Practice Final Exam Problems

(1) Consider a 3-period binomial interest rate model with

$$R_0 = .08, \quad R_1(H) = .085, \quad R_1(T) = .075,$$

$$R_2(H,H) = .09, \quad R_2(H,T) = R_2(T,H) = .08, \quad R_2(T,T) = .07.$$

The risk-neutral measure is a binomial product measure with probability of heads equal to .5.

(a) Compute the bond prices $B_{0,1}$, $B_{0,2}$, and $B_{0,3}$.
(b) Find the forward interest rate $F_{0,2}$.
(c) Find the 3-period swap rate SR_3
(d) Let V be a zero coupon bond with face value 100, and maturity 3. Find the arbitrage-free price at time 0 of a European put option on V with exercise date 2 and strike price 93.

(2) Consider a 3-period binomial interest rate model with

$$R_0 = .10, \quad R_1(H) = .12, \quad R_1(T) = .08,$$

$$R_2(H,H) = .14, \quad R_2(H,T) = R_2(T,H) = .10, \quad R_2(T,T) = .06.$$

The risk-neutral measure is a binomial product measure with probability of heads equal to .5 Let V be a coupon bond with coupon rate $q = .1$, maturity 3 and face value 1,000. The bond is putable at time 1 and callable at time 2. More specifically, at time 1 the holder can sell the bond back to the issuer for 1,000 after the coupon is paid (and receive no further payments); if the put option was not exercised at time 1, the issuer can purchase the bond from the holder for 1,000 after the coupon is paid at time 2 (and make no further payments). Find the time-0 price V_0 of the bond. Be sure to explain your reasoning carefully.

(3) Consider a 3-period binomial interest rate model with

$$R_0 = .10, \quad R_1(H) = .12, \quad R_1(T) = .08,$$

$$R_2(H,H) = .14, \quad R_2(H,T) = R_2(T,H) = .10, \quad R_2(T,T) = .06.$$

The risk-neutral measure is a binomial product measure with probability of heads equal to .5.

Let V be a security that pays $\frac{100}{D_3}$ at time 3.

(a) Find the arbitrage-free price V_0 of V at time 0.

(b) Let $\mathrm{For}_{0,3}$ and $\mathrm{Fut}_{0,3}$ denote the forward and futures prices at time 0 for delivery of V at time 3. Without doing explicit computations, what can you say about the relationship between $\mathrm{For}_{0,3}$ and $\mathrm{Fut}_{0,3}$? Explain.

(c) Compute the forward price $\mathrm{For}_{0,3}$. (You may take it for granted that $B_{0,3} = .75231$.)

(d) Determine the futures price process $(\mathrm{Fut}_{n,3})_{0\le n\le 3}$.

(4) Consider an N-period binomial interest rate model with interest rate process $(R_n)_{0\le n\le N-1}$. The risk-neutral measure is a binomial product measure with probability of heads equal to .5. Let V be a PO strip maturing at time N (i.e., a principal only strip from a pool of mortgages maturing at time N.) Let $m \in \{1, 2, \cdots, N-1\}$ be given and let $\mathrm{For}_{0,m}$ and $\mathrm{Fut}_{0,m}$ be the forward and futures prices at time 0 for delivery of V at time m. What would you expect to be the relationship between $\mathrm{For}_{0,m}$ and $\mathrm{Fut}_{0,m}$? Explain.

The next problem is concerned with an N-period Ho-Lee model with

$$R_n(\omega_1, \cdots, \omega_n) = \Lambda_n + \sigma M_n(\omega_1, \cdots, \omega_n),$$

where $\Lambda_0, \cdots, \Lambda_{N-1}, \sigma > 0$ are constants and $M_0 = 0$,

$$M_n(\omega_1, \cdots, \omega_n) = \#H(\omega_1, \cdots, \omega_n) - \#T(\omega_1, \cdots, \omega_n).$$

The risk-neutral measure is assumed to be a binomial product measure with probability of heads equal to .5.

(5) Here we look at calibration of the model above using bond prices in the simple case when $N = 2$. Let $B_{0,i}^{market}$ denote observed market prices and $B_{0,i}^{model}$ denote prices predicted by the model of a zero-coupon bond with face value 1 and maturity i.

(a) Show that for any choice of $\sigma > 0$, and any observed values of $B_{0,i}^{market}$, $i = 1, 2$ it is possible to choose Λ_0, Λ_1 so that we have

$$B_{0,i}^{market} = B_{0,i}^{model}, \quad i = 1, 2.$$

(Your formula for Λ_1 should involve sigma.)

(b) Assume that $0 < B_{0,2}^{market} < B_{0,1}^{market} < 1$. Do all choices of σ lead to positive interest rates?

(c) Suppose that $B_{0,1}^{market} = .9445$ and $B_{0,2}^{market} = .8883$. Find R_0, $R_1(H)$, and $R_1(T)$ for each of the choices $\sigma = .01$ and $\sigma = .005$.

(6) Consider an N-period binomial model with interest rate process $(R_n)_{0\leq n\leq N-1}$, let $A > 0$ be a given constant and let V be a security that pays the amount

$$V_N(\omega) = A(1 - R_{N-1}(\omega))$$

at time N.

(a) Let $\text{For}_{0,N}$ be the forward price at time 0 for delivery of V at time N. Find a formula that expresses $\text{For}_{0,N}$ in terms of A and the forward interest rate $F_{0,N-1}$.

(b) Let $\text{Fut}_{n,N}$ denote the futures prices at time n for delivery of V at time N. Assume that the interest rate process has the special form

$$R_n(\omega) = \Lambda_n + \sigma M_n(\omega),$$

where $\Lambda_0, \Lambda_1, \cdots \Lambda_{N-1}, \sigma > 0$ are constants and

$$M_n(\omega) = \#H(\omega_1, \omega_2, \cdots, \omega_n) - \#T(\omega_1, \omega_2, \cdots, \omega_n),$$

and that the risk-neutral measure is a binomial product measure with probability of heads equal to .5. Find formulas expressing $\text{Fut}_{n,N}(\omega_1, \cdots, \omega_n)$ in terms of $A, \Lambda_0, \Lambda_1, \cdots, \Lambda_{N-1}, \sigma, \#H(\omega_1, \cdots, \omega_n), \#T(\omega_1, \cdots, \omega_n)$ for $n = 0, 1, 2, \cdots, N-1$. (Your formula for $\text{Fut}_{n,N}$ might not involve all of those quantities.)

(7) Consider a pool of mortgages with maturity 2. Assume the one-period mortgage rate is $Y = 0.10$ and the pool size is $P = 10,000,000$.

(a) In the absence of prepayments, compute the level monthly payment, as well as the principal payments, interest payments and remaining balance for times $n = 1, 2$.

(b) Repeat the above calculation, but now assume prepayments at a one period CPR of 10%.

(c) For the interest rate model of Problem 1, price the IO MBS security assuming no prepayments and 10% CPR.