Week #12 Homework: Due on Wednesday, November 14.

1. Consider an N-period interest rate model with risk neutral probabiity measure $\tilde{\mathbb{P}}$ and interest rate process $\{R_n\}_{n=0}^{N-1}$. Let n and m be integers satisfying $1 \leq n \leq m \leq N$. Show that
$$\frac{\widetilde{\mathbb{E}}_n[D_m R_{m-1}]}{\widetilde{\mathbb{E}}_n[D_m]} = F_{n,m-1}$$

2. In practice, floating rate bonds often have caps and floors on the rate for the coupon payments. Consider a 10-period binomial model with interest rate process $\{R_n\}_{n=0}^{9}$. In this model there is a bond with maturity 10 and face value \$1000. At times $n = 1, 2, \ldots, 10$ the bond makes coupon payments of $1000Q_n$, where
$$Q_n = \begin{cases} .02 & R_{n-1} < .02 \\ R_{n-1} & .02 \leq R_{n-1} \leq .08 \\ .08 & R_{n-1} > .08, \end{cases}$$
and at $n = 10$, after the coupon payment of $1000Q_{10}$, the bond also repays the face value of 1000.

   Let V be an interest rate cap that pays $1000(R_{n-1} - .02)^+$ at times $n = 1, 2, \ldots, 10$ and W an interest rate cap that pays $1000(R_{n-1} - .08)^+$ at times $n = 1, 2, \ldots, 10$.

   Suppose that $V_0 = 111.95$, $W_0 = 0.44$, $B_{0,10} = .56045$, and $\sum_{i=1}^{10} B_{0,i} = 7.3669$. What is the price of the capped and floored floater described above at time $n = 0$.

3. Consider a 3-period Ho-Lee model
$$R_n(\omega_1 \ldots \omega_n) = a_n + b \cdot \#H(\omega_1 \ldots \omega_n),$$
with $a_n = .045 - .005n$ and $b = .01$. The risk-neutral measure in this model is a binomial product measure with probability of heads (and probability of tails) equal to $1/2$.

   In this problem there are three coupon bonds. All three bonds have maturity 3 and one period coupon rate $q = .05$, and face value \$100. The first bond is callable at time 1. The second bond is callable at time 2. The third bond is not callable.

   If a bond is called at time n, the issuer must pay the holder the coupon for time n, and the face value of the bond (i.e. \$105 for the bonds in this problem) at time n and then has no further obligations.

   The bonds in this problem have only one call date. In other settings, a callable bond might have more than one call date. The yield to call Y_c of a callable bond is the yield to maturity computed assuming that the bond is called at the earliest opportunity. The yield to worst Y_w of a callable bond is the lowest possible yield to maturity obtained by considering all possible scenarios for cash flows of the bond. That is, the yield to worst is determined by computing a yield to each possible call date, and a yield assuming the bond is never called, and returning the minimum of these yields. All of the yields are computed using the arbitrage free price of the bond, computed by backward induction. In this problem, all yields are quoted using a one-period compounding convention.

(a) Compute the time-0 price and the yield to maturity of the third bond.

(b) Compute the time-0 price, the yield to call, and the yield to worst of the second bond.

(c) Compute the time-0 price, the yield to call, and the yield to worst of the first bond.

4. An inverse floater is a bond with a variable coupon rate, but unlike a typical float note, the coupon rate moves in the opposite direction of the spot rates. Typically, the rates will have a cap and a floor.

   Consider a 10 period Boack-Derman-Toy model with

$$R_n(\omega_1 \ldots \omega_n) = a_n b_n^{\#H(\omega_1 \ldots \omega_n)},$$

   where $a_n = .06(.9)^n$ and $b_n = 1.2$ for $n = 0, 1, \ldots, 9$. The risk-neutral measure in this model is a binomial product measure with probability of heads (and probability of tails) equal to $1/2$.

   Consider an inverse floater with maturity 10 and face value \$1000. At each of the times $n = 1, 2, \ldots, 10$ this bond makes a coupon payment of $1000Q_n$, where

$$Q_n = \begin{cases} .035 & R_{n-1} > .085 \\ .12 - R_{n-1} & .035 \le R_{n-1} \le .085 \\ .085 & R_{n-1} < .035, \end{cases}$$

   At $n = 10$ the bond also makes a face value payment. Determine the price of this bond at time $n = 0$. Use the number of heads as a state variable.