21-378 **Mathematics of Fixed Income Markets** **D. Handron**

Week #11 Homework: Due on Wednesday, November 7.

1. Consider the following whole-yield model (i.e. a model that describes the evolution of the yield curve, as opposed to a short-rate model, which describes the evolution of one-period interest rates). Today ($t = 0$) the yield curve is flat at $y = .05$. One year from today, One year from today ($t = 1$) the yield curve will be flat at $y = .04$ (with probability .5) or flat at $y = .06$ (also with probabiity .5).

   At $t = 0$ three zero coupon bonds are available: a 2-year, a 5-year, and a 10-year bond.

   (a) Determine the prices of the bonds at $t = 0$.

   (b) Determine the possible prices of the bonds at $t = 1$.

   (c) Show that this model admits an arbitrage. [Hint: There is an arbitrage that involves ~~shorting~~ buying $10,000 face of the 2-year bond at $t = 0$.]

2. This problem will require the use of the principal components that you computed for Problem #4 on Homework #10.

   A trader at your firm thinks that the 7-year spot rate is high relative to the 3-year and 20-year rates. He would like to trade a butterfly that will make a profit if the 7-year rate falls relative to the 3- and 20-year rates. He plans to buy $10,000,000 face of a 7-year zero coupon bond. He plans to short a 3-year and 20-year zero coupon bond, but he's not sure how much face of each to sell (there are many possibilities), so he asks you.

   You think the butterfly portfolio should be hedged against changes in the yield curve corresponding to the first and second principal components. With that in mind, how much of the bonds should he sell?

   [Addendum: the current yields for these bonds are $\hat{r}(3) = 2.39\%$, $\hat{r}(7) = 2.68\%$, and$\hat{r}(20) = 2.85\%$.]

3. Consider a 10-period Black-Derman-Toy model

$$R_n(\omega_1 \dots \omega_n) = a_n b_n^{\#H(\omega_1 \dots \omega_n)},$$

   where $a_n = .035(.98)^n$ and $b_n = 1.04$ for $n = 0, 1, \dots, 9$. The risk-neutral measure in this model is a binomial product measure with probability of heads (and probability of tails) equal to 1/2.

   Use backward induction to compute the price at $t = 0$ of a coupon bond with maturity 10, one period coupon rate $q = .035$ and face value $100. Use the number of heads as a state variable.