Week #15 Homework: Due on Friday, December 7.

1. On a previous assignment, you priced a callable bond with a given coupon rate. In practice, when callable bonds are issued, the coupon rate is chosen so that the bond will trade close to par.

   Consider a 3-period Ho-Lee model

   $$R_n(\omega_1 \ldots \omega_n) = a_n + b \cdot \#H(\omega_1 \ldots \omega_n),$$

   with $a_n = .05 - .005n$ and $b = .01$. The risk-neutral measure in this model is a binomial product measure with probability of heads (and probability of tails) equal to 1/2.

   There is a coupon bond with face value \$1000 and maturity 3. The (one period) coupon rate Q has yet to be determined. This bond is callable at face value at time 1 or time 2. If the bond is called at time n, the holder receives \$1000$(1 + Q)$ and then receives no further payments.

   Determine the coupon rate Q for which the price of the callable bond at time zero is \$1000.

2. Consider a 3-period binomial model with interest rate process given by

   $$R_0 = .09, \quad R_1(H) = .10, \quad R_1(T) = .08$$

   $$R_2(HH) = .11, \quad R_2(HT) = R_2(TH)) = .09, \quad R_2(TT) = .07$$

   The risk-neutral measure in this model is a binomial product measure with probability of heads (and probability of tails) equal to 1/2.

   (a) Consider a <u>non-prepayable</u> \$100 mortgage taken out at time 0, with maturity 3 and one period mortgage rate $Y = .10$. The \$100 principal with be repaid with three payments of equal size at times 1, 2, and 3. The interest and principal payments will be separated into two securities. The holder of the IO strip will receive the interest payments, the holder of the PO strip will receive the principal payments. Determine

      i. the monthly payment,
      ii. the amount of interest paid at times 1, 2, and 3,
      iii. the amount of principal paid at times 1, 2, and 3,
      iv. the value of the IO strip at time 0,
      v. the value of the PO strip at time 0.

   (b) Now consider a pool of mortgages also with maturity 3 and (one-period) mortgage rate $Y = .10$. The mortgages in the pool <u>are</u> prepayable. The interest and principal payments (per \$100 of principal at time 0) are given by

ω	I_1	P_1	I_2	P_2	I_3	P_3
HHH	10.00	36.46	6.35	37.83	2.57	25.71
HHT	10.00	36.46	6.35	37.83	2.57	25.71
HTH	10.00	36.46	6.35	37.83	2.57	25.71
HTT	10.00	36.46	6.35	37.83	2.57	25.71
THH	10.00	42.71	5.73	41.64	1.56	15.64
THT	10.00	42.71	5.73	41.64	1.56	15.64
TTH	10.00	42.71	5.73	48.80	0.85	8.48
TTT	10.00	42.71	5.73	48.80	0.85	8.48

where I_n is the interest payment at time n, and P_n is the principal payment at time n. Determine

i. the value of the IO strip at time 0,

ii. the value of the PO strip at time 0.

3. Consider an 11-period Ho-Lee model

$$R_n(\omega_1 \dots \omega_n) = a_n + b \cdot \#H(\omega_1 \dots \omega_n),$$

with $a_n = .0125 - \sigma n$ and $b = 2\sigma$. The risk-neutral measure in this model is a binomial product measure with probability of heads (and probability of tails) equal to 1/2. The parameter $\sigma > 0$ controls the volatility of the interest rate.

Let

$$QP_{n,10} = \widetilde{\mathbb{E}}_n[100(1 - 4R_{10}], \quad n = 0, 1, 2, \dots, 10$$

be the quoted price process for a Eurodollar futures contract with maturity 10.

Let V be a European call option on the Eurodollar futures contract with expiration date 5. The call option pays

$$V_5 = 2500(QP_{5,10} - K)^+$$

at time 5. $K > 0$ is the strike price of the option. Determine the price V_0 of the option at time 0 for all combinations of

$$\sigma \in \{0.0001, 0.0003, 0.001\}$$

and

$$K \in \{94, 95, 96\}$$

4. This problem is concerned with interest rate swaps in a binomial interest rate model. A receiver swap with fixed rate K, initiation date n and maturity m is an agreement to receive the fixed amount K at times $n+1, n+2, \dots, m$ in exhange for the variable amounts $R_n, R_{n+1}, \dots, R_{m-1}$, also at times $n+1, n+2, \dots, m$. Note that the variable amount received at time k is R_{k-1} the rate for investments over the period prior to k.

Let $\mathrm{Swap}_{n,m}^K$ denote the price at time n of the receiver swap described above. The swap rate over the period from n to m, denoted by $\mathrm{SR}_{n,m}$ is the value of K (set at time 0) that makes $\mathrm{Swap}_{n,m}^K = 0$.

A receiver swaption is an option to enter a receiver swap. (We will restrict ourselves to considering European swaptions.). It has an exercise date n. If the option is exercised at n, payments will be exchanged starting at time $n+1$. The fixed rate K and the maturity m, when the last payment is exchanged, are specified in the contract. A swaption will be exercised if and only if the value of the underlying receiver swap is positive on the exercise date n. That is the value of the swap on the exercise date is $(\mathrm{Swap}^K_{n,m})^+$.

(a) Show that $\mathrm{Swap}^K_{n,m} = B_{n,m} - 1 + K\left(\sum_{i=n+1}^m B_{n,i}\right)$, where $B_{k,\ell}$ is the price at time k of a zero coupon bond that pays \$1 at time ℓ. Find an expression for the swap rate $\mathrm{SR}_{n,m}$ in terms of the bond prices $B_{k,\ell}$.

(b) Show that $(\mathrm{Swap}^K_{n,m})^+ = (K - \mathrm{SR}_{n,m})^+ \sum_{i=n+1}^m B_{n,i}$. That is to say, a receiver swaption can be considered as a put option on the swap rate, with strike price equal to the fixed payment of the underlying receiver swap and scaled by the value of an annuity paying \$1 at each of the times $n+1, n+2, \dots, m$.

(c) Consider a 3-period Ho-Lee model

$$R_n(\omega_1 \dots \omega_n) = a_n + b \cdot \#H(\omega_1 \dots \omega_n),$$

with $a_n = .05 - .005n$ and $b = .01$. The risk-neutral measure in this model is a binomial product measure with probability of heads (and probability of tails) equal to $1/2$.

Compute the time 0 price of a swaption to enter a two-period receiver swap at time 1 with fixed rate $K = 0.05$.