha$. Note that $X_\delta = \bigcup_{\alpha<\delta} X_{\alpha+1} - X_\alpha$, and by construction each $|X_{\alpha+1} - X_\alpha| = |Y_{\alpha+1}| < \mu(T)$. So if $\kappa > \mu(T)$ then $|X_\delta| \leq |\delta| \cdot \mu(T) < \kappa$, whereas if $\kappa = \mu(T)$ is regular then $|X_\delta| \leq \sum_{\alpha<\delta} |X_{\alpha+1} - X_\alpha| < \mu(T) = \kappa$. Moreover since $X - X_\delta = \bigcap_{\alpha<\delta} X - X_\alpha$, $X - X_\delta$ is indiscernible over $C \cup \{d_\alpha : \alpha < \delta\}$.

Taking $X' = \bigcup_{\alpha<|D|} X_\alpha$ then gives the desired indiscernible set $X - X'$. □

2.4 Definability of types and Rank of formulas

The idea of definable types is due to Shelah, and has wide applications in stability theory, a few of which will be useful for us. For this section, we will work with a fixed complete theory T and a universal model $\bar{M}$ of T, such that by a set or a tuple (unless otherwise specified) we will mean a set or a tuple in $\bar{M}$ (which is much smaller than the universal model).

Definition 2.4.1. *For p a set of formulas with parameters in m variables and $\phi(x_0, \ldots, x_{m-1}, \bar{y})$ a formula in L, the ϕ-2-**Rank of** p $R^2_\phi(p)$ is either -1, an ordinal or ∞ (where we consider $\alpha < \infty$ for any ordinal α) and defined inductively by:*

- $R^2_\phi(p) \geq 0$ *if p is a type i.e. p is satisfiable.*
- *For a limit ordinal δ, $R^2_\phi(p) \geq \delta$ if for every $\alpha < \delta$, $R^2_\phi(p) \geq \alpha$*
- $R^2_\phi(p) \geq \alpha + 1$ *if for every finite $q \subseteq p$, there is a tuple $\bar{a}$ in the universal model, such that $R^2_\phi(q \cup \{\phi(\bar{x}, \bar{a})\}), R^2_\phi(q \cup \{\neg\phi(\bar{x}, \bar{a})\}) \geq \alpha$*

*When ϕ is clear from context, we will simply write $R(p)$ and call it the **Rank** of p.*

Note. In [Sh 90], Shelah gives a much more general treatment of ranks which is too general for our purposes. In Shelah's notation, the ϕ-2-rank is denoted $R^m(-, \phi, 2)$.

Lemma 2.4.2. *Let $\phi(x_0, \ldots, x_{m-1}, \bar{y})$ be any formula.*

1. *If p_1, p_2 are sets of formulas with parameters in m variables such that $p_1 \models p_2$, then $R^2_\phi(p_1) \leq R^2_\phi(p_2)$*
2. *For any p a set of formulas with parameters in m variables, there is a finite $q \subseteq p$ such that $R^2_\phi(q) = R^2_\phi(p)$*
3. *If $R^2_\phi(p) = \alpha < \infty$, then there is no tuple $\bar{a}$ such that $R^2_\phi(p \cup \{\phi(\bar{x}, \bar{a})^i\}) \geq \alpha$ for $i = 0, 1$*
4. *If $q \subseteq p \in S_{\phi,m}(A)$ is such that $R^2_\phi(q) = R^2_\phi(p)$, then p is the unique extension of q in $S_{\phi,m}(A)$.*

Proof.

1. We will prove the claim by induction on ordinals: the claim is trivial for $\alpha = 0$ and α a limit ordinal; and if $R^2_\phi(p_1) \geq \alpha + 1$, note that by compactness if $p_1 \models p_2$ then for every finite $q_2 \subseteq p_2$ there is a finite $q_1 \subseteq p_1$ such that $q_1 \models q_2$. Thus definition there is a tuple $\bar{a}$ such that $R^2_\phi(q_1 \cup \{\phi(\bar{x}, \bar{a})^i\}) \geq \alpha$ for $i = 0, 1$. Since $q_1 \cup \{\phi(\bar{x}, \bar{a})^i\} \models q_2 \cup \{\phi(\bar{x}, \bar{a})^i\}$ by construction, by the inductive hypothesis $R^2_\phi(q_2 \cup \{\phi(\bar{x}, \bar{a})^i\}) \geq \alpha$. Therefore $R^2_\phi(p_2) \geq \alpha + 1$.
2. By (1), if $q \subseteq p$ then $R^2_\phi(p) \leq R^2_\phi(q)$. Thus if $R^2_\phi(p) = \infty$, then we can take $q \subseteq p$ arbitrarily in p to prove the claim. Otherwise, if $R^2_\phi(p) = \alpha < \infty$, by definition there is some finite $q \subseteq p$ such that there is no tuple $\bar{a}$ with $R^2_\phi(q \cup \{\phi(\bar{x}, \bar{a})\}) \geq \alpha$ and $R^2_\phi(q \cup \{\phi(\bar{x}, \bar{a})\}) \geq \alpha$. This implies that $R^2_\phi(q) \ngeq \alpha + 1$, so again by (1) $R^2_\phi(q) = R^2_\phi(p)$.

3. If such a tuple $\bar{a}$ does exist, then for every finite $q \subseteq p$, $R^2_\phi(q \cup \{\phi(\bar{x},\bar{a})^i\}) \geq \alpha$ by (1), and therefore $R^2_\phi(p) \geq \alpha + 1$, a contradiction.

4. If $R^2_\phi(q) = R^2_\phi(p) = \alpha$, suppose that for an $\bar{a} \in A$, $\phi(\bar{x},\bar{a}) \in p$. Then by (1), $\alpha = R^2_\phi(p) \leq R^2_\phi(q \cup \{\phi(\bar{x},\bar{a})\}) \leq R^2_\phi(q) = \alpha$, and therefore $R^2_\phi(q \cup \{\phi(\bar{x},\bar{a})\}) = \alpha$. Conversely, suppose that $\neg\phi(\bar{x},\bar{a}) \in p$, so again $R^2_\phi(q \cup \{\neg\phi(\bar{x},\bar{a})\}) = \alpha$. But by (3), as $R^2_\phi(q) = \alpha$, there is no $\bar{a}$ such that $R^2_\phi(q \cup \{\phi(\bar{x},\bar{a})\}), R^2_\phi(q \cup \{\neg\phi(\bar{x},\bar{a})\}) \geq \alpha$, and so only one of the two conditions hold. Therefore $p = q \cup \{\phi(\bar{x},\bar{a}) : R^2_\phi(q \cup \{\phi(\bar{x},\bar{a})\}) = \alpha\}$

□

Lemma 2.4.3. *For every $n < \omega$, $\phi(x_0, \ldots, x_{m-1}, \bar{y})$ a formula and p a set of formulas with parameters in m variables, then $R^2_\phi(p) \geq n$ iff the set*

$$\Gamma_\phi(p,n) = \{\psi(\bar{x}_\eta, \bar{a}) : \psi(\bar{x},\bar{a}) \in p, \eta \in 2^n\} \cup \{\phi(\bar{x}_\eta, \bar{y}_{\eta|_k})^{\eta(k)} : \eta \in 2^n, k < n\}$$

is satisfiable.

Proof. For the forward direction, by compactness, it is sufficient to show that for every finite $q \subseteq p$, $\Gamma_\phi(q,n)$ is satisfiable. So WLOG assume that p is finite. Inductively, we will define $\bar{a}_\sigma$ for $\sigma \in 2^{<n}$ such that if $l(\sigma) = k < n$ then $R(p_\sigma) \geq n - k$ where $p_\sigma = p \cup \{\bar{x} = \bar{x}\} \cup \{\phi(\bar{x}, \bar{a}_{\sigma|_j})^{\sigma(j)} : j < k\}$:

- For $k = 0$, the condition is simply $R(p) \geq n$, which is true by assumption.
- If $\bar{a}_\sigma$ has been defined for $\sigma \in 2^k$, $k + 1 \leq n$, then by the inductive hypothesis $R(p_\sigma) \geq n - k$. Therefore (as p_σ is finite) there is a tuple $\bar{a}$ in $\bar{M}$ such that $R(p_\sigma \cup \{\phi(\bar{x},\bar{a})^i\}) \geq n - k - 1$ for $i = 0, 1$. Defining $\bar{a}_{\sigma^\frown i} = \bar{a}$ thus satisfies the inductive hypothesis.

This construction implies that if $\eta \in 2^n$, then $R(p_\eta) \geq 0$ and thus p_η is satisfiable. Let $\bar{c}_\eta \in \bar{M}$ realize p_η, interpret $\bar{x}_\eta$ as $\bar{c}_\eta$ and for $k < n$, interpret $\bar{y}_{\eta|_k}$ as $\bar{a}_{\eta|_k}$. This shows that $\Gamma_\phi(p,n)$ is satisfiable.

For the backwards direction, we proceed by induction downwards from n: for $n = 0$, if $\Gamma_\phi(p, 0)$ is satisfiable then p is satisfiable and thus $R^2_\phi(p) \geq 0$. Then for $n > 0$, if $\Gamma_\phi(p,n)$ is satisfiable, then for any $\eta \in 2^{n-1}$, $p \cup \{\phi(\bar{x}, \bar{y}_{\eta|_k})^{\eta(k)} : k < n-1\} \cup \{\phi(\bar{x}, \bar{y}_\eta)^i\}$ is satisfiable for $i = 0, 1$, and therefore $R^2_\phi(p \cup \{\phi(\bar{x}, \bar{y}_{\eta|_k}) : k < n-1\}) \geq 1$. Repeating this step n times yields that $R^2_\phi(p) \geq n$. □

Lemma 2.4.4. *If $R^2_\phi(\bar{x} = \bar{x}) \geq \omega$, then ϕ is an unstable formula.*

Proof. For every $n < \omega$, note that the set $\Gamma_\phi(\bar{x} = \bar{x}, n)$ in the above lemma is satisfiable iff the set

$$\Gamma_\phi(n) = \{\phi(\bar{x}_\eta, \bar{y}_{\eta|_k})^{\eta(k)} : \eta \in 2^n, k < n\}$$

is satisfiable. But this definition of $\Gamma_\phi(n)$ coincides with the definition in the proof of Lemma 2.1.9, and so by the same proof we see that ϕ is unstable. □

Definition 2.4.5.

- *Given a complete $\phi(\bar{x},\bar{y})$-m-type p over some set A, a tuple $\bar{c}$ and a formula $\psi(\bar{y},\bar{z})$, p is $\psi(\bar{y},\bar{c})$-**defined** if $\phi(\bar{x},\bar{a}) \in p$ iff $\bar{M} \models \psi(\bar{a},\bar{c})$*
- *p is $(\psi(\bar{y},\bar{z}),C)$-**definable** if there is a $\bar{c} \in C$ such that p is $\psi(\bar{y},\bar{c})$-defined.*
- *p is C-**definable** if there is some formula $\psi(\bar{y},\bar{z})$ such that p is (ψ,C)-defined.*
- *If p is a complete Δ-m-type over some set A and C is a set, p is C-**definable** if $p|_\phi$ is C-definable for every $\phi \in \Delta$.*

Lemma 2.4.6. *For any formula $\phi(\bar{x},\bar{y})$ with $l(\bar{x}) = m$, if $R^2_\phi(\bar{x}=\bar{x}) < \omega$ then there is a $\psi(\bar{y},\bar{z})$ such that for every set A (with $|A| > 1$), every $p \in S_{\phi,m}(A)$ is (ψ,A)-definable. In particular, every ϕ-type over A is A-definable.*

Proof. Let A be a set (with $|A| > 1$), and $p \in S_{\phi,m}(A)$. Since $(\bar{x}=\bar{x}) \in p$, by Lemma 2.4.2(1), $R(p) \le R(\bar{x}=\bar{x}) < \omega$. Then by Lemma 2.4.2(2), there is a finite $q \subseteq p$ such that $R(q) = R(p) = k < \omega$. Since q is a ϕ-type over A, we may assume $q = \{\phi(\bar{x},\bar{a}_l)^{\eta(l)} : l < l(\eta)\}$ for some $\eta \in 2^{<\omega}$. Now, for every $\bar{a} \in A$ let $q(\bar{a}) = q \cup \{\phi(\bar{x},\bar{a})\}$. Consider the cases:

- If $\phi(\bar{x},\bar{a}) \in p$, then $q(\bar{a}) \subseteq p$ and thus by $R(p) \le R(q(\bar{a})) \le R(q) = R(p) = k$, so $R(q(\bar{a})) = k$.
- If $\neg\phi(\bar{x},\bar{a}) \in p$, then $R(p) \le R(q \cup \{\neg\phi(\bar{x},\bar{a})\}) \le R(q) = R(p) = k$. By Lemma 2.4.2(3), if $R(q(\bar{a})) \ge k$ then $R(q) \ge k+1$, a contradiction. Thus $R(q(\bar{a})) < k$.

We have thus shown that $R(q(\bar{a})) \ge k$ iff $\phi(\bar{x},\bar{a}) \in p$. But by Lemma 2.4.3, $R(q(\bar{a})) \ge k$ iff $\Gamma_\phi(q(\bar{a}),k)$ is satisfiable. So let $\theta(\bar{x},\bar{c}) = \bigwedge q$ (so that $\bar{c} \in A$), and define

$$\psi(\bar{y},\bar{c}) = \exists_{\sigma \in 2^{<k}} \bar{z}_\sigma \bigwedge_{\eta \in 2^k} (\exists \bar{x}\theta(\bar{x},\bar{c}) \wedge \phi(\bar{x},\bar{y}) \wedge \bigwedge_{0 \le l < k} \phi(\bar{x},\bar{z}_{\eta|_l})^{\eta(l)})$$

Note that in this case $\Gamma_\phi(q(\bar{a}),k)$ is satisfiable iff $\bar{M} \models \psi(\bar{a},\bar{c})$. Thus we have shown that $\bar{M} \models \psi(\bar{a},\bar{c})$ iff $\Gamma_\phi(q(\bar{a}),k)$ is satisfiable iff $R(q(\bar{a})) \ge k$ iff $\phi(\bar{x},\bar{a}) \in p$ i.e. p is (ψ,A)-definable.

To prove the lemma, we need to show that we can choose ψ such that it is independent of p and A. Note that if we can show that there is some finite $\Delta = \{\psi_i(\bar{y},\bar{z}) : i < n\}$ such that for every A (with $|A| > 1$) and $p \in S_{m,\phi}(A)$, p is (ψ_i,A)-definable for some $i < n$, then

$$\psi(\bar{y},\bar{z}_0,\ldots,\bar{z}_{n-1},z',z'_0,\ldots,z'_{n-1}) = \bigwedge_{0 \le l < n} (z' \ne z'_l \to \psi_l(\bar{y},\bar{z}_l))$$

is such that p is (ψ,A)-definable.

So suppose for a contradiction that such a finite Δ does not exist. If $l(\bar{y}) = n$ in $\phi(\bar{x}, \bar{y})$, then let P be a new unary relation symbol, $b_0, \ldots, b_{m-1}$ be new constant symbols and for any $\Delta \subseteq L$ of formulae of the form $\psi(\bar{y}, \bar{z}_\psi)$ with $l(\bar{y}) = n$, define

$$T_\Delta = T \cup \{\neg \exists z_1, \ldots, z_{l(\bar{z}_\psi)}(\bigwedge_{i=1}^{l(\bar{z}_\psi)} P(z_i) \wedge$$
$$(\forall y_1, \ldots, y_n((\bigwedge_{j=1}^{n} P(y_j)) \rightarrow (\phi(\bar{b}, \bar{y}) \leftrightarrow \psi(\bar{y}, \bar{z}))))) : \psi \in \Delta\}$$

i.e. if $M \models T_\Delta$, then $M \models T$ and for every $\bar{m} \in P^M$, there is a $\bar{m}' \in P^M$ such that $M \not\models \phi(\bar{b}, \bar{m}') \leftrightarrow \psi(\bar{m}', \bar{m})$. Now, if Δ is finite, then by assumption there is a set A and a $p \in S_{\phi,m}(A)$ such that p is not (ψ, A)-definable for any $\psi \in \Delta$. So if $M \models T$ with $A \subseteq M$ and $\bar{m} \in M$ realizes p, then defining $P^M = A$ and $\bar{b} = \bar{m}$ satisfies T_Δ. Thus T_Δ is satisfiable for any finite Δ. But then by compactness, if $\Delta_0 \subseteq L$ is the set of all formulae in the form $\psi(\bar{y}, \bar{z})$ with $l(\bar{y}) = n$, then T_{Δ_0} is also satisfiable.

Thus let $N \models T_{\Delta_0}$, and consider the type $p = tp_\phi^N(\bar{b}^N / P^N)$: since $N \models T$, we have shown above that there is some formula ψ which uses neither P nor $\bar{b}$ and such that p is (ψ, P^N)-definable, which contradicts the construction of T_{Δ_0} as $\psi \in \Delta_0$. This completes the proof by contradiction, and thus the lemma is proven. $\square$

Corollary 2.4.7.

1. *Given $\phi(\bar{x}, \bar{y})$ and $k < \omega$, for every $\theta(\bar{x}, \bar{y})$ there is a formula $\psi(\bar{y})$ such that for every tuple $\bar{a}$, $R_\phi^2(\theta(\bar{x}, \bar{a})) \geq k$ iff $\bar{M} \models \psi(\bar{a})$*

2. *Given $\phi(\bar{x}, \bar{y})$ and $\theta(\bar{x}, \bar{z})$, there is a $\psi(\bar{y}, \bar{z})$ such that for every ϕ-type p, if $\theta(\bar{x}, \bar{c}) \in p$ and $R_\phi^2(p) = R_\phi^2(\theta(\bar{x}, \bar{c})) < \omega$ then $p|_\phi$ is $\psi(\bar{y}, \bar{c})$-definable.*

Proof.

1. As in the above proof, consider the formula

$$\psi(\bar{y}) = \exists_{\sigma \in 2^{<k}} \bar{z}_\sigma \bigwedge_{\eta \in 2^k} (\exists \bar{x} \theta(\bar{x}, \bar{y}) \wedge \bigwedge_{0 \leq l < k} \phi(\bar{x}, \bar{z}_{\eta|_l})^{\eta(l)})$$

   Then by the same reasoning, $\bar{M} \models \psi(\bar{b})$ iff $\Gamma_\phi(\theta(\bar{x}, \bar{b}), k)$ is satisfiable iff $R(\theta(\bar{x}, \bar{b})) \geq k$ (by Lemma 2.4.3).

2. Let $k = R(p) = R(\theta(\bar{x}, \bar{c}))$, and let $q(\bar{a}) = \{\theta(\bar{x}, \bar{c}), \phi(\bar{x}, \bar{a})\}$. Then by the same reasoning as the proof of the above lemma, $R(q(\bar{a})) \geq k$ iff $\phi(\bar{x}, \bar{a}) \in p$. The rest of the proof that the $\psi(\bar{y}, \bar{z})$ defined as above gives $p|_\phi$ is $\psi(\bar{y}, \bar{c})$-definable in exactly the same manner.

$\square$

Theorem 2.4.8. *If there is some set A and a m-type $p \in S_m(A)$ which is not A-definable, then T is unstable.*

Proof. Let p, A be as stated. We may assume $|A| > 1$, as any type over a singleton set is trivially A-definable. So if p is not A-definable, then there is some formula $\phi(\bar{x}, \bar{y})$ (with $l(\bar{x}) = m$) such that $p|_\phi$ is not A-definable. Then by the above lemma, $R^2_\phi(\bar{x} = \bar{x}) \geq \omega$, and therefore by Lemma 2.4.4 ϕ is an unstable formula. By Proposition 2.1.11, T is therefore unstable. $\square$

Corollary 2.4.9. *If T is stable, then every complete type is definable over its set of parameters. Even more, for every formula $\phi(\bar{x}, \bar{y})$, there is a formula $\psi_\phi(\bar{y}, \bar{z})$ such that every ϕ-type p over a set A is (ψ_ϕ, A)-definable.*

Proof. ψ_ϕ is the formula constructed in the above lemma. $\square$

We will see in the upcoming section an important use of definable types, but for now we will prove one more useful result:

Lemma 2.4.10. *Let T be stable and $\Delta \subsetneq L$ be finite. Then there exists a finite $\Delta^* \subsetneq L$ such that for any set A, if the sequence $(\bar{a}_\beta : \beta < \alpha)$ satisfies (where $p_\beta = tp_{\Delta^*}(\bar{a}_\beta / A \cup \{\bar{a}_\gamma : \gamma < \beta\})$):*

1. *$p_0 \subseteq p_\beta$*
2. *For every $\phi \in \Delta^*$, $R^2_\phi(p_\beta|_\phi) = R^2_\phi(p_0|_\phi)$*
3. *For every $\phi \in \Delta^*$, $p_\beta|_\phi$ is A-definable.*

Then $(\bar{a}_\beta : \beta < \alpha)$ is a Δ-indiscernible sequence over A.

Proof. Note that as Δ is finite, there is an $n < omega$ such that it suffices to show that $(\bar{a}_\beta : \beta < \alpha)$ is Δ-m-indiscernible over A for all $m \leq n$. So define a sequence of finite sets $\Delta_n, \ldots, \Delta_0$ by:

1. $\Delta \subseteq \Delta_i$
2. Δ_i is closed under permutation of variables i.e. if $\phi(x_0, \ldots, x_{n-1}, \bar{y}) \in \Delta_i$ and $\sigma \in S_n$, then $\phi(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}, \bar{y})$ (we do not require this to be applicable to $\bar{y}$, which is understood to be reserved for parameters).
3. For every $\phi \in \Delta_i$, there is a $\psi \in \Delta i - 1$ such that if p is a ϕ-type with $R^2_\phi(p) = R^2_\phi(p|_A)$, then p is (ψ, A)-definable.

The construction is straightforward: start with $\Delta_n = \Delta$, and for every $\phi \in \Delta$ add in all the permutation of variables into Δ_n. So suppose Δ_i has been defined for $i > 0$, and start with $\Delta_{i-1} = \Delta_i$. Since T is stable, by the above corollary for every $phi \in \Delta_i$ there is a ψ_ϕ such that every ϕ-type is definable by ψ over its set of parameters, so add ψ_ϕ into Δ_{i-1}. Finally, add in all permutation of variables into Δ_{i-1}. Clearly each set is finite.

So let $\Delta^* = \bigcup_{i \leq n} \Delta_i$. Since $\Delta \subseteq \Delta_i$, it suffices to show that if $(\bar{a}_\beta : \beta < \alpha)$ satisfies the assumptions, then it is an Δ_i-i-indiscernible sequence over A for

$i \leq n$. Now by assumption, for $\gamma < \beta < \alpha$ and $\phi \in \Delta^*$, $p_0|_\phi \subseteq p_\gamma|_\phi, p_\beta|_\phi$ and $R^2_\phi(p_\gamma|_\phi) = R^2_\phi(p_0|_\phi) = R^2_\phi(p_\beta|_\phi)$. By Lemma 2.4.2(4), $p_0|_\phi$ has a unique extension in $S_{m,\phi}(A_\gamma)$ of equal rank (where ϕ has m free variables minus parameters), which is thus $p_\gamma|_\phi$. Moreover, note that $p_0|_\phi \subseteq p_\beta|_{\phi,A_\gamma} \subseteq p_\beta|_\phi$, and so $R(p_\beta|_\phi) = R(p_0|_\phi) \geq R(p_\beta|_{\phi,A_\gamma}) \geq R(p_\beta|_\phi)$. Therefore $R(p_\beta|_{\phi,A_\gamma}) = R(p_\gamma|_\phi)$, and thus by uniqueness $p_\beta|_{\phi,A_\gamma} = p_\gamma|_\phi$. Since this holds for every $\phi \in \Delta^*$, in particular this implies that $tp_{\Delta_i}(\bar{a}_\gamma/A_\gamma) \subseteq tp_{\Delta_i}(\bar{a}_\beta/A_\beta)$ for every $\gamma < \beta < \alpha$ and $i \leq n$.

Now, for a fixed $\beta < \alpha$ and $i < n$, suppose that $\bar{b}, \bar{c} \in A$ are such that $tp_{\Delta_i}(\bar{b}/A) = tp_{\Delta_i}(\bar{c}/A)$. Then for every $\phi \in \Delta_{i+1}$, note that $\psi_\phi \in \Delta_i$. Thus there is a tuple $\bar{d} \in A$ such that $\phi(\bar{x}, \bar{b}) \in tp(\bar{a}_\beta/A)$ iff $\bar{M} \models \psi_\phi(\bar{b}, \bar{d})$ iff $\bar{M} \models \psi_\phi(\bar{c}, \bar{d})$ iff $\phi(\bar{x}, \bar{c}) \in tp(\bar{a}_\beta/A)$ i.e. $tp(\bar{a}_\beta/A)$ does not (Δ_i, Δ_{i+1})-split over A. Therefore the sequence $\Delta_0, \ldots, \Delta_{n+1}$, A and $(\bar{a}_\beta : \beta < \alpha)$ satisfies the assumptions of Lemma 2.3.11, which implies that $(\bar{a}_\beta : \beta < \alpha)$ is a Δ_i-i-indiscernible sequence over A. This completes the proof. □

Proposition 2.4.11. *If T is stable, Δ is finite, λ is regular, $I = \{\bar{a}_\alpha : \alpha < \lambda\}$ with $l(\bar{a}_\beta) = m$ and $|A| < \lambda$, then there is a $J \subseteq I$, $|J| = \lambda$, which is a Δ-indiscernible set over A.*

Proof. Assume WLOG that Δ is closed under permutation of variables (which is possible while keeping Δ finite). Let Δ^* be as constructed in the lemma above, and enumerate $\Delta^* = \{\phi_i : i < n\}$. Note that as Δ^* is finite, for any set B with $|B| < \lambda$, $|S_{\Delta^*,m}(B)| < \lambda$: otherwise as there is a single formula θ which corresponds to the finite set Δ^* (such that for every set B, $S_{\Delta^*,m}(B)$ is naturally bijective with $S_{\theta,m}(B)$), this implies that there is an unstable formula (by Proposition 2.1.3), and thus contradicts that T is stable (by Proposition 2.1.11).

Now, consider all the complete Δ^*-m-types p with parameters in a subset of the universal model $\bar{M}$ and such that $|p| < \lambda$. Note that any $p \in S_{\Delta^*,m}(A)$ satisfies that $|p| < \lambda$ (by the above observation), and as λ is regular there must be some $p \in S_{\Delta^*,m}(A)$ which is realized λ many times in I. So of the types which are realized λ times in I, let p_0 be a type such that $(R^2_{\phi_i}(p_0|_{\phi_i}) : i < n)$ is minimal under the lexicographical ordering (which is a well-ordering since ranks are either $-1, \infty$ or an ordinal and n is finite). Thus let $A_0 \supseteq A$ be such that p_0 is a type over A_0 and $|A_0| < \lambda$.

We will define $(\bar{b}_\alpha : \alpha < \lambda) \subseteq I$ inductively: suppose that $(\bar{b}_\beta : \beta < \alpha)$ has been defined, and let $A_\alpha = A_0 \cup \{\bar{b}_\beta : \beta < \alpha\}$. So $|A_\alpha| < \lambda$, which implies that $|S_{\Delta^*,m}(A_\alpha)| < \lambda$. Then by regularity of λ, there is a type $p_\alpha \in S_{\Delta^*,m}(A_\alpha)$ with $p_\alpha \supseteq p_0$ which is realized λ many times in I. Note that $|p_\alpha| < \lambda$, so for $i < n$, since $p_\alpha \supseteq p_0$, by Lemma 2.4.2(1), $R^2_{\phi_i}(p_\alpha|_{\phi_i}) \leq R^2_{\phi_i}(p_0|_{\phi_i})$. But by the minimality of $(R^2_{\phi_i}(p_0|_{\phi_i}) : i < n)$ amongst complete types with cardinality $< \lambda$, by induction along $i < n$ we see that $R^2_{\phi_i}(p_0|_{\phi_i}) \leq R^2_{\phi_i}(p_\alpha|_{\phi_i})$and therefore $R^2_{\phi_i}(p_0|_{\phi_i}) = R^2_{\phi_i}(p_\alpha|_{\phi_i})$. Then let $\bar{b}_\alpha \in I - A^m_\alpha$ realize p_α.

This construction ensures that $(\bar{b}_\alpha : \alpha < \lambda)$ and A satisfies the assumptions of the above lemma, and thus $(\bar{b}_\alpha : \alpha < \lambda)$ is a Δ-indiscernible sequence over

A. By Corollary 2.3.8, since Δ is closed under permutation of variables, the stability of T ensures that the sequence is in fact a Δ-indiscernible set over A. $\square$

2.5 Two Cardinal Theorem for Nonalgebraic Formula

In general, a Two Cardinal theorem refers to a theorem which gives the conditions under which a model M has two sets $\phi(M)$ and $\psi(M)$ such that $|\phi(M)| \neq |\psi(M)|$. This is especially important when ϕ and ψ are nonalgebraic formulas, since compactness guarantees that $\phi(M), \psi(M)$ can be arbitrarily large provided that M is sufficiently large. Readers interested can consult [Ho 97] for the case when T is countable, or [Sh 90] for a very general (but difficult to prove) version. For our purposes, we will prove a particular two cardinal theorem due to Harnik, with the help of some lemmas:

Lemma 2.5.1. *Suppose T is stable and $p \in S_1(C)$ for a set C. Then there is a $D \subseteq C$, $|D| < \mu(T)$ such that p does not split over D (cf. Definition 2.1.4). In particular if T is λ-stable and λ_0 the least such that $2^{\lambda_0} > \lambda$, then there is a $D \subseteq C$ with $|D| < \lambda_0$ such that p does not split over D.*

Proof. The proof is morally the same as Lemma 2.3.12. Suppose for a contradiction that p, C is a counter-example to the claim, so that for every $D \subseteq C$, if $|D| < \mu(T)$ then p splits over D. Then for $\nu \in 2^{<\mu(T)}$, we will construct a set $D'_\nu \subseteq C$ ($|D'_\nu| < \mu(T)$) and an elementary map f_ν with domain D'_ν inductively on the length of ν. For the base case, let $D_{<>} = f_{<>} = \emptyset$; for ν of limit length let $D'_\nu = \bigcup_{\alpha<l(\nu)} D_{\nu|_\alpha}$, $f'_\nu = \bigcup_{\alpha<l(\nu)} f_{\nu|_\alpha}$.

For successor stages, suppose D'_ν, f_ν is defined. Since $|D'_\nu| < \mu(T)$ by the inductive hypothesis, p splits over D'_ν. This implies that there exists $\bar{d}_0, \bar{d}_1 \in C - D'_\nu$ such that $tp(\bar{d}_0/D'_\nu) = tp(\bar{d}_1/D'_\nu)$ and a $\phi(x, \bar{y}) \in L$ such that $\phi(x, \bar{d}_0), \neg\phi(x, \bar{d}_1) \in p$. So define $D'_{\nu\frown i} = D'_\nu \cup \{\bar{d}_i\}$, and note that

$$q = \{\phi(\bar{x}, f_\nu(\bar{d}')) : \phi(\bar{x}, \bar{d}') \in p|_{D'_\nu}\}$$

is a type (as f is an elementary map) and thus realized by some tuple $\bar{d}$. Then by defining $f_{\nu\frown 0}(\bar{d}_0) = f_{\nu\frown 1}(\bar{d}_1) = \bar{d}$, we can guarantee that $f_{\nu\frown i}$ is an elementary map extending f_ν with domain $D'_{\nu\frown i}$. Moreover, as $\bar{d}_i$ is finite $|D'_\nu| < \mu(T)$ implies that $|D_{\nu\frown i}| < \mu(T)$, and also implies that for ν of length a limit ordinal, $|D_\nu| = |\bigcup_{\alpha<l(\nu)} D_{\nu|_\alpha}| = |\bigcup_{\alpha<l(\nu)} D_{\nu|_{\alpha+1}} - D_{\nu|_\alpha}| \leq \aleph_0 \cdot l(\nu) < \mu(T)$. This completes the inductive construction.

As in Lemma 2.3.12, let

$$p_{\nu\frown i} = \{\phi(x, f_{\nu\frown i}(\bar{d})) : \phi(x, \bar{d}) \in p|_{D'_{\nu\frown i}}\}$$

which is a complete 1-type over $f_{\nu\frown i}(D_{\nu\frown i})$ by virtue of $f_{\nu\frown i}$ being an elementary map. Since by construction $f_{\nu\frown 0}(D_{\nu\frown 0}) = f_{\nu\frown 1}(D_{\nu\frown 1})$, taking $D_\nu = f_\nu(D_\nu)$ for

$\nu \in 2^{<\lambda_0}$ implies that D_ν, p_ν satisfies the assumptions of Lemma 2.1.14, which contradicts the definition of $\mu(T)$.

For the case of T being λ-stable, note that the proof applies Mutatis mutandis by replacing $\mu(T)$ with λ_0. $\square$

The following is a generalization of λ-saturation, which we will need for this proof.

Definition 2.5.2. *For a complete theory T, given a model M and a (possibly partial) 1-type $\Gamma(x)$ over $C \subseteq M$ with $|\Gamma(x)| < \lambda$, C is λ-**compact with respect to** $\Gamma(x)$ if for every $\Delta(x)$ which is a (possibly partial) 1-type over C with $|\Delta(x)| < \lambda$ and $\Gamma(x) \subseteq \Delta(x)$, $\Delta(x)$ is realized by some $c \in C$.*
*C is λ-**compact** if for every $\Gamma(x)$ which is a (possibly partial) 1-type over C, if $|\Gamma(x)| < \lambda$ then $\Gamma(x)$ is realized in C.*

Note. For $\lambda > |T|$ and a model M, being λ-compact is equivalent to being λ-saturated.

Lemma 2.5.3. *For a stable theory T and a $\lambda > |T|$, suppose $(C_\alpha : \alpha < \delta)$ is an increasing chain of sets in some model of T, $\Gamma(x)$ a 1-type over C_0 with $|\Gamma(x)| < \lambda$, and every C_α is λ-compact with respect to $\Gamma(x)$ with $\Gamma(C_\alpha) = \Gamma(C_0)$. Then $C_\delta = \bigcup_{\alpha<\delta} C_\alpha$ is λ-compact with respect to $\Gamma(x)$. Moreover, $\Gamma(C_\delta) = \Gamma(C_0)$.*

Proof. WLOG we assume that δ is a regular limit cardinal: the lemma is trivially true for δ a successor, and if $cf(\delta) < \delta$ then we can simply consider $(C_{\alpha_i} : \alpha_i \in W)$ where $W \subsetneq \delta$ is a cofinal subset of length $cf(\delta)$.

Note if $\delta \geq \lambda$ and regular, then any type $\Delta(x) \supseteq \Gamma(x)$ over C_δ with $|\Delta(x)| < \lambda$ is in fact a type over C_α for some $\alpha < \delta$, and so is therefore realized by assumption. Thus we may also assume that $\delta < \lambda$.

So suppose $\Delta(x)$ is a type over C_δ, $\Gamma(x) \subseteq \Delta(x)$ and $|\Delta(x)| < \lambda$. It suffices to show that this type is realized in C_δ. Now, let $q \in S_1(C_0)$ be any complete 1-type such that $q \cup \Delta(x)$ is consistent (i.e. q is some completion of $\Delta(x)$ over C_0), and so by the lemma above, there is a $D \subseteq C_0$, $|D| < \mu(T) \leq |T|^+ \leq \lambda$ (by Proposition 2.1.16), such that q does not split over D. Now, for $\alpha < \delta$ let $\Delta_\alpha(x) = \Delta(x)|_{C_\alpha}$, so that $\Delta(x) = \bigcup_{\alpha<\delta} \Delta_\alpha(x)$. Note that this ensures that for every α, $\Gamma(x) \subseteq \Delta_\alpha(x)$ (since $\Gamma(x)$ is a type over $C_0 \subseteq C_\alpha$).

Now, for $\xi < \lambda$, inductively define $a_\xi \in C_0$ such that if (by the division theorem for ordinals) $\xi = \delta \cdot \beta + \alpha$ for an $\alpha < \delta$, then a_ξ realizes the type $\Delta_\alpha(x) \cup q|_{D \cup \{a_\eta : \eta < \xi\}}$: note this is a type over C_α, and as $|D|, |\Delta(x)| < \lambda$, $\lambda > |T|$ and $\Gamma(x) \subseteq \Delta_\alpha(x)$, this type is realized in C_α since C_α is assumed to be λ-compact with respect to $\Gamma(x)$. But then $C_\alpha \models \Gamma(a_\xi)$, and by assumption $\Gamma(C_\alpha) = \Gamma(C_0)$, which implies that $a_\xi \in C_0$ as desired.

Note as q is a complete type over C_0, if we define $p_\xi = tp(a_\xi / D \cup \{a_\eta : \eta < \xi\})$ then $p_\xi = q|_{D \cup \{a_\eta : \eta < \xi\}}$. But since by definition q does not split over D, p_ξ does not split over D. In addition, if $\eta < \xi$ then $p_\eta \subseteq p_\xi$, and so the conditions for Proposition 2.3.9 and it's corollary are satisfied. Therefore $I = \{a_\xi : \xi < \lambda\}$ is an indiscernible set over D. Moreover, if $E \subseteq C_\delta$ is the set of elements which appear as parameters in $\Delta(x)$, then as $|\Delta(x)| < \lambda$, $|E| < \lambda$ as well. Now, as $\mu(T) \leq |T|^+ \leq \lambda$, either $\lambda > \mu(T)$ or $\lambda = \mu(T) = |T|^+$ is regular, so by Corollary 2.3.14 there is an $I' \subsetneq I$, $|I'| < \lambda$ such that $I - I'$ is an indiscernible set over $D \cup E$.

But by construction, for any $\alpha < \delta$ the type $\Delta_\alpha(x)$ is realized λ many times in I, and so is realized by an element of $I - I'$. Indiscernibility of $I - I'$ over $D \cup E$ (as $\Delta(x)$ is a type over E) then implies that every element of $I - I'$ realizes $\Delta_\alpha(x)$. Since this holds for every $\alpha < \delta$, therefore every $a \in I - I' \subseteq C_\delta$ realizes $\Delta(x)$. Thus C_δ is λ-compact with respect to $\Gamma(x)$. Finally, $\Gamma(C_\delta) = \Gamma(\bigcup_{\alpha<\delta} C_\alpha) = \bigcup_{\alpha<\delta} \Gamma(C_\alpha) = \Gamma(C_0)$ by assumption. □

Lemma 2.5.4. *For a stable theory T, $\lambda > |T|$, C a set in some model that is λ-compact with respect to $\Gamma(x)$, $\Gamma(x)$ a type over C with $|\Gamma(x)| < \lambda$, and b realizes a λ-isolated type over C (see Definition 2.2.2), then $C \cup \{b\}$ is λ-compact with respect to $\Gamma(x)$ and $\Gamma(C \cup \{b\}) = \Gamma(C)$.*

Proof. Let $\Sigma(x_0)$ be a type over C such that $|\Sigma(x_0)| < \lambda$ and $\Sigma(x_0)$ isolates $tp(b/C)$.

So suppose $\Delta(x_0)$ is a type over $C \cup \{b\}$ with $\Delta(x_0) \supseteq \Gamma(x_0)$ and $|\Delta(x_0)| < \lambda$. To show that $C \cup \{b\}$ is λ-compact with respect to $\Gamma(x_0)$, it suffices to show that $\Delta(x_0)$ is realized in C. Note that we can consider $\Delta(x_0) = \Delta'(x_0, b)$, where $\Delta'(x_0, x_1)$ is a 2-type over C. So define:

$$\begin{aligned}\Delta''(x_0) = \{\exists x_1 \psi(x_0, x_1) : \ & \psi(x_0, x_1) \text{ is a finite conjunction of formulas from} \\ & \Delta'(x_0, x_1) \cup \Sigma(x_1)\}\end{aligned}$$

Note by definition of $\Delta(x_0)$ and $\Delta'(x_0, x_1)$, $\Gamma(x_0) \cup \Delta''(x_0)$ is satisfiable and thus a type over C. Moreover, $|\Delta''(x_0)| < \lambda$, and therefore by the λ-compactness of C with respect to $\Gamma(x_0)$, there is a $c_0 \in C$ which realizes $\Gamma(x_0) \cup \Delta(x_0)$. Then by definition of $\Delta''(x_0)$, there is some model $M \supseteq C$ with a $c_1 \in M$ such that $M \models \Delta'(c_0, c_1) \cup \Sigma(c_1)$. But as $\Sigma(x_0)$ isolates the type $tp(b/C)$, thus $tp(c_1/C) = tp(b/C)$. But as $M \models \Delta'(c_0, c_1)$, in some larger model N with $b \in N$, $N \models \Delta'(c_0, b)$ and therefore $N \models \Delta(c_0)$. This implies that $\Delta(x_0)$ is realized in C, as desired.

Finally, suppose b realizes $\Gamma(x)$. So $\Sigma(x) \cup \Gamma(x)$ is a type over C, and moreover $|\Sigma(x) \cup \Gamma(x)| < \lambda$. Therefore by the λ-compactness of C, there is a $c \in C$ which realizes this type and so $tp(c/C) = tp(b/C)$. But as $(x = c) \in tp(c/C)$, this implies that $b \in C$. Thus $\Gamma(C \cup \{b\}) = \Gamma(C)$. □

Lemma 2.5.5. *For a stable theory T, $\lambda > |T|$, suppose C is a set in some model that is λ-compact with respect to $\Gamma(x)$, $\Gamma(x)$ a type over C with $|\Gamma(x)| < \lambda$, and M is λ-constructible over C (see Definition 2.2.2). Then M is λ-compact with respect to $\Gamma(x)$, and $\Gamma(M) = \Gamma(C)$.*

Proof. Let $\{m_\alpha : \alpha < \kappa\}$ be a λ-construction of M over C, and let $M_\alpha = C \cup \{m_\beta : \beta < \alpha\}$. By the above lemma we see that if M_α is λ-compact with respect to $\Gamma(x)$ and $\Gamma(M_\alpha) = \Gamma(C)$, then this also holds for $M_{\alpha+1}$. For limits, we can use Lemma 2.5.3 to show that this holds similarly. Therefore this holds for $M = M_\kappa$. □

Lemma 2.5.6. *For a stable theory T, $\lambda > |T|$, if there are λ-saturated models $M_0 \precneqq M_1$, and $\Gamma(x)$ is a 1-type over M_0 with $|\Gamma(x)| < \lambda$ and $\Gamma(M_0) = \Gamma(M_1)$, then there is a λ-saturated $M_2 \succneqq M_1$ with $\Gamma(M_2) = \Gamma(M_0)$.*

Proof. Fix a universal model $\bar{M}$, and let $c_0 \in M_1 - M_0$. As T is stable, by Corollary 2.4.9, for every formula $\phi(x, \bar{y})$, there is a defining formula $\psi_\phi(\bar{y}, \bar{z})$ and a $\bar{a}_\phi \in M_0$ such that for every $\bar{b} \in M_0$, $M_0 \models \phi(c_0, \bar{b})$ iff $M_0 \models \psi_\phi(\bar{b}, \bar{a}_\phi)$. So define

$$q = \{\phi(x, \bar{b}) : \phi \in L, \bar{b} \in M_1, M_1 \models \psi_\phi(\bar{b}, \bar{a}_\phi)\}$$

Thus q is a complete 1-type over M_1 with $tp(c_0/M_0) \subseteq q$. Moreover, if $tp(\bar{b}_0/M_0) = tp(\bar{b}_1/M_0)$, $\phi(x, \bar{b}_0) \in q$ iff $M_1 \models \psi_\phi(\bar{b}_0, \bar{a}_\phi)$ iff $M_1 \models \psi_\phi(\bar{b}_1, \bar{a}_\phi)$ iff $\phi(x, \bar{b}_1) \in q$ i.e. q does not split over M_0.

Let $c_1 \notin M_1$ be an element of the universal model which realizes q. Since $c_0 \in M_1 - M_0$ and by assumption $\Gamma(M_1) = \Gamma(M_0)$, thus $M_1 \not\models \Gamma(c_0)$. But as $\Gamma(x)$ is a type over M_0 and $tp(c_0/M_0) \subseteq q$, if c_1 realizes q then c_1 does not realize $\Gamma(x)$ i.e. $\Gamma(M_1 \cup \{c_1\}) = \Gamma(M_1) = \Gamma(M_0)$.

Claim. $M_1 \cup \{c_1\}$ is λ-compact with respect to $\Gamma(x)$.

Let $\Delta(x)$ be a 1-type over $M_1 \cup \{c_1\}$ with $\Gamma(x) \subseteq \Delta(x)$ and $|\Delta(x)| < \lambda$. Thus every formula in $\Delta(x)$ is of the form $\phi(c_1, x, \bar{b})$ with $\bar{b} \in M_1$. Now consider $\Delta'(x) = \{\psi_\phi(x, \bar{b}, \bar{a}_\phi) : \phi(c_1, x, \bar{b}) \in \Delta(x)\}$: since each ψ_ϕ with $\bar{a}_\phi$ defines ϕ, $m \in M_1$ satisfies $\Delta'(x)$ iff m satisfies $\Delta(x)$. Note that $|\Delta'(x)| < \lambda$, so as M_1 is λ-saturated, it suffices to prove that $\Delta'(x)$ is a type with respect to M_1 to show that $\Delta(x)$ is realized in M_1 and thus $M_1 \cup \{c_1\}$ is λ-compact with respect to $\Gamma(x)$.

So for a contradiction, assume that $\Delta'(x)$ is not a type with respect to M_1. By compactness, there is a $k < \omega$ and for $i < k$, $\phi(c_1, x, \bar{b}_i) \in \Delta(x)$ such that $\bar{M} \models \neg\exists x \bigwedge_{i<k} \psi_{\phi_i}(x, \bar{b}_i, \bar{a}_{\phi_i})$. But since $\Delta(x)$ is a type extending $\Gamma(x)$, if $\gamma(x)$ is a finite conjunction of formulas in $\Gamma(x)$, then $\bar{M} \models \exists x(\bigwedge_{i<k} \phi(c_1, \bar{b}_i, \bar{a}_{\phi_i}) \wedge \gamma(x))$. Denoting this formula by $\gamma^*(c_1, \bar{b}_0, \ldots, \bar{b}_{k-1})$, note that this is equivalent to $\bar{M} \models \psi_{\gamma^*}(\bar{b}_0, \ldots, \bar{b}_{k-1})$.

So consider the set

$$\{\neg\exists x \bigwedge_{i<k} \psi_{\phi_i}(x, \bar{y}_i, \bar{a}_{\phi_i})\} \cup \{\psi_{\gamma^*}(\bar{y}_0, \ldots, \bar{y}_{k-1}) : \gamma(x) \text{ is a finite conjunction of formulas from } \Gamma(x)\}$$

Which is a set of formulas with parameters in M_0 and is satisfied by $\bar{b}_0 \frown \cdots \frown \bar{b}_{k-1}$ i.e. a type over M_0. Moreover, since $|\Gamma(x)| < \lambda$, clearly this set also has cardinality $< \lambda$, and so as M_0 is λ-saturated there are $\bar{a}_0 \frown \cdots \frown \bar{a}_{k-1} \in M_0$ satisfying:

$$M_0 \models \neg\exists x \bigwedge_{i<k} \psi_{\phi_i}(x, \bar{a}_i, \bar{a}_{\phi_i}) \tag{2.1}$$

$$M_0 \models \psi_{\gamma^*}(\bar{a}_0, \ldots, \bar{y}_{k-1}) \tag{2.2}$$

The second equation implies that $M_0 \models \gamma^*(c_0, \bar{a}_0, \ldots, \bar{a}_{k-1})$ i.e.

$$M_0 \models \exists x(\bigwedge_{i<k} \phi_i(c_0, x, \bar{a}_i) \wedge \gamma(x))$$

As ψ^*_γ defines $\gamma(x)$ relative to the type $tp(c_0/M_0)$. But by compactness, this implies that $\Gamma(x) \cup \{\bigwedge_{i<k} \phi_i(c_0, x, \bar{a}_i)\}$ is a satisfiable 1-type. Again, the λ-saturation of M_1 gives us an $m \in M_1$ which satisfies this type, and as $m \in \Gamma(M_1) = \Gamma(M_0)$, in fact $m \in M_0$. So $M_0 \models \bigwedge_{i<k} \phi_i(c_0, m, \bar{a}_i)$, and therefore $M_0 \models \bigwedge_{i<k} \psi_{\phi_i}(m, \bar{a}_i, \bar{a}_{\phi_i})$, contradicting (2.1). This proves that $\Delta'(x)$ is satisfiable, which completes the proof of the claim that $M_1 \cup \{c_1\}$ is λ-compact with respect to $\Gamma(x)$.

To complete the proof of the lemma, let M_2 be a λ-constructible λ-prime model over $M_1 \cup \{c_1\}$ (which is constructed in Theorem 2.2.7). By the previous lemma, $\Gamma(M_2) = \Gamma(M_1 \cup \{c_1\}) = \Gamma(M_0)$, and as M_2 is λ-prime over $M_1 \cup \{c_1\}$, in particular $M_1 \precneqq M_2$ (as $c_1 \notin M_1$) and M_2 is λ-saturated. □

Corollary 2.5.7. *If $M_0 \precneqq M_1$ are λ-saturated models of a stable T ($\lambda > |T|$), $\Gamma(x)$ is a 1 type over M_0 with $|\Gamma(x)| < \lambda$ and $\Gamma(M_0) = \Gamma(M_1)$, then for any infinite cardinal κ, there is a model M with cardinality $\geq \kappa$ elementarily extending M_1 with $\Gamma(M) = \Gamma(M_1) = \Gamma(M_0)$.*

Proof. Given any ordinal α, define inductively M_i for $i < \alpha$ such that $M_{i+1} \succneqq M_i$ and $\Gamma(M_i) = \Gamma(M_0)$: M_0, M_1 are given as in the claim, and if M_i is defined, let M_{i+1} be the model gotten by applying the above lemma to the pair M_0, M_i. For a limit $\delta < \alpha$, let $A_\delta = \bigcup_{i<\delta} M_i$, which by Lemma 2.5.3 is λ-compact with respect to $\Gamma(x)$ with $\Gamma(A_0) = \Gamma(M_0)$ by the inductive hypothesis. So taking the λ-constructible λ-prime model over A_δ (constructed in Theorem 2.2.7) to be M_δ, Lemma 2.5.5 shows that M_δ satisfies the inductive hypothesis. This allows us to construct arbitrarily large elementary extensions which satisfies the claim. □

Lemma 2.5.8. *For T a complete theory and $\phi(x)$ a nonalgebraic formula in T (i.e. for every $n < \omega$, $(\exists^{\geq n} x\phi(x)) \in T$) and $\lambda \geq |T|$, if there are models $M \precneqq N$ of T with $\phi(M) = \phi(N)$, then there are λ^+-saturated models $A \precneqq B$ with $\phi(A) = \phi(B)$.*

Proof. Let P be a new relation symbol of arity 1, and let L^+ be the expanded language with P. We form a L^+-expansion N^+ of N by interpreting $P^{N^+} = M$. Note that:

- $N^+ \models \exists x \neg P(x)$
- $N^+ \models \forall x(\phi(x) \to P(x))$
- For every L-formula $\psi(x, \bar{y})$ with $\bar{y}$ of length n,

$$N^+ \models \forall \bar{y}(\exists x \psi(x, \bar{y}) \wedge \bigwedge_{0 \leq i < n} P(y_i)) \to \exists z(\psi(z, \bar{y}) \wedge P(z))$$

i.e. P^{N^+} satisfies the Tarski-Vaught test with respect to L.

So let

$$T^+ = T \cup \{(\exists x \neg P(x)), (\forall x(\phi(x) \to P(x)))\}$$
$$\cup \{\forall \bar{y}(\exists x \psi(x, \bar{y}) \wedge \bigwedge_{0 \leq i < n} P(y_i)) \to \exists z(\psi(z, \bar{y}) \wedge P(z)) : \psi(x, \bar{y}) \in L\}$$

So that for every $C \models T^+$, $P^C|_L \preceq_L C|_L$, and in particular $P^C|_L \models T$. Also, by compactness, if $(c_i : i < 2^\lambda)$ are new constants, then $T^+ \cup \{P(c_i) : i < 2^\lambda\}$ is satisfiable. So by the Löwenheim-Skolem theorems let C be a model of this theory with $|C| = |P^C| = 2^\lambda$

Claim. If $M_0 \models T^+$ and $|M_0| = |P^{M_0}| = 2^\lambda$, then there is a $M_1 \succeq_{L^+} M_0$, $|M_1| = |P^{M_1}| = 2^\lambda$ such that for every set $A \subseteq M_0$ and $B \subseteq P^{M_0}$ with $|A| = |B| = \lambda$, M_1 realizes every $p \in S_1^{M_0|_L}(A)$ and $P^{M_1}|_L$ realizes every $q \in S_1^{P^{M_1}|_L}(B)$ i.e. M_1 realizes every complete L-type of M_0 with cardinality λ and P^{M_1} realizes every complete L-type of P^{M_0} with cardinality λ.

We construct B in a similar fashion to Proposition 1.5.8: let $(c_i : i < 2^\lambda), (d_i : i < 2^\lambda)$ be new constant symbols, and for every $q \in S_1^{P^{M_0}|_L}(B)$ with $B \subseteq P^{M_0}, |B| = \lambda$, let $q' = q \cup \{P(x)\}$ (recall that q is a type in the language L). List $\bigcup\{S_1^{M_0|_L}(A) : A \subseteq M_0, |A| = \kappa\}$ by $(p_i(x) : i < 2^\kappa)$ (as in Proposition 1.5.8, since each $|S_1^{M_0|_L}(A)| \leq 2^\kappa$ and $|\{A \subseteq M_0 : |A| = \lambda\}| = (2^\lambda)^\lambda = 2^\lambda$, there are 2^λ many such types). Similarly, list $\bigcup\{S_1^{P^{M_0}|_L}(B) : B \subseteq P^{M_0}, |B| = \lambda\}$ by $(q_i(x) : i < 2^\lambda)$. Then by compactness

$$Th_{M_0}(M_0) \cup \bigcup\{p_i(c_i) : i < 2^\lambda\} \cup \bigcup\{q_i'(d_i) : i < 2^\lambda\}$$

is satisfiable, and therefore (again by the Löwenheim-Skolem theorems) is satisfied by some L^+-structure M_1 with $|M_1| = |P^{M_1}| = 2^\lambda$ ($|P^{M_1}| = 2^\lambda$ as $P^{M_0} \subseteq P^{M_1}$).

Let $C_0 = C$, and for $i < \lambda^+$, if C_i is defined let C_{i+1} be the model M_1 constructed in the above claim with $M_0 = C_i$. For a limit $\delta < \lambda^+$, let $C_\delta = \bigcup_{i<\delta} C_i$, and let $D = \bigcup_{i<\lambda} C_i$.

Claim. Both $D|_L$ and $P^D|_L$ are λ^+-saturated models of T, with $|D| = |P^D| = 2^\lambda$.

Note that as $D \models T^+$, $P^D \precneqq_L D$ as well. The cardinality condition is clear by construction (since $P^D = \bigcup_{i<\lambda^+} P^{C_i}$). For saturation, this is again similar to the case in Proposition 1.5.8: For any $A \subseteq D$ with $|A| \leq \lambda$, since λ^+ is regular $A \subset C_i$ for some $i < \lambda^+$. Then any $p \in S_1^{C_i|_L}(A)$ is realized in C_{i+1}. Similarly, for any $B \subseteq P^D$ with $|B| \leq \lambda$, B is contained in some P^{C_i}. Since every $q \in S_1^{P^{C_i}|_L}(B)$ is realized in $P^{C_i+1}|_L$, P^D is also saturated.

Thus both $D|_L$ and $P^D|_L$ are saturated models of T i.e there are models $A \precneqq B$ of T with $|A| = |B| = 2^\lambda$, $\phi(A) = \phi(B)$ and both M and N are λ^+-saturated. $\square$

Theorem 2.5.9. *If T is a complete stable theory, $\phi(x)$ is a nonalgebraic formula in T and there are models $M \subsetneq N$ with $\phi(M) = \phi(N)$, then there is a model A such that $|A| > |\phi(A)|$.*

Proof. By the above lemma, we may assume that M, N are $|T|^+$-saturated. Then Corollary 2.5.7 gives us a model $A \supsetneq M$ with $|A| > |M|$ and such that $\phi(A) = \phi(M)$, so that $|A| > |\phi(A)|$. □

2.6 Vaught's Two Cardinal theorem for cardinals far apart

The two cardinal theorem in the previous section applies for a single nonalgebraic formula, but we will also need one for nonalgebraic types. The following theorem is commonly attributed to Vaught, although our approach is based on a type-omitting theorem by Morley. Again, we will be working with a fixed universal model $\bar{M}$ of a complete theory T in the language L. We first need the following lemma by Shelah:

Lemma 2.6.1. *For any set A, if $\mu = \beth_{(2^{|T|})^+}(|A|)$ then for any linear order I with $|I| = \mu$ and $(a_i : i \in I) \subseteq \bar{M}$, there is a A-indiscernible sequence $(b_j : j < \omega) \subseteq \bar{M}$ of distinct elements such that for every $\zeta \in [\omega]^n$, there is a $\xi \in [I]^n$ with $tp(b_{\zeta(0)} \dots b_{\zeta(n-1)}/A) = tp(a_{\xi(0)} \dots a_{\xi(n-1)}/A)$*

Proof. Note that for every $n < \omega$, $|S_n(A)| \leq 2^{|T|+|A|}$ since $|L_A| = |T| + |A|$. So let $\lambda = \sup\{|S_n(A)| : n < \omega\}$. Also, since $\mu = \beth_1(\beth_{2^{|T|}}(|A|))$, by König's theorem we have $cf(\mu) > \beth_{2^{|T|}}(|A|) > 2^{|T|+|A|} \geq \lambda$. Lastly, note that by the Erdös-Rado theorem (Theorem A.0.2), for every $\kappa < \mu$ and $n < \omega$ there is a $\kappa' = \beth_n(\kappa + \lambda)^+ < \mu$ such that for any $f : [\kappa']^{n+1} \longrightarrow \lambda$, there is a $Y \subseteq \kappa'$ with $|Y| \geq \kappa$ and f is constant on $[Y]^{n+1}$.

We will construct a sequence of types $p_1(x_1) \subsetneq p_2(x_1, x_2) \subsetneq \dots$ with $p_n \in S_n(A)$ which satisfies: for every $\kappa < \mu$, there is a $I' \subsetneq I$ with $|I'| = \kappa$ such that for every $i_1 < \dots < i_n \in I'$, $tp(a_{i_1} \dots a_{i_n}/A) = p_n$. For simplicity, we can take $p_0 = T$.

So suppose that p_n has been constructed to satisfy the inductive hypothesis. Given a $\kappa < \mu$, choose κ' as given above. Note that the inductive hypothesis guarantees that there is a $I' \subseteq I$ with $|I'| = \kappa'$ such that for every $(i_1, \dots, i_n) \in [I']^n$, $tp(a_{i_1} \dots a_{i_n}/A) = p_n$. Then, as $|S_n(A)| \leq \lambda$, the Erdös-Rado theorem gives a $I'' \subseteq I'$ with $|I''| = \kappa$ such that for all $i_1, \dots, i_{n+1} \in I''$, $tp(a_{i_1} \dots a_{i_{n+1}}/A) = p_{n+1,\kappa}$ for some choice of $p_{n+1,\kappa} \in S_{n+1}(A)$. Now, as $|S_{n+1}(A)| \leq \lambda < cf(\mu)$, there is a cofinal set $W \subseteq \mu$ such that for all $\kappa_0, \kappa_1 \in W$, $p_{n+1,\kappa_0} = p_{n+1,\kappa_1}$. Thus we choose this to be p_{n+1}, and the cofinality of W in μ guarantees that the inductive hypothesis is met.

Finally, let $p = \bigcup_{n<\omega} p_n$, and, letting $(b_n : n < \omega)$ be new constants, consider the set

$$\{\phi(b_{n_1}, \dots, b_{n_k}) : \phi(x_1, \dots, x_k) \in p, n_1 < \dots < n_k < \omega\} \cup \{b_n \neq b_m : n < m < \omega\}$$

This set is satisfiable by compactness, and the realizations $(b_n : n < \omega)$ is an indiscernible sequence over A by construction. Moreover, by construction each p_n is the type of some subsequence of $(a_i : i \in I)$ over A, and so the lemma is proved. $\square$

Theorem 2.6.2 (Vaught's Two Cardinal theorem for cardinals far apart). *Suppose T is a complete theory, and $\Sigma(x)$ is a 1-type of T. Denoting $\mu = (2^{|T|})^+$, if there is a $\lambda \geq |T|$ and a model $M \models T$ with $|M| \geq \beth_\mu(\lambda)$, $|\Sigma(M)| = \lambda$, then for every $\kappa \geq |T|$ there is a model $N \models T$ with $|N| = \kappa$ and $|\Sigma(M)| \leq |T|$.*

Proof. Let P be a new unary relation symbol, and $(c_i : i < \lambda)$ new constant symbols. Let $L' = L_{Sk} \cup \{P\} \cup \{c_i : i < \lambda\}$ (where L_{Sk} is the Skolemized language of L), and expand M to a L'-structure by interpreting $P^M = \Sigma(M) = \{c_i^M : i < \lambda\}$ (so that $\{c_i : i < \lambda\}$ enumerates $\Sigma(M)$). Our strategy will be to create a set $p \subseteq L'$ in ω-many variables as in the lemma above, while ensuring that the Skolem hull of a sequence which satisfies p does not realize $\Sigma(x)$ too many times. We will ensure the latter clause by restricting the number of Skolem terms which map into the set defined by P.

Let $(a_\alpha : \alpha < |M|)$ be an enumeration of M, and so as in the proof of the above lemma we can define $p_1(x_1) \subseteq p_2(x_1, x_2) \subseteq \ldots$ with $p_n \in S^M_{L',n}(P(M))$. Note that since we enumerate $P(M)$ by the constants $(c_i : i < \lambda)$, p_n can be considered as a L'-type over $\emptyset$. Defining $p_\omega = \bigcup_{n<\omega} p_n(x_1, \ldots, x_n)$, clearly by compactness p_ω is satisfiable.

Note that for every formula $\phi(x) \in \Sigma(x)$, $M \models \forall x P(x) \to \phi(x)$ i.e. the formula $P(x)$ isolates the type $\Sigma(x) \cup \{P(x)\}$. Also, even if $\Sigma(x) \cup \{\neg P(x)\}$ is satisfiable, it is a 1-type that is omitted by M. So for now we will first assume that $\Sigma(x) \cup \{\neg P(x)\}$ is indeed a 1-type of $Th_{L'}(M)$.

Claim. For any term $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in L_{Sk} and any $j_1, \ldots, j_m \in \lambda$, $P(t(x_1, \ldots, x_n, c_{j_1}, \ldots, c_{j_m})) \in p_n(x_1, \ldots, x_n)$ iff there is a $j_0 < \lambda$ such that $(t(x_1, \ldots, x_n, c_{j_1}, \ldots, c_{j_m}) = c_{j_0}) \in p_n(x_1, \ldots, x_n)$.

If $a_{\alpha_1}, \ldots, a_{\alpha_n}$ is such that $p_n(x_1, \ldots, x_n) = tp^N_{L'}(a_{\alpha_1} \ldots a_{\alpha_n}/P(M))$, then

$$\begin{aligned}
(P(t(x_1, \ldots, x_n, &c_{j_1}, \ldots, c_{j_m}))) \in p_n(x_1, \ldots, x_n) \\
&\Leftrightarrow M \models P(t(a_{\alpha_1}, \ldots, a_{\alpha_n}, c_{j_1}, \ldots, c_{j_m})) \\
&\Leftrightarrow \text{for some } j_0 < \lambda, M \models t(a_{\alpha_1}, \ldots, a_{\alpha_n}, c_{j_1}, \ldots, c_{j_m}) = c_{j_0} \\
&\Leftrightarrow (t(x_1, \ldots, x_n, c_{j_1}, \ldots, c_{j_m}) = c_{j_0}) \in p_n(x_1, \ldots, x_n)
\end{aligned}$$

Now, let $(d_i : i < \kappa)$ be new constant symbols and define the set

$$p_\kappa = \{\phi(d_{i_1}, \ldots, d_{i_n}) : \phi(x_1, \ldots, x_n) \in p_\omega, i_1 < \cdots < i_n < \kappa\}$$

Note that the set $p_\kappa \cup Th_{L'_M}(M)$ is satisfiable by compactness (since a finite subset of p_κ is a finite subset of p_ω by a change of variables). Let N model this theory, so that $(d_i^N : i < \kappa)$ is a L'-indiscernible sequence (as asserted by p_κ).

Claim. There exists a $I \subseteq \lambda$, $|I| \leq |T|$ such that for every term $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in L_{Sk}, $i_1 < \cdots < i_n < \kappa$ and $j_1, \ldots, j_m \in I$, if $N \models t(d_{i_1}, \ldots, d_{i_n}, c_{j_1}, \ldots, c_{j_m}) = c_{j_0}$ then $j_0 \in I$.

We will inductively construct I_n for $n < \omega$: let $I_0 = \emptyset$, and given I_n with $|I_n| \leq |T|$ define the set I_{n+1} by

$$\begin{aligned} I_{n+1} = \{j < \lambda : \text{ There is a term } & t(x_1, \ldots, x_n, y_1, \ldots, y_m) \text{ in } L_{Sk} \\ & \text{and } j_1, \ldots, j_m \in I_n \text{ such that} \\ & (t(x_1, \ldots, x_n, c_{j_1}, \ldots, c_{j_m}) = c_j) \in p_\omega\} \end{aligned}$$

Note that there are only $|L_{Sk}| = |L| = |T|$ many terms in L_{Sk}, and if $|I_n| \leq |T|$ then there are $\leq |T|$ many choices of $j_1, \ldots, j_m$, so that inductively indeed $|I_{n+1}| \leq |T|$. Now, let $I = \bigcup_{n<\omega} I_n$, so that $|I| \leq |T|$. Moreover, given any term t in L_{Sk}, $j_1, \ldots, j_m \in I$ and $i_1 < \cdots < i_n < \kappa$, since $m < \omega$ there is some $k < \omega$ such that $j_1, \ldots, j_m \in I_k$. Then as $N \models p_n(d_{i_1}, \ldots, d_{i_n})$, if $N \models t(d_{i_1}, \ldots, d_{i_n}, c_{j_1}, \ldots, c_{j_m}) = c_{j_0}$ then by construction $j_0 \in I_{k+1} \subseteq I$.

Combining the above claims, by reindexing I an initial segment of $|T| \leq \lambda$ we may assume WLOG that if there is a term $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in L_{Sk}, $i_1 < \cdots < i_n < \kappa$ and $j_1 < \cdots < j_m < I$ such that $N \models P(t(d_{i_1}, \ldots, d_{i_n}, c_{j_1}, \ldots, c_{j_m}))$, then there is a $j_0 < I$ such that for every $\zeta \in [\kappa]^n$,

$$N \models t(d_{\zeta(0)}, \ldots, d_{\zeta(n-1)}, c_{j_1}, \ldots, c_{j_m}) = c_{j_0} \wedge P(c_{j_0})$$

So now, consider the L_{Sk}-structure $A = \langle \{d_i^N : i < \kappa\} \cup \{c_j^N : j < I\} \rangle_{Sk}$, where the Skolemization is only done relative to the original language L. Of course, as $A \preceq_L N$, $A \models T$ as $Th_L(M) = T$. Moreover:

Claim. For every $a \in A$, $tp^A_{L'}(a/\emptyset)$ is realized in M.

Since $a \in A$, by construction there is a term t in L_{Sk}, $i_1 < \cdots < i_n < \kappa$ and $j_1 < \cdots < j_m < I$ such that $A \models t(d_{i_1}, \ldots, d_{i_n}, c_{j_1}, \ldots, c_{j_m}) = a$. So if $(a_{\alpha_1}, \ldots, a_{\alpha_n}) \in M$ is such that $tp^M_{L'}(a_{\alpha_1} \ldots a_{\alpha_n}/\emptyset) = p_n(x_1, \ldots, x_n)$, then for every formula $\phi(x) \in L'$,

$$\phi(x) \in tp^A_{L'}(a/\emptyset) \Leftrightarrow \phi(t(x_1, \ldots, x_n, c_{j_1}, \ldots, c_{j_m})) \in tp^M_{L'}(a_{\alpha_1} \ldots a_{\alpha_n}/\emptyset)$$

which implies that $tp^A_{L'}(a/\emptyset) = tp^M_{L'}(t^M(a_{\alpha_1}, \ldots, a_{\alpha_n}, c^M_{j_1}, \ldots, c^M_{j_m})/\emptyset)$

In particular, since M omits the type $\Sigma(x) \cup \{\neg P(x)\}$, A also omits this type. On the other hand, as $P(x)$ isolates the type $\Sigma(x) \cup \{P(x)\}$, thus for every $a \in A$, $A \models \Sigma(a)$ iff $A \models P(a)$. In particular, for every $j < \lambda$, $N \models P(c_j)$ and we see that for every $j < I$, $A \models \Sigma(c_j)$. But on the other hand, if $a \in A$ is such that $A \models P(a)$, then there is a term t in L_{Sk}, $i_1 < \cdots < i_n < \kappa$ and $j_1 < \cdots < j_m < I$ such that $A \models t(d_{i_1}, \ldots, d_{i_n}, c_{j_1}, \ldots, c_{j_m}) = a$. So by the first claim above, there is a $j_0 \in \lambda$ such that $N \models a = c_{j_0}$. But then by the construction of I, we have $j_0 \in I$, and thus $A \models \Sigma(a)$ iff there is a $j < I$ such that $A \models a = c_j$. Therefore we have $A \models T$ with $|A| = \kappa \geq |T|$ and $\Sigma(A) = \{c^A_j : j < I\}$, so that $|\Sigma(A)| = |I| \leq |T|$. □

Remark. Although we refer to the above theorem as Vaught's two cardinal theorem for cardinals far apart, the reader should note that there is a significant difference between what is stated here and Vaught's original claim: Vaught initially gave the proof in [Va 65] while restricting $\Sigma(x)$ to a single formula $\phi(x)$ and where T is a countable theory, although his proof quickly generalizes to uncountable theories. Moreover, his result showed that one could have the size of $\phi(M)$ be any cardinal χ with $\kappa \geq \chi \geq |T|$, which is a stronger result than what we have proven here (although it is not difficult to change the above proof so that this result could be achieved). Morley gave an alternative proof for the countable case in [Mo 65a], which also generalized Vaught's result to any 1-type $\Sigma(x)$ of T, and he stated without proof that the result holds similarly for uncountable theories (to be precise, he stated that his main result holds for an uncountable theory, but his proof of Vaught's theorem was almost identical to the proof of his main result). It has since been pointed out to the author that in [Sh 90], Shelah proved a even more general theorem from which the full generalization of both Morley's and Vaught's result could be recovered. It is however unnecessary for us to do so for this exposition, and we will omit Shelah's longer proof here.

2.7 Ehrenfeucht-Mostowski Models

The construction of what we now call Ehrenfeucht-Mostowski models was first introduced in [E 56], where in contrast to saturated models, these models realize very few types. Following [Ho 97], we will introduce them using a category-theoretical setup:

Definition 2.7.1. *For a linear order X and a $Y \subseteq X$, for $i, j \in X$ we say that $i \equiv j \mod Y$ or $i \equiv_Y j$ if for every $k \in Y$, $i < k$ iff $j < k$.*

Definition 2.7.2. *A **Ehrenfeucht-Mostowski functor (EM functor)** in the language L is a functor from the category of linear orders (with order embeddings as morphisms) to the category of L-structures (with L-embeddings as morphisms) satisfying the following conditions:*

- *For every linear ordering η, $\eta \subseteq F(\eta)$ and $\langle \eta \rangle_L = F(\eta)$*
- *For every order embedding $f : \eta \longrightarrow \xi$, $F(f) : F(\eta) \longrightarrow F(\xi)$ is an L-embedding which extends f*

*For a theory T, a L-structure M is an **Ehrenfeucht-Mostowski model (EM model) of** T if $M \models T$ and there is some linear ordering η and a EM functor F such that $F(\eta) = M$.*

Remark. We require $\eta \subseteq F(\eta)$ as sets, but the ordering on η does not have to be related to the interpretations on $F(\eta)$ in anyway.

Lemma 2.7.3. *For any two linear orders η, ξ and $\bar{a} \in [\eta]^k$, $\bar{b} \in [\xi]^k$, there is a linear ordering ξ with order embeddings $f : \eta \longrightarrow \zeta$, $g : \xi \longrightarrow \zeta$ such that $f(a(i)) = g(b(i))$ for $i < k$.*

Proof. Consider the signature $\{<\}$ where $<$ is a relation symbol of arity 2. Then any linear order is a structure of this language of orders which satisfies the formula

$$\forall x, y(x = y \land x \not< y \land y \not< x) \lor (x < y \land x \neq y \land y \not< x) \lor (y < x \land x \neq y \land x \not< y)$$

Then $Th_\eta(\eta) \cup Th_\xi(\xi) \cup \{a(i) = b(i) : i < k\}$ is satisfiable by compactness, and thus by some linear order ζ. The interpretations a^ζ, b^ζ for each $a \in \eta, b \in \xi$ gives the desired embeddings. □

Proposition 2.7.4. *Suppose $\phi(x_0, \ldots, x_{n-1}) \in L$ is quantifier-free, F is an EM functor and η, ξ are linear orders. Then for every $\bar{a} \in [\eta]^n, \bar{b} \in [\xi]^n$, $F(\eta) \models \phi(\bar{a})$ iff $F(\xi) \models \phi(\bar{b})$.*

Proof. By the above lemma, let ζ be a linear ordering and $f : \eta \longrightarrow \zeta$, $g : \xi \longrightarrow \zeta$ be order embeddings with $f(\bar{a}) = g(\bar{b})$. Note then that $F(f), F(g)$ are embeddings of the language. Since ϕ is quantifier-free, by Proposition 1.2.8, $F(\eta) \models \phi(\bar{a})$ iff $F(\zeta) \models \phi(f(\bar{a}))$ iff $F(\zeta) \models \phi(g(\bar{b}))$ iff $F(\xi) \models \phi(\bar{b})$. □

Corollary 2.7.5. *For every quantifier-free ϕ, η is ϕ-indiscernible in $F(\eta)$.*

Proof. Let $\xi = \eta$ in the above proposition. □

This suggests the following definitions:

Definition 2.7.6. *Given a L-structure M and η a linear ordering contained in M, $Th_L(M, \eta) = \{\phi(x_0, \ldots, x_{n-1}) \in L : \text{for every } \bar{a} \in [\eta]^n, M \models \phi(\bar{a})\}$.*
For a EM functor F in L, $Th(F) = \{\phi(x_0, \ldots, x_{n-1}) \in L : \text{for every linear order } \eta, \phi \in Th(F(\eta), \eta)\}$.

Corollary 2.7.7. *If F is a EM functor in L and for some linear order η, $Th_L(F(\eta))$ has Skolem functions, then $Th(F)$ is a complete theory. Thus for any linear order ξ, $F(\eta) \equiv F(\xi)$, with ξ indiscernible in $F(\xi)$ i.e. $Th(F) = Th(F(\eta), \eta)$.*

Proof. If $Th(F(\eta))$ has Skolem functions, then by Proposition 1.3.7 every formula $\phi(x_0, \ldots, x_{n-1}) \in L$ is equivalent (relative to $Th_L(F(\eta))$ to some quantifier-free formula $\psi(x_0, \ldots, x_{n-1}) \in L$. Now for any $\bar{a} \in [\eta]^n$ and $\bar{b} \in [\zeta]^n$ for a linear order ζ, by the above proposition $F(\eta) \models \psi(\bar{a})$ iff $F(\zeta) \models \psi(\bar{b})$, and so either $\psi \in Th(F)$ or $\neg\psi \in Th(F)$ i.e. $Th(F)$ is complete. In particular this shows that for any linear order ξ, $F(\eta) \equiv F(\xi)$, and again by Proposition 1.3.7 ξ is indiscernible in $F(\xi)$. □

To prove the existence of EM functors, we will need some tools:

Definition 2.7.8. *$T \subseteq L$ is **=-closed** if it is a set of closed atomic formulas satisfying:*

- *For every closed term t of L, $(t = t) \in T$*

- *For every atomic $\phi(x) \in L$ with one free variable, if $(s = t) \in T$ then $\phi(s) \in T$ iff $\phi(t) \in T$.*

Lemma 2.7.9. *Suppose T is a $=$-closed set of formulas. Then there is a L-structure M such that:*

- *For every closed atomic ϕ, $M \models \phi$ iff $\phi \in T$*
- *For every $m \in M$, there is a closed term t in L such that $M \models m = t$*

Proof. Let X be the set of closed terms in L, and define the equivalence relation $\sim$ on X by $s \sim t$ iff $(s = t) \in T$ ($\sim$ is both symmetric and transitive as T is $=$-closed). Let M be the equivalence classes of X, and we interpret M into a L-structure:

- For a constant symbol c, $c^M = c_\sim$
- For a function symbol f, $f^M(\bar{t_\sim}) = [f(\bar{t})]_\sim$
- For a relation symbol P, $P^M(\bar{t_\sim})$ iff $P(\bar{t}) \in T$

We leave it to the reader to verify that f^M and P^M are well-defined because T is $=$-closed. Note by induction on term complexity, for every closed term t, $t^M = t_\sim$. Thus for closed terms s, t, $(s = t) \in T$ iff $M \models s = t$. This similarly holds for all closed atomic formulas. Thus M satisfies the desired properties. $\square$

Lemma 2.7.10. *Suppose M is an L-structure with $\omega \subseteq M$, $\langle \omega \rangle = M$ and for every atomic $\phi \in L$, ω is ϕ-indiscernible in M. Then there exists an EM functor F in L with $F(\omega) \cong M$.*

Proof. First, for every linear order η we will construct a $F(\eta)$. Let us define $T(\eta) \subseteq L_\eta$ by the following: for every closed atomic $\phi \in L_\eta$, note that there is a $\psi(x_0, \ldots, x_{n-1}) \in L$ and an $\bar{a} \in [\eta]^n$ such that $\phi = \psi(\bar{a})$. Then $\phi \in T(\eta)$ iff for every $\bar{b} \in [\omega]^n$, $M \models \psi(\bar{b})$.

Claim. For any linear ordering η, $T(\eta)$ is $=$-closed.

Clearly, for a term $t(x_0, \ldots, x_{n-1})$ of L, for every $\bar{b} \in [\eta]^n$, $M \models t(\bar{b}) = t(\bar{b})$ and so $(t(\bar{b}) = t(\bar{b})) \in T(\eta)$. Next, for any atomic $\phi(x) \in L_\eta$ with one free variable, let $\psi(x, \bar{y}) \in L$ and $\bar{c} \in [\eta]^{<\omega}$ be such that $\phi(x) = \psi(x, \bar{c})$. If s, t are terms in L with $\bar{c} \in \eta$ such that $(s(\bar{c}) = t(\bar{c})) \in T(\eta)$, then $\phi(s(\bar{c})) \in T(\eta)$ iff for every $\bar{d} \in [\omega]^{<\omega}$,

$$M \models \phi(s(\bar{d})) \wedge (s(\bar{d}) = t(\bar{d})) \Leftrightarrow M \models \phi(t(\bar{d})) \wedge (s(\bar{d}) = t(\bar{d})$$

iff $\phi(t(\bar{c})) \in T(\eta)$. Thus $T(\eta)$ is $=$-closed.

Thus by the above lemma there is an $N \models T(\eta)$ such that for every $n \in N$, there is a closed term t in L_η such that $N \models n = t$. Note that for any $a < b$, $a, b \in \eta$, since $M \not\models a = b$, $a = b \notin T(\eta)$ and thus as guaranteed by Lemma 2.7.9, $N \models a \neq b$. Therefore we can identify η with $\{a^N : a \in \eta\}$, such that $\eta \subseteq N$ and $\langle \eta \rangle = N$.

Now, if ξ is a linear order and $f : \eta \longrightarrow \xi$ is an order embedding, note that for every atomic $\phi(x_0, \ldots, x_{n-1}) \in L$ and $\bar{a} \in [\eta]^n$, $F(\eta) \models \phi(\bar{a})$ iff $M \models \phi(\bar{b})$ for every $\bar{b} \in [\omega]^n$ iff $F(\xi) \models \phi(f(\bar{a}))$ (as f preserves strict ordering). In particular, for any term t in L and $\phi(x) \in L$ with one free variable, $F(\eta) \models \phi(t(\bar{a}))$ iff $F(\xi) \models \phi(t(f(\bar{a})))$. Then as $\langle \eta \rangle = F(\eta)$, the map $F(f) : t^{F(\eta)}(\bar{a}) \longmapsto t^{F(\xi)}(f(\bar{a}))$ is an L-embedding which extends f. Further, clearly if $f : \eta \longrightarrow \xi$ and $g : \xi \longrightarrow \zeta$ are order embeddings, then $F(f \circ g) = F(f) \circ F(g)$ i.e. F is a EM functor.

Finally, consider the map $i : F(\omega) \longrightarrow M$ defined by $i(t^{F(\omega)}(\bar{a})) = t^M(\bar{a})$ for any term $t(x_0, \ldots, x_{n-1})$ and $\bar{a} \in [\omega]^n$. We claim that this is well-defined: $F(\omega) \models t(\bar{c}) = s(\bar{c})$ with $\bar{c}$ an increasing sequence in ω iff for every $\bar{d}$ an increasing sequence of the same length, $M \models t(\bar{d}) = s(\bar{d})$, which, as ω is ϕ-indiscernible in M, is equivalent to $M \models t(\bar{c}) = s(\bar{c})$. Thus i is also a L-embedding, and as ω is a generates M, i is also surjective i.e. an isomorphism. □

Theorem 2.7.11 (Ehrenfeucht-Mostowski Theorem). *Let T be a theory in L with infinite models and Skolem functions. Then there is a EM functor F in L with $Th(F) \supseteq T$.*

Proof. Suppose M is a model of T with $\omega \subseteq M$ (since M is infinite, we can simply take any countably infinite subset of M and identify it as ω). Let $\bar{c} = (c_i : i < \omega)$ be new symbols, and define $T' \subseteq L_{\bar{c}}$ to contain:

1. T
2. For every $\phi(x_0, \ldots, x_{n-1}) \in L$ and $\bar{a}, \bar{b} \in [\bar{c}]^n$, the formula $\phi(\bar{a}) \leftrightarrow \phi(\bar{b})$
3. For every $\phi(x_0, \ldots, x_{n-1}) \in Th(M, \omega)$, the formula $\phi(c_0, \ldots, c_{n-1})$

Claim. T' is satisfiable

We will show that T' is finitely satisfiable. Let $U \subseteq T'$ be finite, then let $\phi_0(x_0, \ldots, x_{k-1}), \ldots, \phi_{m-1}(x_0, \ldots, x_{k-1}) \in L$ and $\bar{c}|_k$ be such that $\phi_0(\bar{c}|_k), \ldots, \phi_{k-1}(\bar{c}|_k)$ are precisely the formulas in $U \backslash T$ (we may add redundant variables to each ϕ_i to ensure they follow this form). We then define the following equivalence relation on $[\omega]^k$:

$$\bar{a} \sim \bar{b} \text{ iff for every } i < k, M \models \phi_i(\bar{a}) \Leftrightarrow M \models \phi_i(\bar{b})$$

Note that there are only 2^k equivalence classes of $\sim$. Thus by Ramsey's theorem (see Appendix A, Theorem A.0.1) there is an infinite subset $W = \{w_i : i < \omega\} \subseteq \omega \subseteq M$ such that every $\bar{a} \in [W]^k$ belongs in the same equivalence class. So interpret $c_i^A = w_i$ for $i < k$, which guarantees that $A \models U$. Thus by compactness, T' is satisfiable.

Let $N \models T'$. Note that as $(v_0 \neq v_1) \in Th(M, \omega)$, for any $i < j < \omega$, $T' \models c_i \neq c_j$ and thus $c_i^N \neq c_j^N$ i.e. we can identify ω with $(c_i^N : i < \omega)$ such that $\omega \subseteq N$. Take the L-reduct $N|_L$, and define $A = \langle \omega \rangle \subseteq N|_L$ (recall that as $T \subseteq T'$ has Skolem functions, $A \preceq N|_L$). The definition of T' above guarantees that ω is indiscernible in N and thus in A, so by the above lemma, there is a EM functor F in L such that $A \cong F(\omega)$.

It remains to show that $T \subseteq Th(F)$. Note that by definition $T \subseteq T'$, so $N|_L \models T$ and therefore $F(\omega) \models T$. But since T has Skolem functions, by Proposition 1.3.7, for every $\phi \in L$ there is a quantifier-free $\psi \in L$ such that $T \models \phi \leftrightarrow \psi$. By Lemma 2.7.5 we know that ω is ψ-indiscernible in $F(\omega)$ for every quantifier-free ψ, and thus ω is in fact indiscernible in $F(\omega)$. Therefore for every $\phi \in L$, either $\phi \in Th(F(\omega), \omega)$ or $\neg\phi \in Th(F(\omega), \omega)$. But by Lemma 2.7.7 every EM model from F is elementarily equivalent, and therefore $Th(F)$ is complete. Since $F(\omega) \models Th(F)$, this implies that $T \subseteq Th(F)$. $\square$

Since the above theorem guarantees the existence of EM functors for a theory with Skolem functions, in practice we will often say that "M is an EM model of T" when we actually mean "M is an EM model of T_{Sk}". The most important use of EM models is the following property:

Proposition 2.7.12. *Suppose T is a theory in L with Skolem functions and for some cardinal κ and some EM functor F, $F(\kappa) \models T$. Then for every $X \subseteq F(\kappa)$, $F(\kappa)$ realizes at most $|X| + |L|$ many complete 1-types over X.*

Proof. Note that by definition of EM functors, for every $a \in F(\kappa)$ there is a term $t_a(\bar{x})$ and a $\bar{b}_a \in [\kappa]^{<\omega}$ such that $F(\kappa) \models a = t_a(\bar{b}_a)$. Let $X \subseteq F(\kappa)$, and let $W = \bigcup\{\text{Rang } \bar{b}_a : a \in X$, so $|W| \leq |X| + \aleph_0$. However, the indiscernibility of κ in $F(\kappa)$ guarantees that for every term $s(\bar{x})$ of L and $\bar{c} \in [\kappa]^{<\omega}$, $tp^{F(\kappa)}(s^{F(\kappa)}(\bar{c})/X)$ is determined completely by s and the equivalence class under $\equiv_W$ (see Definition 2.7.1) of each c_i. Since there are $\leq |W| + \aleph_0$ many choices of a sequence of equivalence classes of $\equiv_W$ and $|L|$ many terms, this gives an upper bound of $|W| + |L| + \aleph_0 \leq |X| + |L|$ many 1-types over X which are realized in $F(\kappa)$. $\square$

Chapter 3

Shelah's Categoricity theorem

3.1 Notation and premises

$\alpha, \beta, \gamma, i, j$ will denote (von Neumann) ordinals, λ, μ, κ will denote infinite cardinals (as initial ordinals). δ will denote a limit ordinal, and k, l, m, n will denote natural numbers. α^β will denote the set of sequences of length β with value in α, and $\alpha^{<\beta} = \bigcup_{\gamma<\beta} \alpha^\beta$. ν, η will denote sequences in α^β, and $l(\nu)$ denotes the length of the sequence. σ will denote a permutation of a finite set.

For a linearly ordered set X, $[X]^\beta$ will denote strictly increasing sequences in X^β. $[X]^{<\beta} = \bigcup_{\gamma<\beta}[X]^\gamma$, and ζ, ξ will denote sequences in $[X]^\beta$.

Fix a signature such that the number of relation symbols is at most equal to the number of constant and function symbols, and let L be the language of this signature. t, s, τ will denote terms of L, $\phi, \psi, \theta, \xi, \rho$ will denote formulas in L, and x, y, z will denote variables in L. ϕ^0 will denote ϕ, and ϕ^1 will denote $\neg\phi$. Δ, Γ, Σ will denote sets of formulas in L.

Definition 3.1.1. *A theory T is **λ-categorical** if there is only one model (up to isomorphism) of T of cardinality λ.*

Let T be a complete theory with infinite models which is λ-categorical for a $\lambda > |T| = |L|$, and let T_{Sk} denote the Skolemization of T in the language L_{Sk} which exists by Lemma 1.3.2. Fix a universal model $\bar{M}$ of T that is $\bar{\kappa}$-saturated. By models M, N we will mean elementary submodels of $\bar{M}$ much smaller than $\bar{\kappa}$, and by sets A, B, C we will mean subsets of $\bar{M}$ much smaller than $\bar{\kappa}$. We will also (by an abuse of notation) understand M, N to be subsets of $\bar{M}$. a, b, c, d, e will denote elements of $\bar{M}$, and by $\models \phi(\bar{a})$ we mean $\bar{M} \models \phi(\bar{a})$. For any $\bar{a} \in A^{<\omega}$, we will (by an abuse of notation) write $\bar{a} \in A$.

By a Δ-n-type we will mean a Δ-n-type over some set A with respect to $\bar{M}$, and for any a and set A, $tp_\Delta(a/A) = tp_\Delta^{\bar{M}}(a/A)$ and $S_{\Delta,n}(A) = S_{\Delta,n}^{\bar{M}}(A)$. An Δ-n-indiscernible sequence (or set) over A will be a sequence(or set respectively) in $\bar{M}$ that is Δ-n-indiscernible over A with respect to $\bar{M}$. As usual, we omit Δ when $\Delta = L$, and we replace $\{\phi\}$ by ϕ. p, q will denote Δ-n-types.

3.2 Preliminaries

Proposition 3.2.1. *If T is λ-categorical for a $\lambda > |T|$, then for every $\mu < \lambda, \mu \geq |T|$, T is μ-stable. In particular, T is stable.*

Proof. By Theorem 2.7.11, there is a EM functor F in L_{Sk} with $Th(F) \supseteq T_{Sk}$, and by Proposition 2.7.12, $F(\lambda)$ is a model of T such that for every $C \subseteq F(\lambda)$, $F(\lambda)$ realizes $|C| + |T|$ many complete 1-types over C. So suppose for a contradiction that T is not μ-stable, and let A be a set such that $|A| \leq \mu < |S_1(A)|$. Then by applying the Downward Löwenheim-Skolem theorem on the universal model, there is a model M with cardinality λ which contains A and realizes $\mu^+ \leq \lambda$ many 1-types over A. But by λ-categoricity $M \cong F(\lambda)$, so there is a set $B \subseteq F(\lambda)$ with $|B| < \mu^+$ such that $F(\lambda)$ realizes μ^+ many 1-types over B, a contradiction. Thus T is μ-stable, and therefore stable. □

Note. One can in fact show that T is μ-stable for every $\mu \geq |T|$ i.e. T is superstable (see [Sh 90], for example).

Definition 3.2.2. *A model M is* ***locally saturated*** *if for every finite Δ and p a Δ-m-type over M with $|p| < |M|$, p is realized in M.*

Proposition 3.2.3. *If T is λ-categorical for a $\lambda > |T|$, then every model of T is locally saturated.*

Proof. We first show that every model of cardinality λ is locally saturated: Given a finite Δ, note that for any set A and $m < \omega$, $|S_{m,\Delta}(A)| = |A| + \aleph_0$. So let M_0 be a model with $|M_0| = \lambda$, and let $(M_i : i \leq \lambda)$ be an elementary chain where M_{i+1} realizes all every $S_{m,\Delta}(M_i)$ and $|M_{i+1}| = \lambda$ i.e. M_{i+1} is a model of size λ realizing (for new constants $(c_i : i < \lambda)$)

$$Th_M(M) \cup \bigcup \{p_i(c_i) : p_i \in S_{m,\Delta}(M)\}$$

So for any $\mu < \lambda$, any Δ-m-type p over M_{μ^+} with $|p| = \mu$ is a type over M_i for some $i < \mu^+$ (since μ^+ is regular), and therefore realized in M_{μ^+}. Since T is λ-categorical and $|M_{\mu^+}| = \lambda$, this implies that for any model M with $|M| = \lambda$ and any p a Δ-m-type over M with $|p| = \mu$, p is realized in M. But this holds for every $\mu < \lambda$, so in fact any model of cardinality λ is locally saturated.

So now consider the more general case, and assume for a contradiction that M is a model of T which is not locally saturated: so let Δ, m is finite and p a Δ-m-type with $|p| < |M|$ such that p is not realized in M. We may assume that Δ is in fact a singleton $\{\phi\}$: if $\Delta = \{\phi_i(\bar{x}, \bar{y}_i) : i < |\Delta|\}$, then consider

$$\phi(\bar{x}, y, y_{0,0}, \ldots, y_{|\Delta|-1,0}, y_{0,1}, \ldots, y_{|\Delta|-1,1}, \bar{y}_0, \ldots, \bar{y}_{|\Delta|-1}) = \bigwedge_{i<|\Delta|, j<2} y = y_{i,j} \to \phi_i(\bar{x}, \bar{y}_i)^j$$

So that every Δ-m-type corresponds naturally to an unique ϕ-m-type, and a Δ-m-type is realized in M iff the corresponding ϕ-m-type is realized in M.

Now, if $p = \{\phi(\bar{x}, \bar{a}_i) : i < |p|\}$, let $A = \bigcup_{i<|p|} \bar{a}_i$ so that p is a type over A. Of course $|A| \leq |p| \cdot \aleph_0 < |M|$, so by Proposition 2.4.11, for any finite Δ', and $n < \omega$, there is a Δ'-n-indiscernible set over A of size $|A|^+ \leq |M|$ in M (since T is stable by the previous proposition). Also, by the above proposition T is stable, so by Corollary 2.4.9 there is a ψ_ϕ such that any ϕ-type over any set B is (ψ_ϕ, B)-definable. So let $\bar{a}_\phi$ be such that $\psi_\phi(\bar{y}, \bar{a}_\phi)$ defines (possibly some completion of) p.

Next, we wish to construct a $|T|^+$-saturated model N with the following properties:

- There are distinct $(\bar{b}_i : i, |T|^+)$ such that $q = \{\phi(\bar{x}, \bar{b}_i) : i < |T|^+\}$ is a type over N which is omitted by N
- N contains a sequence $(c_j : j < |T|^+)$ which is indiscernible over $B = \{\bar{b}_i : i < |T|^+\}$

To construct N, let P be a new unary relation symbol, and let$(\bar{b}_i : i < |T|^+), (c_j : j < |T|^+)$ be new constant symbols with $|\bar{b}_i| = n$. Then consider the set of formulas which is the union of:

1. T
2. $\{c_i \neq c_j \wedge \bar{b}_i \neq \bar{b}_j : i < j < |T|^+\}$
3. $\{\psi_\phi(\bar{b}_i, \bar{a}_\phi) \wedge \bigwedge_{k<n} P(b_{k,i}) : i < |T|^+\}$
4. $\{\exists \bar{x} \bigwedge_{l<m} \phi(\bar{x}, \bar{b}_{i_m}) : m < \omega, i_0, \ldots, i_m < |T|^+\}$
5. $\{\forall \bar{x} \exists y_1, \ldots, y_n \bigwedge_{k<n} P(y_k) \wedge \neg(\phi(\bar{x}, \bar{y}) \leftrightarrow \psi_\phi(\bar{y}, \bar{a}_\phi))\}$
6. $\{\varphi(c_{\zeta(0)}, \ldots, c_{\zeta(m-1)}, \bar{b}_{i_0}, \ldots, \bar{b}_{i_n}) \leftrightarrow \varphi(c_{\xi(0)}, \ldots, c_{\xi(m-1)}, \bar{b}_{i_0}, \ldots, \bar{b}_{i_n}) : \varphi(x_0, \ldots, x_{m-1}, \bar{y}_0, \ldots, \bar{y}_n) \in L, \zeta, \xi \in [|T|^+]^m, i_0, \ldots, i_n < |T|^+\}$

Suppose that N_0 satisfies this set. Note that (2) ensures that the new constants have distinct interpretations; (3) ensures that all the $\bar{b}_i$ satisfies $\psi_\phi(\bar{b}_i, \bar{a}_\phi)$ and have $\bar{b}_i \in P(N_0)$; (4) ensures (by compactness) that q is a ϕ-type of $N_0|_L$; (5) ensures that the ϕ-type defined by $\psi_\phi(x, \bar{a}_\phi)$ is omitted in N_0; and (6) ensures that $(c_i^{N_0} : i < |T|^+)$ is an L-indiscernible set over B.

To show that such an N_0 exists, note that by expanding the interpretation of M by $P^M = A$, for every finite subset of the formulas, we can interpret the $\bar{b}_j$'s as one of the $\bar{a}_i \in A$ and c_j's to be an element in a Δ'-indiscernible sequence over A for an appropriate finite Δ'. This satisfies the finite subset, and so by compactness the set of formulas is satisfiable. Then by the Upward Löwenheim-Skolem theorem there is a model N_0 of cardinality $2^{|T|}$, and by the method of Proposition 1.5.8 there is a $|T|^+$-saturated extension N of N_0 with $|N| = 2^{|T|}$. Note that the type $\{\phi(\bar{x}, \bar{b}_i) : i < |T|^+\}$ can be omitted by ensuring (5) is satisfied at every stage of the construction.

Let $B = \{\bar{b}_i : i < |T|^+\}$, $\mu = \beth_{(2^{|T|})^+}(|T|^+)$ and again let $(c_i : |T|^+ \leq i < \mu)$ be new symbols. Since

$$Th_N(N) \cup \{\phi(c_{\zeta(0)}, \dots, c_{\zeta(n-1)}, \bar{b}) : \bar{b} \in B, n < \omega, \zeta \in [\mu]^n, N \models \phi(c_0, \dots, c_{n-1}, \bar{b})\}$$

is satisfiable by compactness, there is an interpretation of $(c_i : i < \mu)$ such that it is an indiscernible sequence over B. Denoting this indiscernible sequence by I, let N' be a $|T|^+$-prime $|T|^+$-constructible model over $B \cup I$ (which exists by Theorem 2.2.7). Notice that since every finite sequence in I realizes the same type as a sequence in $(c_i : i < |T|^+)$ and N is $|T|^+$-saturated but omits the type $\{\phi(x, \bar{b}_i) : i < |T|^+\}$, N' also omits this type (as N' is $|T|^+$-constructible over $B \cup I$).

Finally, let L^+ be the language consisting only of L, another new relation symbol R and new constants identifying the tuple $\bar{a}_\phi$. Defining $R^{N'} = B$ and $T^+ = Th_{L^+}(N')$, note that the 1-type $\{R(x)\}$ is satisfied $|B| = |T|^+$ times in $N'|_{L^+}$ but $|N'| = \beth_{(2^{|T|})^+}(|T|^+)$, and so by Theorem 2.6.2 there is a model M' of T^+ with cardinality λ and $|R(M')| \leq |T|$. But since N' omits q, $N'|_{L^+} \models \forall \bar{x} \exists y_1, \dots, y_n \bigwedge_{k<n} R(y_k) \wedge \psi_\phi(\bar{y}, \bar{a}_\phi) \wedge \neg\phi(\bar{x}, \bar{y})$, and therefore M' also omits q. Therefore $M'|_L$ is a model of T with cardinality λ which omits the type $\{\phi(\bar{x}, \bar{b}_i) : i < |T|^+\}$, and is thus not locally saturated. This contradicts the first part of the proof, where we showed that every model of T with cardinality λ is locally saturated. Thus there does not exist any model of T which is not locally saturated. $\square$

Corollary 3.2.4. *If T is λ-categorical in a $\lambda > |T|$, $M \precneqq N$ are models of T, $\bar{a} \in M$ and $\phi(x, \bar{y})$ is such that $\phi(x, \bar{a})$ is nonalgebraic, then there is a $c \in N - M$ such that $\models \phi(c, \bar{a})$. Additionally, $|\phi(M, \bar{a})| = |M|$.*

Proof. Note that for any model M_1 with $\bar{a} \in M_1$, by compactness

$$p = \{\phi(x, \bar{a}) \wedge x \neq c : c \in \phi(M_1, \bar{a})\}$$

is a type over M_1 which is not realized in M_1. As M_1 is locally saturated by the above proposition, $|p| = |M_1|$ which implies that $|\phi(M_1, \bar{a})| = |M_1|$.

Now, assume for a contradiction that $M, N, \phi(x, \bar{a})$ is a counterexample to the corollary. Then by Theorem 2.5.9, there is a model A such that $|A| > |\phi(A)|$, a contradiction. $\square$

3.3 Degree of a formula

The concept of Morley rank is used in the proof of Morley's categoricity theorem: any λ-categorical theory is $\aleph_0$-stable and therefore has a well-defined Morley rank i.e. is totally transcendental, which implies the existence of a strongly minimal formula. Unfortunately, for an uncountable language even if the theory is λ-categorical it is not necessarily totally transcendental. Shelah, however, introduced the degree as a weaker measure of complexity:

Definition 3.3.1. *A set of formulas $\Gamma = \{\phi_i(x, \bar{a}_i) : i < \kappa\}$ with parameters is m-**almost contradictory** if for every $\Sigma \subseteq \Gamma$ with $|\Sigma| = m$, $\models \neg\exists x \bigwedge_{i\in\Sigma} \phi_i(x, \bar{a}_i)$. Γ is **almost contradictory** if it is m-almost contradictory for some $m < \omega$.*

Definition 3.3.2. *For every formula $\phi(x, \bar{a})$ with parameters, the degree $D(\phi(x, \bar{a}))$ is defined to be either -1, an ordinal or ∞ (where we consider $\alpha < \infty$ for every ordinal α):*

- *$D(\phi(x, \bar{a})) \geq 0$ if $\models \exists x\phi(x, \bar{a})$*
- *For a limit ordinal δ, $D(\phi(x, \bar{a})) \geq \delta$ if for every $\alpha < \delta$, $D(\phi(x, \bar{a})) \geq \alpha$*
- *$D(\phi(x, \bar{a})) \geq \alpha + 1$ if there is a formula $\psi(x, \bar{y})$ and sequences $\bar{c}_i$ for $i < |T|^+$ such that*
    1. *For every $i < |T|^+$, $D(\phi(x, \bar{a}) \wedge \psi(x, \bar{c}_i)) \geq \alpha$*
    2. *$\{\psi_i(x, \bar{c}_i) : i < |T|^+\}$ is almost contradictory*

Note. Shelah generalizes both the concept of rank and degree in [Sh 90], and the degree here corresponds to what is denoted $D^1(-, L, |T|^+)$.

Lemma 3.3.3.

1. *If $\models \forall x\phi(x, \bar{a}) \to \psi(x, \bar{b})$, then $D(\phi(x, \bar{a})) \leq D(\psi(x, \bar{b}))$*
2. *If $tp(\bar{a}/\emptyset) = tp(\bar{b}/\emptyset)$, then $D(\phi(x, \bar{a})) = D(\phi(x, \bar{b}))$*
3. *There is an ordinal $\alpha_0 < (2^{|T|})^+$ such that there is no $\phi(x, \bar{a})$ with $D(\phi(x, \bar{a})) = \alpha_0$. Thus if $D(\phi(x, \bar{a})) \geq \alpha_0$, then $D(\phi(x, \bar{a})) > \alpha_0$*

Proof. For (1), we will prove the following claim: for every α, if $\models \forall x\phi(x, \bar{a}) \to \psi(x, \bar{b})$ and $D(\phi(x, \bar{a})) \geq \alpha$, then $D(\psi(x, \bar{b})) \geq \alpha$:

- For $\alpha = 0$, if $D(\phi(x, \bar{a})) \geq 0$ then there is an c such that $\models \phi(x, \bar{a})$. Then by assumption $\models \psi(c, \bar{b})$ and therefore $D(\psi(x, \bar{b})) \geq 0$
- If the claim holds for all $\alpha < \delta$ for a limit δ, then trivially it holds also for δ
- Suppose $D(\phi(x, \bar{a})) \geq \alpha + 1$, then there is a $\varphi(x, \bar{y})$ and sequences $\bar{c}_i$ for $i < |T|^+$ such that $\{\varphi(x, \bar{c}_i) : i < |T|^+\}$ is almost contradictory and $D(\phi(x, \bar{a}) \wedge \varphi(x, \bar{c}_i)) \geq \alpha$. Note that

  $$\models \forall x\phi(x, \bar{a}) \wedge \varphi(x, \bar{c}_i) \to \psi(x, \bar{b}) \wedge \varphi(x, \bar{c}_i)$$

  and so by the inductive hypothesis, $D(\psi(x, \bar{b}) \wedge \varphi(x, \bar{c}_i)) \geq \alpha$. By definition, thus $D(\psi(x, \bar{b})) \geq \alpha + 1$

We will use a similar induction for (2): for every ordinal α, if $tp(\bar{a}/\emptyset) = tp(\bar{a}/\emptyset)$ then $D(\phi(x, \bar{a})) \geq \alpha$ iff $D(\phi(x, \bar{a})) \geq \alpha$.

1. If $D(\phi(x,\bar{a})) \geq 0$, then $(\exists y \phi(y,\bar{x})) \in tp(\bar{a}/\emptyset) = tp(\bar{b}/\emptyset)$ and so $\models \exists y \phi(y,\bar{b})$ i.e. $D(\phi(x,\bar{b})) \geq 0$

2. The claim is trivial by definition for limit cases

3. If $D(\phi(x,\bar{a})) \geq \alpha + 1$, then there exists $\psi(x,\bar{y})$ and $\bar{c}_i$ for $i < |T|^+$ such that $D(\phi(x,\bar{a}) \wedge \psi(x,\bar{c}_i)) \geq \alpha$ and $\{\psi(x,\bar{c}_i) : i < |T|^+\}$ is n-almost contradictory for some $n < \omega$. We will then define $\bar{d}_i$ inductively for $i < |T|^+$: suppose that $\bar{d}_j$ has been defined for all $j < i$ such that

   $$tp(\bar{d}_j/\{\bar{b}\} \cup \{\bar{d}_k : k < j\}) = \{\phi(\bar{x},\bar{b},\bar{d}_{k_0},\dots,\bar{d}_{k_{n-1}}) : \\ \phi(\bar{x},\bar{a},\bar{c}_{k_0},\dots,\bar{c}_{k_{n-1}}) \in tp(\bar{c}_j/\{\bar{a}\} \cup \{\bar{c}_k : k < j\})\} \tag{3.1}$$

   i.e. $f_i : \bar{a} \mapsto \bar{b}, \bar{c}_j \mapsto \bar{d}_j$ for $j < i$ is an elementary map. Then this ensures that

   $$\{\phi(\bar{x},\bar{b},\bar{d}_{k_0},\dots,\bar{d}_{k_{n-1}}) : \phi(\bar{x},\bar{a},\bar{c}_{k_0},\dots,\bar{c}_{k_{n-1}}) \in tp(\bar{c}_i/\{\bar{a}\} \cup \{\bar{c}_j : j < i\})\}$$

   is a type over $\{\bar{b}\} \cup \{\bar{d}_j : i < j\}$, and define $\bar{d}_i$ to realize this type. This construction implies that $\{\psi(x,\bar{d}_i) : i < |T|^+\}$ is also n-almost contradictory, and also that $tp(\bar{a} \frown \bar{c}_i/\emptyset) = tp(\bar{b} \frown \bar{d}_i/\emptyset)$. Then by the inductive hypothesis, $D(\phi(x,\bar{a}) \wedge \psi(x,\bar{d}_i)) \geq \alpha$, and therefore $D(\phi(x,\bar{b})) \geq \alpha + 1$

Finally, for (3), note that the number of complete n-types over $\emptyset$ is at most $2^{|T|}$, so there are at most $|T| \cdot 2^{|T|} = 2^{|T|}$ choices of a formula $\phi(x,\bar{y})$ and a complete type over $\emptyset$. By (2), the degree of $\phi(x,\bar{a})$ is determined only by $\phi(x,\bar{y})$ and $tp(\bar{a}/\emptyset)$, and so the range of D has cardinality $2^{|T|}$. Therefore there is some $\alpha_0 \in (2^{|T|})^+$ which is not in the range of D. □

For the next few results, let $\alpha_0 < (2^{|T|})^+$ be a fixed ordinal not in the range of D.

Lemma 3.3.4. *If $D(x = x) > \alpha_0$ and $\mu = |T|^+$, then for $\nu \in \mu^{<\omega}$ there are formulas $\phi_\nu(x,\bar{a}_\nu)$ with parameters satisfying, for every $\nu \in \mu^{<\omega}$:*

1. *For every $k < l(\nu)$, $\models \forall x \phi_\nu(x,\bar{a}_\nu) \to \phi_{\nu|_k}(x,\bar{a}_{\nu|_k})$*
2. *$D(\phi_\nu(x,\bar{a}_\nu)) > \alpha_0$*
3. *$\{\phi_{\nu\frown i}(x,\bar{a}_{\nu\frown i}) : i < \mu\}$ is almost contradictory*
4. *For every $i < \mu$, $\phi_{\nu\frown i} = \phi_{\nu\frown 0}$*

Proof. We will define $\phi_\nu(x,\bar{a}_\nu)$ inductively: let $\phi_{<>}(x,\bar{a}_{<>}) = (x = x)$. If $\phi_\nu(x,\bar{a}_\nu)$ is defined satisfying the above conditions, then as $D(\phi_\nu(x,\bar{a}_\nu)) \geq \alpha_0 + 1$, there is a $\psi(x,\bar{y})$ and $\bar{c}_i$ for $i < |T|^+ = \mu$ such that $D(\phi_\nu(x,\bar{a}_\nu) \wedge \psi(x,\bar{c}_i)) \geq \alpha_0$ and $\{\psi(x,\bar{c}_i) : i < \mu\}$ is almost contradictory. Defining $\phi_{\nu\frown i}(x,\bar{a}_{\nu\frown i}) = \phi_\nu(x,\bar{a}_\nu) \wedge \psi(x,\bar{c}_i)$ then satisfies all the above conditions. □

Theorem 3.3.5. *If T is λ-categorical for a $\lambda > |T|$, then $D(x = x) < \infty$. Thus by Lemma 3.3.3(1), every formula has degree $< \infty$.*

Proof. Consider T_{Sk} which is the Skolemization of T in the language L_{Sk}, so by Theorem 2.7.11 let M^* be the EM model of T_{Sk} over the linearly ordered set $\{y_i : i < \lambda\}$. Note by Corollary 2.7.7, $\{y_i : i < \lambda\}$ is an indiscernible sequence. Moreover as M^* is an EM model, by definition $M^* = \langle y_i : i < \lambda \rangle_{Sk}$. Therefore for every $\bar{a} \in M^*$, there is a finite sequence of terms $\bar{\tau} \in L_{Sk}$ and $i_1 < \cdots < i_n < \lambda$ such that $M^* \models \bar{a} = \bar{\tau}(y_{i_1}, \ldots, y_{i_n})$.

So assume for a contradiction that $D(x = x) = \infty$. By the above lemma, there are formulas $\phi_\nu(x, \bar{a}_\nu)$ for $\nu \in \mu^{<\omega}$ ($\mu = |T|^+$ as in the above lemma) which satisfies those conditions. By appending unnecessary constants, we may assume that for $\nu \neq \eta, \bar{a}_\nu \neq \bar{a}_\eta$. Now, for every limit $\delta < \mu$ with $cf(\delta) = \omega$, choose a $\eta_\delta \in [\mu]^\omega$ with $\delta = \sup\{\eta_\delta(n) : n < \omega\}$ i.e. η_δ is an ω-length strictly increasing sequence in μ which is a cofinal subset of δ. Let $W = \{\eta_\delta : \delta < \mu, cf(\delta) = \omega\}$, and note that $|W| = \mu$ as the limit ordinals δ with cofinality ω is a cofinal subset of μ. Then, for $\eta \in W$, define $p_\eta = \{\phi_{\eta|_n}(x, \bar{a}_{\eta|_n}) : n < \omega\}$ (which is a satisfiable 1-type by the previous lemma) and let c_η realize p_η. Finally, define $A = \bigcup\{\bar{a}_\nu : \nu \in \mu^{<\omega}\} \cup \{c_\eta : \eta \in W\}$.

Since $|W| = \mu$, we see that $|A| \leq \mu = |T|^+ \leq \lambda$, and therefore by the Löwenheim-Skolem theorems there is a model M with $|M| = \lambda$ and $A \subseteq M$. Since T is λ-categorical, necessarily $M \cong M^*|_L$, and so we may assume WLOG that $A \subseteq M^*$. So for every $\bar{a}_\nu$, fix a particular finite sequence of terms $\bar{\tau}_\nu$ in L_{Sk} and a $\zeta_\nu \in [\lambda]^{n_\nu}$ such that $M^* \models \bar{a}_\nu = \bar{\tau}_\nu(y_{\zeta_\nu(1)}, \ldots, y_{\zeta_\nu(n_\nu)})$. Now, we shall define, for $n < \omega$, sets $X_n \subseteq \lambda$ which satisfies the following:

1. $X_0 = \emptyset$, and for all $n < \omega$, $|X_n| \leq |T|$
2. For every $n < \omega$, there is an $\nu \in \mu^{<\omega}$ such that $\bar{a}_\nu \in \langle X_{n+1} \rangle_{Sk} - \langle X_n \rangle_{Sk}$
3. For every $n < \omega$ and $\nu, \sigma \in \mu^{<\omega}$, if $\bar{a}_\nu \in \langle X_n \rangle_{Sk}$ and $\max \nu \geq \max \sigma$, then $\bar{a}_\sigma \in \langle X_n \rangle_{Sk}$
4. If $\bar{a}_\nu \in \langle X_n \rangle_{Sk}$ but there exists an $i < \mu$ such that $\bar{a}_{\nu^\frown i} \notin \langle X_n \rangle_{Sk}$, then there are infinitely many $j < \mu$, such that $\bar{\tau}_{\nu^\frown i} = \bar{\tau}_{\nu^\frown j}$, $\zeta_{\nu^\frown i}(k) \equiv \zeta_{\nu^\frown j}(k) \mod X_n$ (see Definition 2.7.1) for each $k \leq n_{\nu^\frown i} = n_{\nu^\frown j}$, and such that $\bar{a}_{\nu^\frown j} \in \langle X_n \rangle_{Sk}$

The construction is by induction on $n < \omega$: suppose that X_k, for $k \leq n$, has been constructed to satisfy the above conditions, with the sole exception that we weaken (2) for the case of $k = n$ and instead require that there is some $\nu \in \mu^{<\omega}$ such that $\bar{a}_\nu \notin \langle X_n \rangle_{Sk}$. We construct X_{n+1} in several stages:

1. First, let $X'_n = X_n \cup \{\zeta_\nu(1), \ldots, \zeta_\nu(n_\nu)\}$ so that $\bar{a}_\nu \in \langle X'_n \rangle_{Sk}$. Of course, $|X'_n| = |X_n| + k \leq |X_n| + \aleph_0 \leq |T|$
2. Next, consider λ / X_n, which is the set of equivalence classes of $\lambda \mod X_n$: it is immediate that $|\lambda / X_n| \leq |X_n| + \aleph_0$. So for every $\mathscr{C} \in \lambda / X_n$, let $\mathscr{C}' = \mathscr{C}$ if it is finite, or choose $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| = \aleph_0$. Then define

$X_n^{(2)} = X'_n \cup \bigcup\{\mathscr{C}' : \mathscr{C} \in \lambda/X_n\}$. Note as there are $|X_n| + \aleph_0 \leq |T|$ choices of $\mathscr{C} \in \lambda/X_n$ and each $|\mathscr{C}'| \leq \aleph_0$, $|X_n^{(2)}| \leq |T|$.

3. We then use a supplementary induction. Let $Y_0 = X_n^{(2)}$, and for $l < \omega$, we will construct $Y_{l+1} \supseteq Y_l$ with $|Y_l| \leq |T|$:

   (a) If Y_l is defined and $l+1$ is even, let $\beta = \sup\{(\max\nu)+1 : \bar{a}_\nu \in \langle Y_l\rangle_{Sk}\}$. Note that as $|Y_l| \leq |T|$, $|\langle Y_l\rangle_{Sk}| \leq |T| < |T|^+ = \mu$, and therefore $\beta < \mu$ (since μ is regular). So define $Y_{l+1} = Y_l \cup \{\zeta_\sigma(k) : \sigma \in \beta^{<\omega}, k \leq n_\sigma\}$ which ensures that for every σ with $\max\sigma < \beta$, $\bar{a}_\sigma \in \langle Y_{l+1}\rangle_{Sk}$. Since $|\beta^{<\omega}| = |\beta| + \aleph_0$, $|Y_{l+1}| \leq |Y_l| + (\aleph_0 \cdot (|\beta| + \aleph_0)) \leq |T|$

   (b) If Y_l is defined and $l+1$ is odd, consider the all the pairs $(\nu, i) \in \mu^{<\omega} \times \mu$ such that:

   - $\bar{a}_\nu \in \langle Y_l\rangle_{Sk}$
   - $\bar{a}_{\nu\frown i} \notin \langle Y_l\rangle_{Sk}$
   - There are only finitely many $j < \mu$ such that $\bar{\tau}_{\nu\frown i} = \bar{\tau}_{\nu\frown j}$ and for each $k \leq n_{\nu\frown i} = n_{\nu\frown j}$, $\zeta_{\nu\frown i}(k) \equiv_{X_n} \zeta_{\nu\frown j}(k)$

   Then let Y_{l+1} contain Y_l and $\zeta_{\nu\frown i}(k)$ for every such pair (ν, i), so that $\bar{a}_{\nu\frown i} \in \langle Y_{l+1}\rangle_{Sk}$. Note that as $|L_{Sk}| = |T|$, given a particular $\nu \in \mu^{<\omega}$ there are only $|T|$ choices for $\bar{\tau}_{\nu\frown i}$, and at most $|X_n| + \aleph_0 \leq |T|$ choices of each equivalence class $\zeta_{\nu\frown i}(k)/X_n$. So consider the function $i \mapsto (\bar{\tau}_{\nu\frown i}, (\zeta_{\nu\frown i}(1)/X_n, \ldots, \zeta_{\nu\frown i}(n_{\nu\frown i})/X_n))$: this function partitions μ into $|T|$ many equivalence classes, and we are requiring Y_{l+1} to contain $\zeta_{\nu\frown i}(1), \ldots, \zeta_{\nu\frown i}(n_{\nu\frown i})$ iff i belongs to a finite equivalence class. Thus $|Y_{l+1}| \leq |Y_l| + |T| \cdot \aleph_0 \leq |T|$

4. Finally, let $X_{n+1} = \bigcup_{l<\omega} Y_l$. Since each $|Y_l| \leq |T|$, $|X_{n+1}| < |T|$

Claim. X_{n+1}, constructed in this way, satisfies the conditions for the inductive construction.

That condition (1) is satisfied has been shown along the way in the construction. The first step guarantees that (2) is satisfied with respect to X_n, and if $\bar{a}_\nu \in \langle X_{n+1}\rangle_{Sk}$, then $\bar{a}_\nu \in \langle Y_l\rangle_{Sk}$ for some $l < \omega$, so step 3a guarantees that condition (3) is met. For condition (4), assume that $\bar{a}_\nu \in \langle X_{n+1}\rangle_{Sk}$, $i < \mu$ is such that $\bar{a}_{\nu\frown i} \notin \langle X_{n+1}\rangle_{Sk}$. If there does exists infinitely many $j < \mu$ with $\bar{\tau}_{\nu\frown i} = \bar{\tau}_{\nu\frown j}$, $\zeta_{\nu\frown i}(k) \equiv_{X_n} \zeta_{\nu\frown j}(k)$ for each $k \leq n_{\nu\frown i} = n_{\nu\frown j}$, then step 2 guarantees that at least countably infinitely many $\bar{a}_{\nu\frown j}$ is in $\langle X_{n+1}\rangle_{Sk}$ as every equivalence class of λ/X_n is represented either completely or countably many times in X_{n+1}. On the other hand, if $i < \mu$ is such that there does not exist infinitely many such $j < \mu$, then as $\bar{a}_\nu \in \langle Y_l\rangle_{Sk}$ for some $l < \omega$, step 3b guarantees that $\bar{a}_{\nu\frown i} \in \langle Y_{l+1}\rangle_{Sk}$, which then trivially ensures that condition (4) is satisfied. Finally, since $|X_{n+1}| \leq |T|$, $|\langle X_{n+1}\rangle_{Sk}| = |T| < |T|^+ = \mu$, so there must exist some $\nu \in \mu^{<\omega}$ such that $\bar{a}_\nu \notin \langle X_{n+1}\rangle_{Sk}$ (since for $\nu \neq \sigma$, $\bar{a}_\nu \neq \bar{a}_\sigma$).

So let $X = \bigcup_{n<\omega} X_n$, and define $\delta = \sup\{\nu(n) : n < l(\nu), \bar{a}_\nu \in \langle X\rangle_{Sk}\}$. Note that condition (2) and (3) of the above construction guarantees that for every

$n < \omega$ and $\nu \in \mu^{<\omega}$ such that $\bar{a}_\nu \in \langle X_n \rangle_{Sk}$, there is a σ with $\max \nu < \max \sigma$ such that $\bar{a}_\sigma \in \langle X_{n+1} \rangle_{Sk}$, which implies that δ is a limit ordinal with cofinality ω (since $X = \bigcup_{n<\omega} X_n$). Therefore, by the definition of W there is an $\eta \in W$ with η a cofinal increasing sequence in δ. Then by definition of A, there is a $c_\eta \in A \subseteq M^*$ which realizes the type p_η. So let τ be a term in L_{Sk} and $i(1) < \cdots < i(n) < \lambda$ be such that $M^* \models c_\eta = \tau(y_{i(1)}, \ldots, y_{i(n)})$.

Now, for $l \leq n$, let $j(l) = \inf\{j \in X : j \geq i(l)\}$, and let $k_0 < \omega$ be the least such that $j(1), \ldots, j(n) \in X_{k_0}$ and $\bar{a}_{\eta|_1} \in \langle X_{k_0} \rangle_{Sk}$ (a k_0 exists satisfying this last condition by the definition of δ as the supremum of ν with $\bar{a}_\nu \in \langle X \rangle_{Sk}$, η as a cofinal increasing sequence in δ and by condition (3) of the construction of X_n). Then let $k < \omega$ be maximal such that $\bar{a}_{\eta|_k} \notin \langle X_{k_0+1} \rangle_{Sk}$: since $\sup\{\eta(n) : n < \omega, \bar{a}_{\eta|_n} \in \langle X_{k_0+1} \rangle_{Sk}\} < \delta$ by definition of δ because of conditions (2) and (3), and η is an increasing cofinal sequence in δ, it cannot be the case that $\bar{a}_{\eta|_k} \in \langle X_{k_0+1} \rangle_{Sk}$ for all $k < \omega$, and therefore such a maximal k exists.

Denote $\eta|_k = \nu$ and $i = \eta(k)$. Since $\bar{a}_\nu \in \langle X_{k_0+1} \rangle_{Sk}$ but $\bar{a}_{\nu\frown i} \notin \langle X_{k_0+1} \rangle_{Sk}$, by condition (4) of the construction of X_{k_0+1}, there exists infinitely many $\beta < \mu$ such that $\bar{\tau}_{\nu\frown\beta} = \bar{\tau}_{\nu\frown i}$, $\zeta_{\nu\frown\beta}(l) \equiv_{X_{k_0}} \zeta_{\nu\frown i}$ for $l \leq n_{\nu\frown\beta} = n_{\nu\frown i}$ and $\bar{a}_{\nu\frown\beta} \in \langle X_{k_0+1} \rangle_{Sk}$. Further, for each β as above and any $l \leq n_{\nu\frown i}$, (assuming WLOG that $\zeta_{\nu\frown\beta}(l) < \zeta_{\nu\frown i}(l)$) suppose for a contradiction that there exists an $m \leq n$ such that $\zeta_{\nu\frown\beta}(l) \leq i(m) < \zeta_{\nu\frown i}(l)$: then as $j(m) = \inf\{j \in X : j(m) \geq i(m)\} \in X_{k_0}$ by definition of k_0, this implies that $\zeta_{\nu\frown\beta}(l) \not\equiv_{X_{k_0}} \zeta_{\nu\frown i}(l)$, contradicting the definition of β. Similar reasoning shows that there is no $m \leq n$ such that $\zeta_{\nu\frown\beta}(l) < i(m) \leq \zeta_{\nu\frown i}(l)$. This shows that in fact $\zeta_{\nu\frown\beta}(l) \equiv \zeta_{\nu\frown i}(l)$ mod $X_{k_0} \cup \{i(1), \ldots, i(n)\}$.

But as M^* is an EM model over $\{y_i : i < \lambda\}$, the indiscernibility of $\{y_i : i < \lambda\}$ thus implies that for each β, $\bar{a}_{\nu\frown i}$ and $\bar{a}_{\nu\frown\beta}$ realize the same L_{Sk}-type over $\{y_\alpha : \alpha \in X \text{ or } \alpha = i(l), l \leq n\}$, and in particular the same type over c_η. Thus, since c_η realizes p_η, there are infinitely many $\beta < \mu$ such that $M^* \models \phi_{\nu\frown i}(c_\eta, \bar{a}_{\nu\frown\beta})$. By property (4) of the previous lemma, for every $\beta < \mu$, $\phi_{\nu\frown i} = \phi_{\nu\frown\beta}$, so in fact for infinitely many $\beta < \mu$, $M^* \models \phi_{\nu\frown\beta}(c_\eta, \bar{a}_{\nu\frown\beta})$. But $\{\phi_{\nu\frown i}(x, \bar{a}_{\nu\frown i}) : i < \mu\}$ is almost contradictory by property (3) of the previous lemma, a contradiction. This completes the proof by contradiction, from which we conclude that $D(x = x) < \infty$. □

3.4 Weakly Minimal formulas

In the proof of Morley's categoricity theorem, one shows that T being categorical in some uncountable cardinal implies that T is totally transcendental (i.e. the Morley rank of $(x = x)$ is less than ∞), from which one then proves that T has a strongly minimal formula. Having proved that $D(x = x) < \infty$, we now proceed to find a weakly minimal formula, which is a generalization of strongly minimal formulas.

We will now use freely that T has these properties, which we proved in the previous sections:

- T is λ-categorical for some $\lambda > |T|$

- T is $|T|$-stable
- Every model of T is locally saturated
- For models $M \precneqq N$ and a nonalgebraic formula $\theta(x, \bar{a})$ with parameters $\bar{a} \in M$, $\theta(N, \bar{a}) \nsubseteq M$
- $D(x = x) < \infty$

Definition 3.4.1. *Given a formula $\phi(x, \bar{y})$, a formula with parameters $\psi(x, \bar{a})$ is **ϕ-minimal** if there is no $\bar{b}$ such that $|\{c \in \bar{M} :\models \psi(c, \bar{a}) \wedge \phi(c, \bar{b})\}| \geq \aleph_0$ and $|\{c \in \bar{M} :\models \psi(c, \bar{a}) \wedge \neg\phi(c, \bar{b})\}| \geq \aleph_0$*
*Given a formula with parameters $\theta(x, \bar{a})$, $\psi_0(x, \bar{a}_1), \ldots, \psi_n(x, \bar{a}_n)$ is a **ϕ-partition of** $\theta(x, \bar{a})$ if:*

1. *$\models \forall x \theta(x, \bar{a}) \leftrightarrow \bigvee_{i \leq n} \psi_i(x, \bar{a}_i)$*
2. *For $i \neq j$, $\models \neg\exists x \psi_i(x, \bar{a}_i) \wedge \psi_j(x, \bar{a}_j)$*
3. *Each $\psi_i(x, \bar{a}_i)$ is satisfiable, nonalgebraic and ϕ-minimal*

Lemma 3.4.2.

1. *If $\models \theta(c, \bar{a})$ and $\psi_i(x, \bar{a}_i)$, $i \leq n$ is a ϕ-partition, then there is some $i \leq n$ such that $\models \theta(c, \bar{a}) \wedge \psi_i(c, \bar{a})$*
2. *If $\theta(x, \bar{a})$ has a ϕ-partition, then it has a ϕ-partition of the form $\psi(x, \bar{a}_1), \ldots, \psi(x, \bar{a}_n)$*

Proof.

1. This is trivial by definition.
2. Suppose $\psi_i(x, \bar{a}_i)$, $i \leq n$ is a ϕ-partition of $\theta(x, \bar{a})$. Then define

$$\psi(x, \bar{y}_1, \ldots, \bar{y}_n, z, z_1, \ldots, z_n) = \bigvee_{i \leq n} (z = z_i \rightarrow \psi_i(x, \bar{y}_i))$$

   Let $b_1, \ldots, b_n$ be distinct elements, and define $\bar{a'}_i = \bar{a}_1 \frown \cdots \frown \bar{a}_n \frown b_i \frown b_1 \frown \cdots \frown b_n$. Then clearly $\psi(x, \bar{a'}_i)$ is a ϕ-partition of $\theta(x, \bar{a})$.

□

Remark. If $\theta(x, \bar{a})$ has a ϕ-partition, then we may refer to it as if it is unique and denote the ϕ-partition by $\psi_\phi(x, \bar{a}_i)$.

Definition 3.4.3. *The formula $\theta(x, \bar{a})$ is **weakly minimal** if for every formula $\phi(x, \bar{y})$, there is a ϕ-partition of $\theta(x, \bar{a})$.*
*A set B **partitions** $\theta(x, \bar{a})$ if $\bar{a} \in B$ and for every ϕ, $\theta(x, \bar{a})$ has a ϕ-partition $\psi(x, \bar{a}_i)$ with $\bar{a}_i \in B$.*
*A type p is **minimal** if there is no $\phi(x, \bar{a})$ such that both $p \cup \{\phi(x, \bar{a})\}$ and $p \cup \{\neg\phi(x, \bar{a})\}$ are nonalgebraic types.*

Lemma 3.4.4.

1. *If $\theta(x, \bar{a})$ is weakly minimal and $\bar{a} \in M$ for a model M, then there is some $A \subseteq M$, $|A| \leq |T|$ such that A partitions $\theta(x, \bar{a})$.*

2. *If A partitions $\theta(x, \bar{a})$ and $p \in S_1(A)$ is such that $\theta(x, \bar{a}) \in p$, then p is minimal.*

Proof.

1. Suppose $\theta(x, \bar{a})$ is weakly minimal, so in particular for every formula $\phi(x, \bar{y})$, there are $n < \omega$, $l_1, \ldots, l_n \in 2$ and $k_1, \ldots, k_n < \omega$ such that

$$\begin{aligned}\bar{M} \models \exists \bar{y}_1, \ldots, \bar{y}_n (\theta(x, \bar{a}) \leftrightarrow \bigvee_{i \leq n} \psi(x, \bar{y}_i)) \\ \wedge (\bigwedge_{i \neq j} \neg \exists z \psi(z, \bar{y}_i) \wedge \psi(z, \bar{y}_j)) \\ \wedge (\bigwedge_{i \leq n} \exists^{\leq k_i} z \psi(z, \bar{y}_i) \wedge \phi(z, \bar{y}_i)^{l_i})\end{aligned}$$

   Since this is a formula with parameters $\bar{a} \in M$, by the Tarski-Vaught test this formula holds in M (as $M \precneqq \bar{M}$). The choice of $\bar{y}_1, \ldots, \bar{y}_n$ in M then gives the desired ϕ-partition of $\theta(x, \bar{a})$. But since for each formula ϕ, the ϕ-partition $\psi_\phi(x, \bar{a}_i)$ only requires finitely many elements $\bar{a}_1, \ldots, \bar{a}_n$ in M, taking the union of these parameters over all ϕ together with $\bar{a}$ gives a set of cardinality at most $|T|$.

2. Since p is a type, let c be such that $\models p(c)$. So $\models \theta(c, \bar{a})$, and therefore $\models \bigvee_{i \leq n} \psi_\phi(c, \bar{a}_i)$, with $\bar{a}_i \in A$ as A partitions $\theta(x, \bar{a})$. If $i \leq n$ is such that $\models \psi_\phi(c, \bar{a}_i)$, then as p is a complete type over A, $\psi_\phi(x, \bar{a}_i) \in p$. So for any $\bar{b}$, there is an $n < \omega$ such that either $\models \exists^{\leq n} x \psi_\phi(x, \bar{a}_i) \wedge \phi(x, \bar{b})$ or $\models \exists^{\leq n} x \psi_\phi(x, \bar{a}_i) \wedge \neg\phi(x, \bar{b})$. But as $p \cup \{\phi(x, \bar{b})^i\} \models \psi_\phi(x, \bar{a}_i) \wedge \phi(x, \bar{b})^i$, this implies that either $p \cup \{\phi(x, \bar{b})\}$ or $p \cup \{\neg\phi(x, \bar{b})\}$ is algebraic.

$\square$

Theorem 3.4.5. *If $|M_0| > |T|$, $\theta(x, \bar{a})$ is a nonalgebraic formula such that $D(\theta(x, \bar{a}))$ is minimal amongst nonalgebraic formulas and $\bar{a} \in M_0$, then $\theta(x, \bar{a})$ is weakly minimal.*

Proof. Given a fixed ϕ, to show that there is a ϕ-partition of $\theta(x, \bar{a})$, we will try to define inductively a $\bar{a}_\nu$ for every $\nu \in 2^{<\omega}$: suppose that $\bar{a}_\eta$ is defined, and there exists a $\bar{b}$ such that $\{\theta(x, \bar{a})\} \cup \{\psi(x, \bar{a}_{\eta|_l})^{\eta(l)} : l < l(\eta)\} \cup \{\phi(x, \bar{b})^i\}$ is nonalgebraic for both $i = 0, 1$, then let $\bar{a}_{\eta^\frown i} = \bar{b}$. Otherwise $\bar{a}_{\eta^\frown i}$ is undefined.

Note that if we can only define $\bar{a}_\nu$ for $\nu \in S \subsetneq 2^{<\omega}$, S a finite subset, then a ϕ-partition exists: enumerate $S' = \{\nu_i : i < n\}$ where S' are the leaves of S, and define

$$\psi_i(x, \bar{a}_i) = \theta(x, \bar{a}) \wedge \bigwedge_{j < l(\nu_i)} \phi(x, \bar{a}_{\eta|_j})^{\eta(j)}$$

By construction, clearly each $\psi_i(x, \bar{a}_i)$ is ϕ-minimal and nonalgebraic. Additionally, if $i \neq j$, then $\nu_i \neq \nu_j$ since they are both leaves in S and thus $\psi_i(x, \bar{a}_i)$ and $\psi_j(x, \bar{a}_i)$ is contradictory by construction. Finally, if $\models \theta(c, \bar{a})$ then we can always trace a path in S to a particular leaf $\nu_i \in S'$ such that $\models \psi_i(c, \bar{a}_i)$ by construction. Therefore $\psi_i(x, \bar{a}_i)$ partitions $\theta(x, \bar{a})$, and trivially if $\models \psi_i(c, \bar{a}_i)$ then $\models \theta(c, \bar{a})$. Thus there is a ϕ-partition of $\theta(x, \bar{a})$.

So now, assume that we can define infinitely many $\bar{a}_\nu$. Hence by König's lemma, there is a $\eta \in 2^\omega$ such that for every $l < \omega$, $\bar{a}_{\nu|_l}$ is defined. WLOG we may assume that for infinitely many $l < \omega$, $\nu(l) = 0$, so for $k < \omega$ let l_k be the k-th l such that $\eta(l) = 0$, and let $\bar{a}_k = \bar{a}_{\eta|_{l_k}}$. Thus by construction, for every $n < \omega$, $\{\theta(x, \bar{a})\} \cup \{\phi(x, \bar{a}_i) : i < n\} \cup \{\neg\phi(x, \bar{a}_n)\}$ is a nonalgebraic type (since otherwise $\bar{a}_{\eta|_{l_k}}$ would not be defined). Let c_n realize this type i.e. $\models \phi(c_n, \bar{a}_i)$ if $i < n$ and $\models \neg\phi(c_n, \bar{a}_n)$. Now, we define two equivalence classes on $[\omega]^2$: $(n, m) \sim (0, 1)$ iff $\models \phi(c_n, \bar{a}_m) \leftrightarrow \phi(c_0, \bar{a}_1)$. Then by Ramsey's theorem (Appendix A, A.0.1), there is an infinite $W \subseteq \omega$ such that either $(n, m) \sim (0, 1)$ for every $(n, m) \in [W]^2$ or $(n, m) \nsim (0, 1)$ for every $(n, m) \in [W]^2$. In either case, we may replace W by ω by re-indexing, and note that there is some $n < m$ such that $\models \phi(c_n, \bar{a}_m)$ iff for every $n < m$, $\models \phi(c_n, \bar{a}_m)$. But since by definition we have $\models \phi(c_n, \bar{a}_m)$ if $m < n$, if $\models \neg\phi(c_n, \bar{a}_m)$ for $n < m$ then ϕ has the order property and is thus an unstable formula (see Proposition 2.1.3), so by Proposition 2.1.11 this contradicts that T is $|T|$-stable. Therefore we have $\models \phi(c_n, \bar{a}_m)$ iff $m \neq n$.

Suppose for a contradiction that for every $m < \omega$, there is a sequence $\bar{b}_0, \ldots, \bar{b}_{m-1}$ such that for every choice of $w \subseteq m = \{0, \ldots, m-1\}$, $\{\phi(x, \bar{b}_k) : k \in w\} \cup \{\neg\phi(x, \bar{b}_k) : k \in m - w\}$ is satisfiable. Then by compactness, if $(c_i : i < 2^\omega), (\bar{b}_k : k < \omega)$ are new constants and $(W_i : i < 2^\omega)$ enumerates the subsets of ω, the set

$$\begin{aligned} T \cup \bigcup \{ \{\phi(c_i, \bar{b}_k) : k \in W_i\} \cup \{\neg\phi(c_i, \bar{b}_k) : k \in \omega - W\} : i < 2^\omega \} \\ \cup \{c_i \neq c_j : i \neq j\} \cup \{\bar{b}_k \neq \bar{b}_l : k \neq l\} \end{aligned}$$

is satisfiable i.e. there is a sequence $(\bar{b}_k : k < \omega)$ such that for every $W \subseteq \omega$, $\{\phi(x, \bar{b}_k) : k \in W\} \cup \{\phi(x, \bar{b}_k) : k \in \omega - W_i\}$ is satisfiable. This contradicts Corollary 2.1.12 as T is $|T|$-stable, and therefore there is some $m' < \omega$ such that for every $\bar{b}_0, \ldots, \bar{b}_{m'-1}$, there is a $w \subseteq m'$ such that $\{\phi(x, \bar{b}_k) : k \in W\} \cup \{\neg\phi(x, \bar{b}_k) : k \in m' - W\}$ is not satisfiable.

Now, fix a particular $w \subseteq m'$, and consider the following map $[\omega]^{m'} \longrightarrow 2$: $(i_1, \ldots, i_{m'}) \mapsto 0$ iff $\models \exists x (\bigwedge_{k \in w} \phi(x, \bar{a}_{i_k})) \wedge (\bigwedge_{k \in m' - w} \neg\phi(x, \bar{a}_{i_k}))$. Again by Ramsey's theorem there is an infinite $W \subseteq \omega$ such that the mapping is constant on $[W]^{m'}$, so we may again re-index and assume $W = \omega$. But as there are only finitely many $w \subseteq m'$, we can reiterate this process until we have, for every $w \subseteq m'$ and $i_1 < \cdots < i_{m'} < \omega$, either:

$$\models \exists x (\bigwedge_{k \in w} \phi(x, \bar{a}_{i_k})) \wedge (\bigwedge_{k \in m' - w} \neg\phi(x, \bar{a}_{i_k})) \tag{3.2}$$

or

$$\models \neg\exists x(\bigwedge_{k\in w}\phi(x,\bar{a}_{i_k}))\wedge(\bigwedge_{k\in m'-w}\neg\phi(x,\bar{a}_{i_k})) \tag{3.3}$$

If there exists some $w_1, w_2 \subseteq m'$ with $|w_1| = |w_2|$ but such that (3.2) holds for w_1 while (3.3) holds for w_2, then the formula

$$\exists x(\bigwedge_{k\in w_1}\phi(x,\bar{a}_{i_k}))\wedge(\bigwedge_{k\in m'-w_1}\neg\phi(x,\bar{a}_{i_k}))$$

would be connected and asymmetric (see Definition 2.3.6) over $(\bar{a}_n : n<\omega)$, and therefore by Proposition 2.3.6 contradicting that T is $|T|$-stable. Thus whether or not (3.2) holds depends solely on $|w|$. But we have defined m' such that there is some $w_0 \subseteq m'$ for which (3.2) fails, and therefore (3.3) holds for every $w \subseteq m'$ with $|w| = |w_0|$ i.e. there does not exist a b such that :

1. $\models \phi(b,\bar{a}_n)$ for at least $|w_0|$ many $n<\omega$; and
2. $\models \neg\phi(b,\bar{a}_n)$ for at least $|m'-w_0|$ many $n<\omega$

So define the formula

$$\psi(x,\bar{b}_n) = \theta(x,\bar{a})\wedge\neg\phi(x,\bar{a}_{n+m'})\wedge\bigwedge_{k<m'}\phi(x,\bar{a}_k)$$

and note that if there is a b such that $\models \psi(b,\bar{b}_n)$ for at least m' many $n<\omega$, then it there are m' many $n<\omega$ and m' many $n'<\omega$ such that $\models \phi(b,\bar{a}_n)\wedge\neg\phi(x,\bar{a}_{n'})$, which contradicts what we proved above. Thus the formulas $\psi(x,\bar{b}_n)$ are m'-almost contradictory, and in particular almost contradictory. However, we have proven above that each $\{\theta(x,\bar{a})\}\cup\{\phi(x,\bar{a}_i) : i<n\}\cup\{\neg\phi(x,\bar{a}_n)\}$ is a nonalgebraic type, and therefore each $\psi(x,\bar{b}_n)$ is satisfiable and nonalgebraic.

Claim. There exists an n_ψ such that for every model M and $\bar{e}\in M$, if $|\psi(M,\bar{e})| \geq n_\psi$ then $|\psi(M,\bar{e})| \geq \aleph_0$

Assume for a contradiction that such an n_ψ does not exist, so that there is a model M such that for every $n<\omega$, there is an $\bar{e}_n \in M$ with $n < |\psi(M,\bar{b}_n)| < \aleph_0$. By the upward Löwenheim-Skolem theorem, we may assume that $|M| > 2^{\aleph_0}$ (since the condition $n<|\psi(M,\bar{b}_n)|<\aleph_0$ can be expressed by a formula in L_M). Let $\mathscr{D}$ be an ultrafilter over ω, and define $N = M^\omega/\mathscr{D}$, $\bar{e} = (\bar{e}_n : n<\omega)/\mathscr{D}$. By Theorem 1.6.8, note that $N \models \psi(b,\bar{e})$ iff there is a $D\in\mathscr{D}$ such that for every $n\in D$, $M\models\psi(b(n),\bar{e}_n)$. Therefore there is some $b' \in \prod_{n\in\omega}\psi(M,\bar{e}_n)$ such that $||b'=b|| = D$, and so $b \in \pi(\prod_{n\in\omega}\psi(M,\bar{e}_n))$. But as each $n<|\psi(M,\bar{e}_n)|<\aleph_0$, so $|\psi(N,\bar{e})| = |\pi(\prod_{n\in\omega}\psi(M,\bar{e}_n))| \leq 2^{\aleph_0} < |M| \leq |N|$ (the last equality by Corollary 1.6.9). This implies that $\{\psi(x,\bar{e})\wedge x\neq a : a\in\phi(N,\bar{e})\}$ is a type with cardinality $<|N|$ and is thus realized in N as it is locally saturated (since $N\models T$), a contradiction.

Lastly, we shall define by induction a sequence $(\bar{b}_\alpha : \alpha<|M_0|)$ such that

- $\models \exists^{\geq n_\psi} x\psi(x,\bar{b}_\alpha)$

- For every $\alpha_1 < \cdots < \alpha_{m'} < |M_0|$, $\models \neg\exists x \bigwedge_{k \le m'} \psi(x, \bar{b}_{\alpha_k})$

For $\alpha < \omega$, we can take $\bar{b}_n$ to be the sequence as defined previously. Then, given $\alpha \ge \omega$, suppose $\bar{b}_\beta$ has been constructed satisfying these conditions for $\beta < \alpha$. So consider the set:

$$p(\bar{y}) = \{\exists^{\ge n_\psi} x\psi(x, \bar{y})\} \cup \{\neg\exists x\psi(x, \bar{y}) \wedge \bigwedge_{k<m'} \psi(x, \bar{b}_{\beta_k}) : \beta_1 < \cdots < \beta_{m'-1} < \alpha\}$$

We claim that $p(\bar{y})$ is satisfiable: for any finite subset $q \subseteq p$, there is some $n < \omega$ such that $\bar{b}_n$ does not appear in q, and therefore by the inductive hypothesis satisfies q. Thus by compactness $p(\bar{y})$ is satisfiable and therefore realized by some $\bar{b}_\alpha$. This construction gives us $(\bar{b}_\alpha : \alpha < |M_0|)$, and as each $\models \exists^{\ge n_\psi} x\psi(x, \bar{b}_\alpha)$, by the definition of n_ψ, each $\psi(x, \bar{b}_\alpha)$ is a nonalgebraic formula.

To complete the proof, let $\beta = D(\theta(x, \bar{a}))$, which is minimal amongst nonalgebraic formulas by assumption. So for every $\alpha < |M_0| \ge |T|$, $D(\psi(x, \bar{b}_\alpha)) \ge \beta$. But by construction $\{\psi(x, \bar{b}_\alpha) : \alpha < |M_0|\}$ is m'-almost contradictory, and as $\theta(x, \bar{a})$ is a conjunct in each $\psi(x, \bar{a}_\alpha)$, by the definition of the degree this implies that $D(\theta(x, \bar{a})) \ge \beta + 1$, a contradiction. □

Corollary 3.4.6. *If T is λ-categorical for some $\lambda > |T|$, then there is a nonalgebraic weakly minimal formula $\theta(x, \bar{a})$*

Proof. We have proven previously that such a T satisfies all the conditions listed at the beginning of this section. Now, since the degree of a nonalgebraic formula is an ordinal, in particular the ordinals which are the degree of some nonalgebraic formula $\phi(x, \bar{b})$ is well-ordered. Therefore there is a minimal degree $\beta \ge 0$ for nonalgebraic formulas, and by the above theorem any formula $\theta(x, \bar{a})$ with $D(\theta(x, \bar{a})) = \beta$ is weakly minimal (as $\bar{a}$ is contained in some model M with $|M| > |T|$ by the upward Löwenheim-Skolem theorem). □

Lemma 3.4.7. *Suppose A partitions the nonalgebraic weakly minimal formula $\theta(x, \bar{a})$, and for every formula ϕ, $\psi_\phi(x, \bar{a}_i)$ is a ϕ-partition with $\bar{a}_i \in A$. If $B \supseteq A$, satisfies:*

1. *For every ϕ and i, $|\{c \in B : \models \psi_\phi(c, \bar{a}_i)\}| > |T|$*
2. *For every formula ψ, if $\bar{b} \in B$ is such that $\models \exists x\psi(x, \bar{b}) \wedge \theta(x, \bar{a})$, then there is a $c \in B$ such that $\models \psi(c, \bar{b}) \wedge \theta(c, \bar{a})$*

Then there is a model $M \supseteq B$ with $\theta(M, \bar{a}) \subseteq B$

Proof. Consider $\mathscr{B} = \{B' : B' \supseteq B, B' \text{ satisfies } (2), b \in B' - B \Rightarrow \not\models \theta(b, \bar{a})\}$: if $\mathscr{C} \subseteq \mathscr{B}$ is a chain (under inclusion), then clearly $\bigcup \mathscr{C} \in \mathscr{B}$. Thus by Zorn's lemma, there is a maximal $B^* \in \mathscr{B}$. We will show that B^* is an elementary substructure of $\bar{M}$, which proves the claim as $\theta(B^*, \bar{a}) = \theta(B, \bar{a}) \subseteq B$.

We will proceed by the Tarski-Vaught test: for a $\bar{b}_1 \in B^*$, suppose $\models \exists x\phi_1(x, \bar{b}_1)$. Pick a formula $\phi(x, \bar{b})$ with $\bar{b} \in B^*$ such that $D(\phi(x, \bar{b})) \ge 0$ is minimal amongst formulas such that $\models \forall x\phi(x, \bar{b}) \to \phi_1(x, \bar{b}_1)$. Since $D(\phi(x, \bar{b})) \ge 0$,

$\phi(x,\bar{b})$ is satisfiable and so let a realize $\phi(x,\bar{b})$. If $\models \theta(a,\bar{a})$, then $\models \exists x\theta(x,\bar{a}) \wedge \phi(x,\bar{b})$, and therefore as $\bar{a} \in B \subseteq B^*$ and B^* satisfies (2), there is some $a' \in B^*$ such that $\models \phi(a',\bar{b}) \wedge \theta(a',\bar{a})$. This implies that $\models \phi_1(a',\bar{b}_1)$, and so by the Tarski-Vaught test we are done. Otherwise, it suffices to show that $B^* \cup \{a\}$ satisfies (2), for then the maximality of B^* in $\mathscr{B}$ guarantees that $a \in B^*$, and again we are done by Tarski-Vaught test.

So for a contradiction, assume that $B^* \cup \{a\}$ does not satisfy (2) i.e. there is a formula $\rho_1(x,a,\bar{c}_1)$ with $\bar{c}_1 \in B^*$ such that $\models \exists x\rho(x,a,\bar{c})$ (where we define $\rho(x,a,\bar{c}) = \rho_1(x,a,\bar{c}_1) \wedge \theta(x,\bar{a})$) but there is no $b \in B^*$ such that $\models \rho(b,a,\bar{c})$. Let d be such that $\models \rho(d,a,\bar{c})$, and as $\models \theta(d,\bar{a})$, by Lemma 3.4.2(1) suppose that $\models \psi_\rho(d,\bar{a}_\rho)$. So $\rho(x,a,\bar{c}) \wedge \theta(x,\bar{a})$ is satisfiable (by d), and by definition of $\psi_\rho(x,\bar{a}_\rho)$ as part of the ρ-partition of $\theta(x,\bar{a})$, either $\psi_\rho(x,\bar{a}_\rho) \wedge \rho(x,a,\bar{c})$ is algebraic or $\psi_\rho(x,\bar{a}_\rho) \wedge \neg\rho(x,a,\bar{c})$ is algebraic.

Claim. $\psi_\rho(x,\bar{a}_\rho) \wedge \rho(x,a,\bar{c})$ is algebraic.

Assuming for a contradiction that $\psi_\rho(x,\bar{a}_\rho) \wedge \neg\rho(x,a,\bar{c})$ is algebraic, then all but finitely many elements of $\psi_\rho(\bar{M},\bar{a}_\rho)$ are in $\rho(\bar{M},a,\bar{c})$. But by assumption $|\psi_\rho(B,\bar{a}_\rho)| > |T|$, and therefore there must be some $b \in B \subseteq B^*$ such that $\models \psi_\rho(b,\bar{a}_\rho) \wedge \rho(b,a,\bar{c})$, which contradicts the definition of $\rho(x,a,\bar{c})$. So it must be the case that $\psi_\rho(x,\bar{a}_\rho) \wedge \rho(x,a,\bar{c})$ is algebraic.

This implies that $\rho(x,a,\bar{c})$ is algebraic as $|\psi_\rho(B,\bar{a}_\rho)| > |T|$ by assumption. Thus there is some $m < \omega$ such that $\models \exists^{\leq m}x\rho(x,a,\bar{c})$. Now, define

$$\chi(z,\bar{b},\bar{c}) = \exists x\phi(x,\bar{b}) \wedge \rho(z,x,\bar{c}) \wedge (\exists^{\leq m}y\rho(y,x,\bar{c}))$$

Choosing $x = a$ shows that $\models \chi(d,\bar{b},\bar{c})$, and by its definition above $d \notin B^*$. So if (again by Lemma 3.4.2(1)) $\models \psi_\chi(d,\bar{a}_\chi)$, then let us define $\chi_1(x,\bar{a}^*) = \chi(x,\bar{b},\bar{c}) \wedge \psi_\chi(x,\bar{a}_\chi)$, so that $\models \chi_1(d,\bar{a}^*)$.

Claim. $\bar{a}^* \in B^*$

Note that we have defined $\bar{a}^* = \bar{b} \frown \bar{c} \frown \bar{a}_\chi$, and by assumption $\bar{a}_\chi \in A \subseteq B \subseteq B^*$ since A partitions $\theta(x,\bar{A})$. Likewise, we defined $\bar{c} = \bar{c}_1 \frown \bar{a}$, with $\bar{c}_1 \in B^*$ by definition and $\bar{a} \in A \subseteq B^*$ by assumption as A partitions $\theta(x,\bar{a})$. Finally, we defined $\bar{b} \in B^*$, and so indeed $\bar{a}^* \in B^*$.

Claim. $\chi_1(x,\bar{a}^*)$ is a nonalgebraic formula

Note by definition of d, $\models \rho(d,a,\bar{c})$, and therefore by definition of ρ, $\models \theta(d,\bar{a})$. Therefore $\models \chi_1(d,\bar{a}^*) \wedge \theta(d,\bar{a})$, and thus $\models \exists\chi_1(x,\bar{a}^*) \wedge \theta(x,\bar{a})$. Since $\bar{a}^* \in B^*$ and B^* satisfies (2), there is a $d_0 \in B^*$ such that $\models \chi_1(d_0,\bar{a}^*) \wedge \theta(d_0,\bar{a})$. Now, suppose for an $n < \omega$, $d_0,\ldots,d_n \in B^*$ has been defined such that for each $i \leq n$, $\models \chi_1(d_i,\bar{a}^*) \wedge \theta(d_i,\bar{a}) \wedge \bigwedge_{j<i} d_i \neq d_j$. Since $d \notin B^*$, $\models \chi_1(d,\bar{a}^*) \wedge \theta(d,\bar{a}) \wedge \bigwedge_{i\leq n} d \neq d_i$, and therefore (again, as B^* satisfies (2)) there is a d_{n+1} such that $\models \chi_1(d_{n+1},\bar{a}^*) \wedge \theta(d_{n+1},\bar{a}) \wedge \bigwedge_{i<n+1} d_{n+1} \neq d_i$. So inductively there are $(d_n : n < \omega) \subseteq B^*$ such that for each $n < \omega$, $\models \chi_1(d_n,\bar{a}^*)$. Therefore $\chi_1(x,\bar{a}^*)$ is a nonalgebraic formula.

Recalling that $\psi_\chi(x,\bar{a}_\chi)$ is χ-minimal, that $\chi_1(x,\bar{a}^*) = \chi(x,\bar{b},\bar{c}) \wedge \psi_\chi(x,\bar{a}_\chi)$ is nonalgebraic implies that $\psi_\chi(x,\bar{a}_\chi) \wedge \neg\chi(x,\bar{b},\bar{c})$ is algebraic. Then, as in the previous case, as $|\psi_\chi(B,\bar{a}_\chi)| > |T|$, all but finitely many of the elements in

$\psi_\chi(B,\bar{a}_\chi)$ realize the formula $\chi_1(x,\bar{a}^*)$ i.e. $|\chi(B,\bar{b},\bar{c})| \geq |T|^+$. So let $(b_k : k < |T|^+)$ be such that for each $k < |T|^+$, $\models \chi(b,\bar{b}_k,\bar{c})$, so that the set

$$\{\phi(x,\bar{b}) \wedge \rho(b_k,\bar{b},\bar{c}) \wedge (\exists^{\leq m} y\rho(y,x,\bar{c})) : k < |T|^+\}$$

is a set of satisfiable formulas with parameters in B^* but the set is $m+1$-almost contradictory. Thus by the definition of degree,

$$D(\phi(x,\bar{b})) > \inf\{D(\phi(x,\bar{b}) \wedge \rho(b_k,\bar{b},\bar{c}) \wedge (\exists^{\leq m} y\rho(y,x,\bar{c}))) : k < |T|^+\}$$

But each $\phi(x,\bar{b}) \wedge \rho(b_k,\bar{b},\bar{c}) \wedge (\exists^{\leq m} y\rho(y,x,\bar{c})) \models \phi(x,\bar{b})$ and $\phi(x,\bar{b}) \models \phi_1(x,\bar{b}_1)$, thus contradicting that $\phi(x,\bar{b})$ has minimal degree. This contradiction shows that $\rho_1(x,a,\bar{c}_1)$ cannot exist, and therefore $B^* \cup \{a\}$ satisfies (2). The maximality of B^* in $\mathscr{B}$ thus implies that $a \in B^*$, and therefore by the Tarski-Vaught test, B^* is an elementary substructure of $\bar{M}$. This completes the proof. □

Theorem 3.4.8. *If $\theta(x,\bar{a})$ is a nonalgebraic weakly minimal formula, A partitions $\theta(x,\bar{a})$ with $|A| = |T|$ and M is a model with $|M| > |T|$ and $M \supsetneq A$ where every nonalgebraic $p \in S_1(A)$ with $\theta(x,\bar{a}) \in p$ is realized $|M|$ times in M, then M is a saturated model.*

Proof. Note that for any formula ϕ, if $\psi_\phi(x,\bar{a}_i)$ is a ϕ-partition of $\theta(x,\bar{a})$ with $\bar{a}_i \in A \subsetneq M$, then by definition for any $m \in M$, if $\models \psi_\phi(c,\bar{a}_i)$ then $\models \theta(c,\bar{a})$. But this implies that $\theta(x,\bar{a}) \in tp(c/A)$, and therefore by assumption $tp(c/A)$ is realized $|M|$ times in M. Since $\psi_\phi(x,\bar{a}_i) \in tp(c/A)$ too, thus $|\{c \in M :\models \psi_\phi(c,\bar{a}_i)\}| = |M| > |T|$ i.e. M satisfies condition (1) of the above lemma.

So suppose for a contradiction that M is not saturated, and let p be a complete type with $|p| < |M|$ which is omitted by M, with $B \subseteq M$ such that $p \in S_1(B)$. Since $|p| < |M| > |T|$, this implies that $|B| < |M|$. Choose a formula $\phi(x,\bar{b})$ with $\bar{b} \in M - B$ such that $p \cup \{\phi(x,\bar{b})\}$ is satisfiable and that $\phi(x,\bar{b})$ has minimal degree amongst such formulas: such a choice always exists as $B \subsetneq M$.

Define $p' = p \cup \{\phi(x,\bar{b})\}$, and let a realize p'. By definition of p, necessarily $a \notin M$. Therefore, if $M \cup \{a\}$ satisfies condition (2) of the previous lemma, then by that lemma there is a $N \succneq M$ with $a \in N$ such that $\theta(N,\bar{a}) = \theta(M \cup \{a\},\bar{a})$. Of course, for any $m \in M$ with $\models \theta(m,\bar{a})$, $\theta(x,\bar{a}) \in tp(m/A)$, and therefore by assumption $tp(m/A)$ is realized $|M|$ times in M, and in particular $\theta(x,\bar{a})$ is realized $|M|$ times in M i.e. $\theta(x,\bar{a})$ is a nonalgebraic formula. Thus by Corollary 3.2.4, if $N \succneq M$ then $\theta(N,\bar{a}) \supsetneq \theta(M,\bar{a})$. Therefore we conclude that either $M \cup \{a\}$ does not satisfy condition (2) of the previous lemma, or $\models \theta(a,\bar{a})$ so that $\theta(N,\bar{a}) = \theta(M \cup \{a\},\bar{a}) \supsetneq \theta(M,\bar{a})$.

If $M \cup \{a\}$ does not satisfy condition (2) of the above lemma, then there exists a formula $\bar{\phi}(x,a,\bar{c}_0)$ with $\bar{c}_0 \in M$ such that $\models \exists x\theta(x,\bar{a}) \wedge \bar{\phi}(x,a,\bar{c}_0)$ but no element of M satisfies this formula. In this case, define $\phi(x,a,\bar{c}) = \bar{\phi}(x,a,\bar{c}_0) \wedge \theta(x,\bar{a})$. Otherwise, if $M \cup \{a\}$ satisfies condition (2) of the above lemma, then as shown above $\models \theta(a,\bar{a})$, so let $\psi(x,a,\bar{c}) = \theta(x,\bar{a}) \wedge x = a$. Thus in either case, $\psi(x,a,\bar{c})$ is such that $\models \exists x\psi(x,a,\bar{c})$ but $\psi(x,a,\bar{c})$ is not satisfied by any element of M. Let d be an element which satisfies this formula.

Define $B' = B \cup A \cup \bar{c}$, so that $B' \subseteq M$ but $|T| \leq |B'| < |M|$, and let $q = tp(d/B')$. q cannot be an algebraic type, since as $B' \subseteq M$, any element which realizes an algebraic type over B' is an element of M, which contradicts that $d \notin M$. However, since $\theta(x, \bar{a}) \in q$ and $A \subseteq B'$ partitions $\theta(x, \bar{a})$, by Lemma 3.4.4(2), $q|_A$ is a minimal type. So for any formula $\chi(x, \bar{e})$, either $q|_A \cup \{\chi(x, \bar{e})\}$ is algebraic or unsatisfiable, or $q|_A \cup \{\neg\chi(x, \bar{e})\}$ is algebraic or unsatisfiable. In particular, if $\chi(x, \bar{e}) \in q$, then as d satisfies the type $q|_A \cup \{\chi(x, \bar{e})\}$, and so by the same reasoning as above it is a nonalgebraic type. This implies that $q|_A \cup \{\neg\chi(x, \bar{e})\}$ is algebraic. Thus, if we define $C = \{c \in M : \models q|_A(c)\}$, for every formula $\chi(x, \bar{e}) \in q$, $\chi(x, \bar{e})$ is realized by all but finitely many members of C. Therefore q is realized by all but $|q| \cdot \aleph_0 = |q| = |B'| < |M|$ members of C.

As q is a complete type over $B' \supseteq A$, $q|_A \in S_1(A)$, and as $\theta(x, \bar{a}) \in q|_A$ (since $\bar{a} \in A$ as A partition $\theta(x, \bar{a})$), by assumption $q|_A$ is realized $|M|$ times in M i.e. $|C| = |M|$. By the above observation, q is realized $|M| - |B'| = |M|$ many times in M, and so as $|M| > |T|$, let $(d_k : k < |T|^+) \subseteq M$ be a sequence of distinct elements in M which realize q.

Now, if $q|_A \cup \{\neg\psi(x, a, \bar{c})\}$ is an algebraic type, then all but finitely many elements of C would satisfy $\neg\psi(x, a, \bar{c})$. But as $|M| = |C|$, this contradicts that for every $m \in M$, $\models \neg\phi(x, a, \bar{c})$. However, $q|_A \cup \{\neg\psi(x, a, \bar{c})\}$ is indeed satisfiable, namely by d. Thus necessarily $q|_A \cup \{\psi(x, a, \bar{c})\}$ is an algebraic type (since $q|_A$ is a minimal type). Therefore there is some $\rho(x, \bar{a}^*) \in q|_A$ such that $\psi(x, a, \bar{c}) \wedge \rho(x, \bar{a}^*)$ is algebraic: if not, then (since $q|_A$ is closed under finite conjugation) the set (using new constants $(c_n : n < \omega)$)

$$\bigcup\{q|_A(c_n) : n < \omega\} \cup \{\psi(c_n, a, \bar{c}) : n < \omega\} \cup \{c_m \neq c_n : n \neq m\}$$

is satisfiable by compactness, and therefore the type is nonalgebraic. Defining $\psi'(x, a, \bar{c}') = \psi(x, a, \bar{c}) \wedge \rho(x, \bar{a}^*)$, this implies that there is an $m < \omega$ such that $\models \exists^{\leq m} x \psi'(x, a, \bar{c}')$. Note that as $\bar{c}' = \bar{c} \frown \bar{a}^*$ with $\bar{a}^* \in A$ and $\bar{c} \in B'$ by definition, $\bar{c}' \in B'$.

Finally, define $p'' = p' \cup \{\psi'(d, x, \bar{c}') \wedge \exists^{\leq m} y \psi'(y, x, \bar{c}')\}$. We have shown that a satisfies p'', and additionally, since $\bar{c}' \in B'$ and each d_k realizes $q = tp(d/B')$, a satisfies each type $p_k = p' \cup \{\psi'(d_k, x, \bar{c}') \wedge \exists^{\leq m} y \psi'(y, x, \bar{c}')\}$ for $k < |T|^+$. Of course, each d_k is distinct, and therefore the set

$$\{\phi(x, \bar{b}) \wedge \psi'(d_k, x, \bar{c}') \wedge \exists^{\leq m} y \psi'(y, x, \bar{c}') : k < |T|^+\}$$

is a $m + 1$-almost contradictory set of satisfiable sentences. By the definition of degree, thus

$$D(\phi(x, \bar{b})) > \inf\{D(\phi(x, \bar{b}) \wedge \psi'(d_k, x, \bar{c}') \wedge \exists^{\leq m} y \psi'(y, x, \bar{c}')) : k < |T|^+\}$$

Since each $p \cup \{\phi(x, \bar{b}) \wedge \psi'(d_k, x, \bar{c}') \wedge \exists^{\leq m} y \psi'(y, x, \bar{c}')\}$ is satisfiable (by a), this contradicts the minimality of $D(\phi(x, \bar{b}))$. Thus the proof is completed. $\square$

Corollary 3.4.9. *For every $\lambda > |T|$, T has a saturated model of cardinality λ.*

Proof. By Corollary 3.4.6, there is a nonalgebraic weakly minimal formula $\theta(x,\bar{a})$, and so for any elementary substructure M_0 of the universal model with $\bar{a} \in M_0$, by Lemma 3.4.4(1) there is an $A \subseteq M_0$ with $|A| \leq |T|$ which partitions $\theta(x,\bar{a})$, and we may in fact take $A = |T|$ by adding in unnecessary elements from the universal model. So by the upward Löwenheim-Skolem theorem, let M be a model with $A \subsetneq M$ and $|M| = \lambda$. Now, since T is $|T|$-stable (by Proposition 3.2.1), $|A| \leq |T|$ implies that $|S_1(A)| \leq |T|$, so for every $p \in S_1(A)$ with p nonalgebraic, let $(c_i^p : i < \lambda)$ be new constants. Then the set

$$Th_M(M) \cup \bigcup \{p(c_i^p) \cup \{c_i^p \neq c_j^p : i < j < \lambda\} : i < \lambda, p \in S_1(A) \text{ a nonalgebraic type}\}$$

is satisfiable by compactness. But since this set has cardinality λ, by the downward Löwenheim-Skolem theorem it has a model N of size λ. Clearly $M \preceq N$ and for each $p \in S_1(A)$ which is nonalgebraic, p is realized $\lambda = |N|$ times in N. Moreover, since the inclusion map $A \hookrightarrow N$ is an elementary map, and the properties of being weakly minimal formula and partition a weakly minimal formula are expressible in the language, even in N $\theta(x,\bar{a})$ is a weakly minimal formula partitioned by A. N thus satisfies the conditions of the above theorem, and is therefore a saturated model of cardinality λ. $\square$

3.5 Finale

Suppose T is a complete theory that is λ-categorical for some $\lambda > |T|$.

Lemma 3.5.1. *Suppose M is a model with $|T| \leq \kappa < |M|$, $\theta(x,\bar{a})$ is nonalgebraic weakly minimal with $A \subsetneq M$ partitioning it, $|A| = |T|$, and $M \preceq N$. Define $B = M \cup \{c \in N : \models \theta(c,\bar{a}), tp(c/A)$ is realized $> \kappa$ times in $M\}$. Then B satisfies the conditions of Lemma 3.4.7*

Proof. We first prove the following claim:

Claim. Suppose b realizes some algebraic $p \in S_1(A)$. Then there is a formula $\phi(x,\bar{a}) \in p$ for every c, if $\models \phi(c,\bar{a})$ then $tp(c/A) = tp(b/A) = p$ i.e. $\phi(x,\bar{a})$ isolates the type p

By compactness, if p is algebraic then there is a finite $q \subsetneq p$ which is algebraic, and as p is closed under conjugation there is some $\phi(x,\bar{a}) \in p$ which is an algebraic formula. So suppose $m < \omega$ is such that $\models \exists^{=m} x\phi(x,\bar{a})$, and let $b_1,\ldots,b_k$ be distinct elements which satisfy $\phi(x,\bar{a})$, with $b = b_1$. Then, for $k \leq m$, let $\psi_k(x,\bar{a}_k) \in p$ be such that $\models \neg\psi_k(b_k,\bar{a}_k)$ if such a formula exists, and otherwise (if $tp(b_k/A) = tp(b_1/A) = p$) let $\psi_k(x,\bar{a}_k) = (x = x)$. Then the formula $\phi(x,\bar{a}) \wedge \bigwedge_{k\leq m} \psi_k(x,\bar{a}_k)$ isolates p.

For condition (1) of Lemma 3.4.7, note that if $\psi_\phi(x,\bar{a}_i)$ is part of the ϕ-partition of $\theta(x,\bar{a})$, then by definition it is nonalgebraic, and thus by Corollary 3.2.4 $|\psi_\phi(M,\bar{a}_i)| = |M| > \kappa \geq |T|$ since $\bar{a}_i \in A \subsetneq M$. Therefore condition (1) is satisfied by B.

So suppose for a contradiction that B does not satisfy condition (2) of Lemma 3.4.7, so that there exists a formula $\phi'(x,\bar{b}',\bar{c})$ with $\bar{b}' \in M$ and $\bar{c} \in B - M$ such that $\models \exists x\phi'(x,\bar{b}',\bar{c}) \wedge \theta(x,\bar{a})$ but for every $b \in B$, $\models \neg(\phi'(b,\bar{b}',\bar{c}) \wedge \theta(b,\bar{a})$. Let $d \notin B$ be such that $\models \phi'(d,\bar{b}',\bar{c}) \wedge \theta(d,\bar{a})$, and by Lemma 3.4.2(1), suppose that $\models \psi_{\phi'}(d,\bar{a}_i)$. Define $\phi(x,\bar{b},\bar{c}) = \phi'(x,\bar{b}',\bar{c}) \wedge \theta(x,\bar{a}) \wedge \psi_{\phi'}(x,\bar{a}_i)$, so that $\bar{b} \in M$.

If $\phi(x,\bar{b},\bar{c})$ is not algebraic, then as $\psi_{\phi'}(x,\bar{a}_i)$ is ϕ'-minimal, $\psi_{\phi'}(x,\bar{a}_i) \wedge \neg\phi'(x,\bar{b}',\bar{c})$ is either unsatisfiable or algebraic. In either case, all but finitely many elements of $\psi_{\phi'}(x,\bar{a}_i)$ satisfy $\phi'(x,\bar{b},\bar{c})$. But as $\psi_{\phi'}(x,\bar{a}_i)$ is nonalgebraic by definition and $\bar{a}_i \in M$, by Corollary 3.2.4 $|\psi_{\phi'}(M,\bar{a}_i)| = |M| > \aleph_0$, and thus there is some $m \in M$ so that $\models \psi_{\phi'}(m,\bar{a}_i) \wedge \phi'(m,\bar{b}',\bar{c})$. But as $\models \psi_{\phi'}(m,\bar{a}_i)$ implies that $\models \theta(m,\bar{a})$, therefore $\models \theta(m,\bar{a}) \wedge \phi'(m,\bar{b},\bar{c})$, contradicting the definition of $\phi'(x,\bar{b}',\bar{c})$. Thus we conclude that $\phi(x,\bar{b},\bar{c})$ is an algebraic formula.

Let $A^* = A \cup \bar{b} \cup \{m \in M : tp(m/A) = p, \theta(x,\bar{a}) \in p, |p(M)| \leq \kappa\} \subseteq M$, so that $|A^*| \leq |A| + \aleph_0 + |S_1(A)| \cdot \kappa \leq \kappa$ (since $|A| = |T| \leq \kappa$ and as T is $|T|$-stable, $|S_1(A)| = |T|$), and let $\bar{c} = (c_0,\ldots,c_n)$. We wish to define a $\bar{c}' = (c_0',\ldots,c_n') \in M$ with $tp(c_0\ldots c_n/A^*) = tp(c_0'\ldots c_n'/A^*)$, and we shall do so by induction with the hypothesis that for a $0 \leq k \leq n$, $tp(c_0\ldots c_{k-1}/A^*) = tp(c_0'\ldots c_{k-1}'/A^*)$, thus eliminating the need to treat the base case separately:

1. If $tp(c_k/A^* \cup \{c_i : i < k\})$ is an algebraic type, by the first claim in this proof, there is a formula $\rho(x,c_0,\ldots,c_{k-1},\bar{e}) \in tp(c_k/A^* \cup \{c_i : i < k\})$ which isolates the type. Of course, since c_k satisfies this sentence, $\models \exists x\rho(x,c_0,\ldots,c_{k-1},\bar{e})$, and so by the inductive hypothesis, $\models \exists x\rho(x,c_0',\ldots,c_{k-1}',\bar{e})$. But this formula has parameters in M, and thus by the Tarski-Vaught test (since $M \precneqq \bar{M}$), there is a $c_k' \in M$ with $\models \rho(c_k',c_0',\ldots,c_{k-1}',\bar{e})$. Now, since $\rho(x,c_0,\ldots,c_{k-1},\bar{e})$ isolates $tp(c_k/A^* \cup c_i : i < k)$, for every formula $\chi(x,c_0,\ldots,c_{k-1},\bar{e}') \in tp(c_k/A^* \cup \{c_i : i < k\})$,

$$\models \forall x\rho(x,c_0,\ldots,c_{k-1},\bar{e}) \to \chi(x,c_0,\ldots,c_{k-1},\bar{e}')$$

   And so again by the inductive hypothesis,

$$\models \forall x\rho(x,c_0',\ldots,c_{k-1}',\bar{e}) \to \chi(x,c_0',\ldots,c_{k-1}',\bar{e}')$$

   This implies that $tp(c_0\ldots c_k/A^*) = tp(c_0'\ldots c_k'/A^*)$, completing the inductive step.

2. If $p_k = tp(c_k/A^* \cup \{c_i : i < k\})$ is a nonalgebraic type, then for a formula $\rho(x,c_0,\ldots,c_{k-1},\bar{e}) \in p_k$, note that as $\models \theta(c_k,\bar{a})$, there is a $\bar{a}^* \in A \subseteq A^*$ such that $\models \psi_\rho(c_k,\bar{a}^*)$ i.e. $\psi_\rho(x,\bar{a}^*) \in p_k|_A$. Since additionally $\theta(x,\bar{a}) \in p_k|_A \in S_1(A)$ and A partitions $\theta(x,\bar{a})$, by Lemma 3.4.4(2) $p_k|_A$ must be a minimal type, so it cannot be that both $p_k|_A \cup \{\rho(x,c_0,\ldots,c_{k-1},\bar{e})\}$ and $p_k|_A \cup \{\neg\rho(x,c_0,\ldots,c_{k-1},\bar{e})\}$ are nonalgebraic types. Of course, $\rho(x,c_0,\ldots,c_{k-1},\bar{e}) \in p_k$ and p_k is nonalgebraic by assumption, so it must be the case that $p_k|_A \cup \{\neg\rho(x,c_0,\ldots,c_{k-1},\bar{e})\}$ is either algebraic or unsatisfiable. By the inductive hypothesis, as $p_k|_A = tp(c_k/A)$, so necessarily $p_k|_A \cup \{\neg\rho(x,c_0',\ldots,c_{k-1}',\bar{e})\}$ is algebraic or unsatisfiable. Thus all but finitely many elements which realize $p_k|_A$ satisfy the formula $\rho(x,c_0',\ldots,c_{k-1}',\bar{e})$. Now, as $|A^*| \leq \kappa \geq |T|$, $|p_k| \leq \kappa$,

and therefore there are at most κ choice of $\rho(x, c_0, \ldots, c_{k-1}, \bar{e}) \in p_k$. Thus there are at most $\kappa \cdot \aleph_0 = \kappa$ elements in M which realize $p_k|_A$ but not $\rho(x, c'_0, \ldots, c'_{k-1}, \bar{e})$ for some $\rho(x, c_0, \ldots, c_{k-1}, \bar{e})$. But as $p_k|_A = tp(c_k/A)$ and $c_k \in B - M$, by definition of B, $p_k|_A$ is realized $> \kappa$ times in M, and so there is some $c'_k \in M$ which realizes $p_k|_A$ and such that for every $\rho(x, c_0, \ldots, c_{k-1}, \bar{e}) \in p_k = tp(c_k/A^* \cup \{c_0, \ldots, c_{k-1}\})$, $\models \rho(c'_k, c'_0, \ldots, c'_{k-1}, \bar{e})$ i.e. $tp(c_0 \ldots c_k/A^*) = tp(c'_0 \ldots c'_k/A^*)$.

Returning back to our main argument, note that $\models \phi(d, \bar{b}, \bar{c})$ and $\phi(x, \bar{b}, \bar{c})$ is an algebraic formula. Thus $tp(d/A^*\bar{c})$ is an algebraic type, so using the same argument as case 1 in the above induction, we can find a $d' \in M$ such that $tp(\bar{c} \frown d/A^*) = tp(\bar{c}' \frown d'/A^*)$. Note if $d' \in A$, then $(x = d') \in tp(\bar{c}' \frown d'/A^*) = tp(\bar{c} \frown d/A^*)$, and so $d = d' \in A^* \subseteq M$, contradicting that $d \notin M$ by definition. So $d' \in M - A^*$, and therefore by definition of A^*, $tp(d'/A) = tp(d/A)$ is realized $> \kappa$ times in M. Therefore by definition of B, $\{c \in N : tp(c/B) = tp(d/B)\} \subseteq B$ (as $B \supseteq M \supseteq A$), but for any such c, $\models \phi'(c, \bar{b}', \bar{c}) \wedge \theta(c, \bar{a})$ as all these parameters are in B, which contradicts the definition of $\phi'(x, \bar{b}', \bar{c})$. This completes the proof by contradiction, so B must satisfy condition (2) of Lemma 3.4.7. $\square$

Theorem 3.5.2. *If T is λ-categorical for some $\lambda > |T|$, then T is μ-categorical for every $\mu > |T|$*

Proof. By Corollary 3.4.9, for every $\mu > |T|$, T has a saturated model of cardinality $|\mu|$. By Lemma 1.5.10, saturated models of the same cardinality are isomorphic, and so it suffices to show that there are no unsaturated models of T with cardinality $> |T|$.

So for a contradiction, assume that M is an unsaturated model with $|M| > |T|$. By Theorem 3.4.6, there is a nonalgebraic weakly minimal formula $\theta(x, \bar{a})$, and so by Lemma 3.4.4(1) there is a set A with $|A| = |T|$ which partitions $\theta(x, \bar{a})$. Since M is not saturated, by Theorem 3.4.8 there must be some nonalgebraic type $p_0 \in S_1(A)$ with $\theta(x, \bar{a}) \in p_0$ which is realized $\leq \kappa$ many in M, where $\aleph_0 \leq \kappa < |M|$. Pick any $\lambda_1 > |M|$, and by the proof of Corollary 3.4.9 there is an $N \supsetneq M$ with $|N| = \lambda_1$ such that N is a saturated model.

As in the previous lemma, define $B = M \cup \{c \in N :\models \theta(x, \bar{a}), tp(c/A) = p, |p(M)| > \kappa\}$.

Claim. $|B| = \lambda_1$

We first show that there is some $p \in S_1(A)$ with $\theta(x, \bar{a}) \in p$ such that $|p(M)| > \kappa$: for otherwise we would have

$$|\theta(M, \bar{a})| \leq \sum_{p \in S_1(A), \theta(x, \bar{a}) \in p} |\{c \in M :\models p(c)\}| \leq \kappa \cdot |S_1(A)| < |M|$$

with the last inequality holding because T is $|T|$-stable (by Proposition 3.2.1) and $|A| = |T| < |M|$. But $\theta(x, \bar{a})$ is a nonalgebraic formula, so this contradicts Corollary 3.2.4.

Next, we shall define inductively $(c_i : i < \lambda_1) \subseteq N$ such that each c_i realizes p: if $(c_i : i < \alpha)$ has been defined, then consider the set $p \cup \{x \neq c_i : i < \alpha\}$:

this is an incomplete type over N with cardinality $< \lambda_1 = |N|$, and as N is saturated, it must realize this type, say by $c_\alpha \in N$. Thus $|p(N)| = \lambda_1$, and as $p(N) \subseteq B$, $|B| = \lambda_1$.

By the previous lemma, B satisfies the conditions of Lemma 3.4.7, so there is a model $M' \supseteq B$ with $\theta(M', \bar{a}) = \theta(B, \bar{a})$. Of course, $|M'| \geq |B| = \lambda_1$, and additionally if $c \in M'$ is such that $\models p_0(c)$, then $\models \theta(c, \bar{a})$, and thus $c \in B$. But by assumption p_0 is realized $\leq \kappa$ times in M, so in fact $c \in M$ by the definition of B. This implies that M' is a model of cardinality λ_1 such that p_0 is realized $\leq \kappa$ times in M'.

Now, $\lambda_1 > |M|$ was chosen arbitrarily in the above prove, so in particular we can choose $M' \supsetneq M$ with $|M'| = \beth_{(2^{|T|})^+}(\kappa)$. Since p_0 is a type over A with $|A| = |T|$, we can add A into the language without increasing the size of the language, so that p_0 can be considered as a type over the empty set. Then by Vaught's Two Cardinal theorem for cardinals far apart (Theorem 2.6.2), there is a model N' of T with cardinality λ where p_0 is realized $\leq |T|$ times. But by Theorem 3.4.9 there is a saturated model of T of cardinality λ, and as p_0 is a nonalgebraic type we have shown above that p_0 must be realized λ times. This contradicts that T is λ-categorical. $\square$

Appendices

Appendix A

Ramsey-type Theorems

The following proof from [Ho 97] uses the compactness theorem to prove this celebrated result of combinatorics:

Theorem A.0.1 (Ramsey's Theorem). *Suppose X is a linearly ordered set of size $\aleph_0$. Then for every $0 < k, n < \omega$ and $f : [X]^k \longrightarrow n$, there is some infinite $Y \subseteq X$ such that f is constant on $[Y]^k$.*

Proof. Note that there is an natural bijections between $[X]^k$, $\{A \subseteq X : |A| = k\}$, $\{B \subseteq\subseteq \omega : |B| = k\}$ and $[\omega]^k$. So WLOG suppose $X = \omega$.

We proceed by induction on k. For $k = 1$, the statement follows by the pigeonhole principle. Then inductively, suppose for every $n < \omega$ and $g : [\omega]^k \longrightarrow n$ there is an infinite $Y \subseteq \omega$ with g constant on $[Y]^k$. For a given $f : [\omega]^{k+1} \longrightarrow n$, let L be the language with the following signature:

- For every $i < \omega$, there is a constant symbol $\bar{i}$.
- There is a function symbol F with arity $k + 1$.
- There is a binary relation symbol $<$.

We define a L-structure W with domain ω by:

- For every $i < \omega$, $\bar{i}^W = i$
- For every increasing $\bar{a} \in [\omega]^{k+1}$, $F^W(\bar{a}) = f(\bar{a})$. For a $\bar{b} \in \omega^{k+1}$ that is not strictly increasing, we let $F^W(\bar{b}) = 0$.
- $<^W$ is simply the usual ordering on ω.

By the Upward Löwenheim-Skolem theorem there is a proper elementary extension V of W. Note that:

1. $W \models \forall x_0, \ldots, x_{n-1} F(x_0, \ldots, x_{n-1}) < \bar{n}$
2. $W \models \forall x, y(x = y \wedge x \not< y \wedge y \not< x) \vee (x < y \wedge x \neq y \wedge y \not< x) \vee (y < x \wedge x \neq y \wedge x \not< y)$

3. For every $m < \omega$, $W \models \exists^{=m} x(x < \bar{m})$

Thus V satisfies those formulas as well. Therefore if $v \in V \backslash W$ then for every $w \in W$, $w <^V v$. So let v be some fixed element in $V \backslash W$.

For $i < \omega$, we inductively choose $m_i \in \omega$ such that

- If $j < i$, then $m_j < m_i$
- For $j_0 < \cdots < j_{k-2} < i$, $V \models F(m_{j_0}, \ldots, m_{j_{k-2}}, m_i) = F(m_{j_0}, \ldots, m_{j_{k-2}}, v)$

Suppose m_j has been chosen for $j < i$. Then for every $j_0 < \cdots < j_{k-2} < i$, let $l(j_0, \ldots, j_{k-2}) = F^V(j_0, \ldots, j_{k-2}, v)$ (note that $F^V(j_0, \ldots, j_{k-2}, v) < n$ by assumption). Define the formula $\phi(x)$ to be

$$\overline{m_{i-1}} < x \wedge \bigwedge_{0 \leq j_0 < \cdots < j_{k-2} < i} F(\overline{m_{j_0}}, \ldots, \overline{m_{j_{k-2}}}, x) = \overline{l(m_{j_0}, \ldots, m_{j_{k-2}})}$$

Note that $\phi(x)$ is a formula in L. So as $V \models \phi(v)$, $V \models \exists x \phi(x)$ and therefore $W \models \phi(m)$ for some $m \in \omega$. If we let $m_i = m$, then m_i satisfies the above conditions.

Let $A = \{m_i : i < \omega\}$, and define $g : [A]^k \longrightarrow n$ by $g(\bar{a}) = f(\bar{a}, b)$ where $(\bar{a}, b) \in [A]^{k+1}$. The above conditions on choosing m_i ensures that g is well-defined. But as A is order-isomorphic to ω, by the induction hypothesis there is an infinite $B \subseteq A$ with g constant on $[B]^k$.

Claim. f is constant on $[B]^{k+1}$.

Let $\bar{c}, \bar{d} \in [B]^{k+1}$, and let $b \in B$ be such that $c_{k-1}, d_{k-1} < b$. Thus by definition of A, $f(\bar{c}) = f(\bar{c}|_k, b) = g(\bar{c}|_k)$ and $f(\bar{d}) = f(\bar{d}|_k, b) = g(\bar{d}|_k)$. But as g is constant on $[B]^k$, $g(\bar{c}|_k) = g(\bar{d}|_k)$. Thus f is constant on $[B]^{k+1}$ and the theorem is proven.

□

The following extension, originally a result in combinatorics, similarly has a model-theoretical proof (due to Simpson):

Theorem A.0.2 (Erdös-Rado Theorem). *Let α be an infinite cardinal, $n < \omega$ and suppose that X is a linearly ordered set with $|X| > \beth_n(\alpha)$. Then for any $f : [X]^{n+1} \longrightarrow \alpha$, there is a $Y \subseteq X$ with $|Y| > \alpha$ and such that f is constant on $[Y]^{n+1}$.*

Remark. Recall that we define $\beth_0(\alpha) = \alpha$, $\beth_{\beta+1}(\alpha) = 2^{\beth_\beta(\alpha)}$ and for a limit δ, $\beth_\delta(\alpha) = \bigcup_{i<\delta} \beth_i(\alpha)$.

Proof. The case is trivial for $n = 0$ by the pigeonhole principle. We then proceed by induction: Suppose that the theorem holds for some n. As in the proof above, we may assume that $X = |X|$ (where $|X|$ is the cardinal number of X as a Von Neumann ordinal), so that $\alpha \subseteq X$. Consider the signature with:

- For every $i < \alpha$, a constant symbol $\bar{i}$.
- A relation symbol R of arity $n + 3$.

Given a $f : [X]^{n+2} \longrightarrow \alpha$, let M be a structure in this signature with domain X by interpreting:

- For every $i < \alpha$, $\bar{i}^M = i$.
- For $\bar{a} \in X^{n+3}$, $\bar{a} \in R^M$ iff $\bar{a}|_{n+2} \in [X]^{n+2}$ and $f(\bar{a}|_{n+2}) = a_{n+2}$.

Claim. There is a $N \precneqq M$ with $|N| = \beth_{n+1}(\alpha)$ such that for every $A \subseteq N$ with $|A| < \beth_n(\alpha)$, if $p \in S_1(A)$ is realized in M then it is realized in N.

We will construct N by an elementary chain: let $N_0 = \langle \beth_{n+1}(\alpha) \rangle_{Sk}$ so that $|N_0| = \beth_{n+1}(\alpha)$. If N_i is defined for some $i < \beth_n(\alpha)^+$, then for every $A \subseteq N_i$ with $|A| < \beth_n(\alpha)$ and $p \in S_1(A)$, if p is realized in M then choose a m_p with $M \models p(m_p)$. Define $B_i = \{m_p \in M : A \subseteq N_i, |A| < \beth_n(\alpha), p \in S_1(A)\}$, and let $N_{i+1} = \langle N_i \cup B_i \rangle_{Sk}$. Note that there are $\beth_{n+1}(\alpha)^{\beth_n(\alpha)}$ choices of A, and for each A there are at most $2^{\beth_n(\alpha)}$ 1-types over A, so

$$|B_i| \leq \beth_{n+1}(\alpha)^{\beth_n(\alpha)} \cdot 2^{\beth_n(\alpha)} = (2^{\beth_n(\alpha)})^{\beth_n(\alpha)} \cdot 2^{\beth_n(\alpha)} = \beth_{n+1}(\alpha)$$

This guarantees that $|N_{i+1}| = \beth_{n+1}(\alpha)$. For a limit $\delta \leq \beth_n(\alpha)^+$, let $N_\delta = \bigcup_{i<\delta} N_i$. Then by construction, $N_{\beth_n(\alpha)^+}$ is the desired elementary substructure.

Let $c \in M \backslash N$. Let us define the sequence $\bar{b} = (b_i : i < \beth_n(\alpha))$ inductively: If $\bar{b}|_i$ is defined, let $b_i \in N$ be such that $tp(b_i/\bar{b}|_i) = tp(c/\bar{b}|_i)$ (b_i is guaranteed to exist by definition of N). Note that as $c \notin N$, every b_i is distinct.

Define the function $g : [\bar{b}]^{n+1} \longrightarrow \alpha$ by the following: for any $\bar{a} \in [\bar{b}]^{n+1}$, let $\bar{a'}$ be such that Rang $\bar{a'}$ = Rang $\bar{a} \cup \{c\}$ (i.e. $\bar{a'}$ is $(\bar{a}, c)$ but rearranged such that $\bar{a'} \in [X]^{n+2}$. Then $g(\bar{a}) = f(\bar{a'})$. By the induction hypothesis, since $|\bar{b}| > \beth_n(\alpha)$, there is a $Y \subseteq$ Rang $\bar{b}$ such that $|Y| > \alpha$ and g is constant on $[Y]^{n+1}$, with say $g([Y]^{n+1}) = \{j\}$ for some $j < \alpha$. Thus for any $(b_{i_0}, \ldots, b_{i_{n+1}}) \in [Y]^{n+2}$, assuming WLOG $i_0 < \cdots < i_{n+1}$, $(\bar{b}|'_{n+1}, j) \in R^M$. But since $tp(c/\{b_{i_0}, \ldots, b_{i_n}\}) = tp(b_{i_{n+1}}/\{b_{i_0}, \ldots, b_{i_n}\})$ by definition, $(b_{i_0} \ldots, b_{i_{n+1}}, j) \in R^M$ i.e. $f(b_{i_0}, \ldots, b_{n+1}) = j$. This proves that f is constant on $[Y]^{n+2}$. $\square$

Appendix B

Historical Remarks

B.1 Chapter 1

Chapter 1 concerns mostly introductory model theory, which by now is well-established and can be found in any textbook on model theory. The overall approach and all the proofs are based on lectures given by Freitag at University of California, Berkeley in Fall 2014, which are in turn loosely based on the exposition given in [Ho 97].

B.1.1 Section 1.1

This section is mostly basic definitions, and the claims are all basic observations. Chang attributes most of these definitions to [Ta 35].

B.1.2 Section 1.2

Again, most of these are basic observations, and Chang attributes most of them to [Ta 35]. In particular, 1.2.13 to 1.2.15 are from [TV 57].

B.1.3 Section 1.3

The idea and basic properties of Skolemization were first given in [Sk 20], and although the history is convoluted, the Downward Löwenheim-Skolem theorem (1.3.6) is generally attributed independently to [Lö 15] and [Sk 20]. However, both papers only proved the case for a countable theory, and Chang attributes both the statement and the proof of the general case to [TV 57].

B.1.4 Section 1.4

The compactness theorem (1.4.7) for countable languages was first given in [Go 30] and the generalization to uncountable languages was given in [Ma 36]. However, their proofs are based on a proof-theoretical approach from Gödel's completeness theorem, which is not the model-theoretical approach we gave. Our proof is based on [He 49], and this idea of building models using constants (commonly known as Henkin constructions) is used in various forms throughout the exposition. Finally, the Upward Löwenheim-Skolem theorem (1.4.10) is commonly attributed to Tarski (neither Löwenheim nor Skolem actually published the upward part of the theorem, and many account suggest that Skolem thought the it was meaningless due to his philosophical denial of uncountable sets), and the statement of the theorem as given here is again attributed to [TV 57].

B.1.5 Section 1.5

The idea of types and saturated models goes back to Hausdorff, although model theory was just beginning when he presented this idea and Hausdorff did not made any connection between his ideas and model theory. The given definitions of saturation and universality, and the results 1.5.8-1.5.10 are all from [MV 62]. In particular, the proof for 1.5.8 is directly based on what was given in [Bu 96], while the proofs for 1.5.9 and 1.5.10 are based on [Ho 97]

B.1.6 Section 1.6

The general idea of ultraproducts and the main theorem (1.6.8) are both from [Ło 55]. The given proof is based on that given in [Ho 97].

B.2 Chapter 2

Globally, many claims and their proofs are based on their exposition in [Sh 90], although they often arise from earlier work of Shelah.

B.2.1 Section 2.1

The idea of λ-stability was first given in [Ro 64], the definition of stable theories in [Sh 69], and the results 2.1.3-2.1.13 are all from [Sh 71], though the proofs given are based on [Sh 90]. 2.1.14-2.1.16 are essentially due to [Sh 69a], although the proofs given here were suggested by [HR 71] and Scanlon (by personal correspondence).

B.2.2 Section 2.2

The idea of λ-prime models and λ-isolated types are from [Sh 69a], which generalize prime models and isolated types introduced in [Mo 65]. Shelah actually proved the main result 2.2.7 in [Sh 69a], but for that prove he used tools which we did not introduce. Our approach of 2.2.5-2.2.8 was suggested in [HR 71].

B.2.3 Section 2.3

The idea of an indiscernible sequence was first given in [EM 56], as is the claim and proof of 2.3.2. The idea of connected and asymmetric was first given in [Eh 57], and 2.3.6-2.3.8 was stated and proved in [Sh 71]. 2.3.9 was suggested in [Ha 75], and 2.3.11 was given in [Sh 90]. 2.3.12 and 2.3.14 were stated and proved in [HR 71].

B.2.4 Section 2.4

Definability and the ϕ-2-rank were introduced in [Sh 71], based on the idea of Morley rank from [Mo 65]. Practically every result in this section is from [Sh 71], and their proofs follows the exposition in [Sh 90].

B.2.5 Section 2.5

Shelah first proved a more general version of the main result 2.5.9 in [Sh 69], although the proof was set-theoretical and required an argument involving the GCH and absoluteness. The approach here was suggested in [Ha 75], where 2.5.1-2.5.7 are stated and proved.

B.2.6 Section 2.6

As mentioned in the remark at the end of the section, the result that we proved differs from [Va 65] in two ways:

1. $\Sigma(x)$ is a single unary predicate in Vaught's proof
2. Vaught proved the theorem for any $\kappa \geq \chi \geq |T|$, with $|M| = \kappa$ and $\Sigma(M) = \chi$

In [Mo 65a], an alternative proof was given where (1) was generalized to any 1-type of T, although Morley only proved the statement for the countable case. It was stated as an exercise in [CK 77] that the proof for countable T generalizes to an uncountable language, although the author has had difficulty verifying this claim. This approach using 2.6.1 is based on an answer given by Haykazyan on mathoverflow.net (http://mathoverflow.net/questions/222504/how-to-extend-morleys-omitting-type-theorem-to-uncountable-languages). 2.6.1 itself was proved in [TZ 12], where it was attributed to Shelah. The author suspects that this attribution is due to Shelah having used a similar technique in [Sh 90] in the proof of a theorem generalizing both Vaught's result and a related result by Morley on omitting types.

B.2.7 Section 2.7

The main results 2.7.11 and 2.7.12 were given in [EM 56]. The approach of the entire section is based on [Ho 97].

B.3 Chapter 3

The entirety of this chapter follows [Sh 74], although we go into more detail in some proofs.

B.4 Appendix A

Ramsey's theorem (A.0.1) was first given in [Ra 30], and the proof here follows the proof from [Ho 97]. The Erdös-Rado theorem (A.0.2) was first given in [ER 56], and the proof here follows that in [CK 77], although Chang and Keisler attributes the proof to Simpson.

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