21-378 **Mathematics of Fixed Income Markets** **D. Handron**

Week #4 Homework: Due on Wednesday, September 19.

1. (a) Suppose the price of the $5\frac{1}{4}$s of May 31, 2020 was 107.07 as of May 31, 2017. What was the yield to maturity of this bond on that date?

   (b) On May 31,2017, Aiofe purchased \$100 face of the $5\frac{1}{4}$s of May 31, 2020. Six months later, just after the coupon payment on November 30, 2017, she sold the bond. The yield to maturity at the time of sale was identical to the yield to maturity on the day she purchased the bond.

   What was the sale price of the bond, and what was her total return from her investment over the six months (under the semiannual compounding convention)?

   (c) Is your answer to part (b) of this problem significant or interesting in some way? Explain why an investor might buy a premium bond when that bond only returns the par value at maturity.

2. Show that if if the yield to maturity of a coupon bond remains unchanged between two successive coupon payments, then this common value represents the yield associated with holding the bond over that period of time.

3. Today is $t = 0$. Let T be a multiple of 6 months.

   $S^{(1)}$ is a security making payments $F_i^{(1)}$ at each time $\frac{i}{2}$ for $i = .5, 1, 1.5, \ldots, T$. The price of $S^{(1)}$ today is $P^{(1)}$ and the (semiannually compounded) yield to maturity of $S^{(1)}$ is $y^{(1)}$.

   $S^{(2)}$ is a security making payments $F_i^{(2)}$ at each time $\frac{i}{2}$ for $i = .5, 1, 1.5, \ldots, T$. The price of $S^{(2)}$ today is $P^{(2)}$ and the (semiannually compounded) yield to maturity of $S^{(2)}$ is $y^{(2)}$.

   Let S be a security that makes payments $F_i = F_i^{(1)} + F_i^{(2)}$ at each time $\frac{i}{2}$ for $i = .5, 1, 1.5, \ldots, T$. The price of S today is $P = P^{(1)} + P^{(2)}$. The (semiannually compounded) yield to maturity of S is y.

   Show that $\min(y^{(1)}, y^{(2)}) \leq y \leq \max(y^{(1)}, y^{(2)})$.

4. Consider two coupon bonds, each making semiannual coupon payments, and having maturity T, a multiple of 6 months. The bonds have coupon rates $q^{(1)}$ and $q^{(2)}$ satisfying $0 < q^{(1)} \leq q^{(2)}$.

   Assume that the spot rate curve is monotonically increasing, i.e. that $\hat{r}(t) \geq \hat{r}(t - .5)$ for $t = .5, 1, 1.5, 2, \ldots, T$. Show that $\hat{r}(T) \geq y^{(1)} \geq y^{(2)}$

5. (Continuously compounded spot and forward rates) Let $\tilde{r}(T)$ denote the spot rate for investments between 0 and time T under the *continuous compounding* convention. (The maturity T need not be a multiple of 6 months.) That is, a deposit of A at time 0 grows to $Ae^{T\tilde{r}(T)}$ at time T. Assume that $\tilde{r}(t)$ is a continuous function of t.

(a) Given $\eta \geq 0$ and $T \geq \eta$, let $\tilde{r}^{for}_{0,\eta,T}$ denote the (continuously compounded) interest rate agreed to at $t = 0$ for an investment of loan to be initiated at $t = \eta$ and settled with a single payment at $t = T$. If A invested at $t = \eta$, the value of the payment at $t = T$ will be $Ae^{(t-\eta)\tilde{r}^{for}_{0,\eta,T}}$

Find an expression for $\tilde{r}^{for}_{0,\eta,T}$ in terms of η, T, $\tilde{r}(\eta)$, and $\tilde{r}(T)$.

(b) Express

$$\lim_{T \to \eta+} \tilde{r}^{for}_{0,\eta,T}$$

in terms of η, $\tilde{r}(\eta)$, and $\tilde{r}'(\eta)$. The value of this limit is called the *instantaneous forward rate at time* η *as seen from time* 0.