Practice Final Exam Problems: Solutions

(1) Consider a 3-period binomial interest rate model with

$$R_0 = .08, \quad R_1(H) = .085, \quad R_1(T) = .075,$$

$$R_2(H,H) = .09, \quad R_2(H,T) = R_2(T,H) = .08, \quad R_2(T,T) = .07.$$

The risk-neutral measure is a binomial product measure with probability of heads equal to .5.

(a) Compute the bond prices $B_{0,1}$, $B_{0,2}$, and $B_{0,3}$.

(b) Find the forward interest rate $F_{0,2}$.

(c) Find the 3-period swap rate SR_3

(d) Let V be a zero coupon bond with face value 100, and maturity 3. Find the arbitrage-free price at time 0 of a European put option on V with exercise date 2 and strike price 93.

Solution

(a)

$$B_{0,1} = \frac{1}{1.08} = .9259,$$

$$B_{0,2} = \frac{1}{2}\left[\frac{1}{1.08(1.085)} + \frac{1}{1.08(1.075)}\right] = .85736,$$

$$\begin{aligned} B_{0,3} &= \tfrac{1}{4}\left[\tfrac{1}{1.08(1.085)(1.09)} + \tfrac{1}{1.08(1.085)(1.08)}\right. \\ &\qquad \left. + \tfrac{1}{1.08(1.075)(1.08)} + \tfrac{1}{1.08(1.075)(1.07)}\right] \\ &= .79390. \end{aligned}$$

(b) Recall that the forward rate $F_{0,2}$ satisfies

$$B_{0,2} - (1 + F_{0,2})B_{0,3} = 0,$$

so that

$$F_{0,2} = \frac{B_{0,2}}{B_{0,3}} - 1 = \frac{.85736}{.79390} - 1 = .079935.$$

(c) The 3-period swap rate is given by

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}} = .079971.$$

(d) Let V_2 denote the price of the coupon bond at time 2. Then we have

$$V_2(H,H) = \frac{100}{1.09} = 91.7431, \quad V_2(T,T) = \frac{100}{1.07} = 93.4579,$$

$$V_2(H,T) = V_2(T,H) = \frac{100}{1.08} = 92.5926.$$

Let us denote by P_n the price of the call option at time n. Observe that

$$P_2(H,H) = 93 - 91.7431 = 1.2569, \;\; P_2(H,T) = P_2(T,H) = 93 - 92.5926 = .4074,$$

$$P_2(T,T) = 0.$$

Using backward induction, we find that

$$P_1(H) = \frac{1}{1.085}[.5(1.2569) + .5(.4074] = .7670, \;\; P_1(T) = \frac{1}{1.075}[.5(.4074)] = .1895,$$

$$P_0 = \frac{1}{1.08}[.5(.7670) + .5(.1895)] = .4428.$$

(2) Consider a 3-period binomial interest rate model with

$$R_0 = .10, \quad R_1(H) = .12, \quad R_1(T) = .08,$$

$$R_2(H,H) = .14, \quad R_2(H,T) = R_2(T,H) = .10, \quad R_2(T,T) = .06.$$

The risk-neutral measure is a binomial product measure with probability of heads equal to .5 Let V be a coupon bond with coupon rate $q = .1$, maturity 3 and face value 1,000. The bond is putable at time 1 and callable at time 2. More specifically, at time 1 the holder can sell the bond back to the issuer for 1,000 after the coupon is paid (and receive no further payments); if the put option was not exercised at time 1, the issuer can purchase the bond from the holder for 1,000 after the coupon is paid at time 2 (and make no further payments). Find the time-0 price V_0 of the bond. Be sure to explain your reasoning carefully.

Solution: Since the bond is callable at time 2, we have

$$V_2 = \min\left\{1,000, \frac{1,100}{1 + R_2}\right\}.$$

Putting in the numbers, we find that

$$V_2(H,H) = 964.91, \quad V_2(H,T) = V_2(T,H) = 1,000, \quad V_2(T,T) = 1,000.$$

Since the bond is putable at time 1, we have

$$V_1 = \max\left\{1,000, \frac{1}{1+R_1}\tilde{\mathbb{E}}_1[100+V_2]\right\}.$$

Putting in the numbers, we have

$$V_1(H) = \max\left\{1,000, \frac{1}{1.12}[100+.5(964.91)+.5(1,000)]\right\} = 1,000,$$

$$V_1(T) = \max\left\{1,000, \frac{1}{1.08}[100+.5(1,000)+.5(1,000)]\right\} = 1,018.52.$$

Finally, we have

$$V_0 = \frac{1}{1.1}[100+.5(1,000)+.5(1018.52)] = 1,008.42.$$

(3) Consider a 3-period binomial interest rate model with

$$R_0 = .10, \quad R_1(H) = .12, \quad R_1(T) = .08,$$

$$R_2(H,H) = .14, \quad R_2(H,T) = R_2(T,H) = .10, \quad R_2(T,T) = .06.$$

The risk-neutral measure is a binomial product measure with probability of heads equal to .5.

Let V be a security that pays $\frac{100}{D_3}$ at time 3.

(a) Find the arbitrage-free price V_0 of V at time 0.

(b) Let $\text{For}_{0,3}$ and $\text{Fut}_{0,3}$ denote the forward and futures prices at time 0 for delivery of V at time 3. Without doing explicit computations, what can you say about the relationship between $\text{For}_{0,3}$ and $\text{Fut}_{0,3}$? Explain.

(c) Compute the forward price $\text{For}_{0,3}$. (You may take it for granted that $B_{0,3} = .75231$.)

(d) Determine the futures price process $(\text{Fut}_{n,3})_{0\le n\le 3}$.

Solution: Observe that

$$V_3(H,H,H) = V_3(H,H,T) = 100(1.10)(1.12)(1.14) = 140.45,$$

$$V_3(H,T,H) = V_3(H,T,T) = 100(1.10)(1.12)(1.10) = 135.52,$$

$$V_3(T,H,H) = V_3(T,H,T) = 100(1.10)(1.08)(1.10) = 130.68,$$

$$V_3(T,T,H) = V_3(T,T,T) = 100(1.10)(1.08)(1.06)125.93.$$

(a) $V_0 = \tilde{\mathbb{E}}\left[D_3 \frac{100}{D_3}\right] = \tilde{\mathbb{E}}[100] = 100.$

(b) Since D_3 and V_3 are negatively correlated under $\tilde{\mathbb{P}}$, we can be sure that $\text{Fut}_{0,3} > \text{For}_{0.3}$.

(c) Recall that

$$\text{For}_{0,3} = \frac{\tilde{\mathbb{E}}[D_3 V_3]}{\tilde{\mathbb{E}}[D_3]},$$

and since V makes only a single payment at time 3, we can write

$$\text{For}_{0,3} = \frac{V_0}{B_{0,3}}.$$

Therefore, we have

$$\text{For}_{0,3} = \frac{100}{.75231} = 132.92.$$

(d) We have $\text{Fut}_{3,3} = V_3$ and

$$\text{Fut}_{n,3} = .5\text{Fut}_{n+1,3} + .5\text{Fut}_{n+1,3} \quad \text{for } n = 0, 1, 2.$$

Putting in the numbers, we obtain

$$\text{Fut}_{3,3}(H, H, H) = \text{Fut}(H, H, T) = \text{Fut}_{2,3}(H, H) = 140.45,$$

$$\text{Fut}_{3,3}(H, T, H) = \text{Fut}_{3,3}(H, T, T) = \text{Fut}_{2,3}(H, T) = 135.52,$$

$$\text{Fut}_{3,3}(T, H, H) = \text{Fut}_{3,3}(T, H, T) = \text{Fut}_{2,3}(T, H) = 130.68,$$

$$\text{Fut}_{3,3}(T, T, H) = \text{Fut}_{3,3}(T, T, T) = \text{Fut}_{2,3}(T, T) = 125.93,$$

$$\text{Fut}_{1,3}(H) = 137.985, \quad \text{Fut}_{1,3}(T) = 128.305, \quad \text{Fut}_{0,3} = 133.145.$$

(4) Consider an N-period binomial interest rate model with interest rate process $(R_n)_{0 \le n \le N-1}$. The risk-neutral measure is a binomial product measure with probability of heads equal to .5. Let V be a PO strip maturing at time N (i.e., a principal only strip from a pool of mortgages maturing at time N.) Let $m \in \{1, 2, \cdots, N-1\}$ be given and let $\text{For}_{0,m}$ and $\text{Fut}_{0,m}$ be the forward and futures prices at time 0 for delivery of V at time m. What would you expect to be the relationship between $\text{For}_{0,m}$ and $\text{Fut}_{0,m}$? Explain.

Solution: If interest rates go down, we expect the price of the PO strip to go up for two reasons: (i) discount factors going up will increase the present value of future payments; and (ii) rates going down will likely increase prepayments and prepayments increase the value of a PO strip. Even if we are in a situation where there is significant burnout and prepayments do not increase, we expect the price of the PO strip to increase if rates go down. In other words, the correlation (under the risk-neutral measure) of the price of the PO strip with the discount process at delivery should be positive. So, we should expect to have

$\mathrm{For}_{0,m} > \mathrm{Fut}_{0,m}$.

The next problem is concerned with an N-period Ho-Lee model with

$$R_n(\omega_1, \cdots, \omega_n) = \Lambda_n + \sigma M_n(\omega_1, \cdots, \omega_n),$$

where $\Lambda_0, \cdots, \Lambda_{N-1}, \sigma > 0$ are constants and $M_0 = 0$,

$$M_n(\omega_1, \cdots, \omega_n) = \#H(\omega_1, \cdots, \omega_n) - \#T(\omega_1, \cdots, \omega_n).$$

The risk-neutral measure is assumed to be a binomial product measure with probability of heads equal to .5.

(5) Here we look at calibration of the model above using bond prices in the simple case when $N = 2$. Let $B_{0,i}^{market}$ denote observed market prices and $B_{0,i}^{model}$ denote prices predicted by the model of a zero-coupon bond with face value 1 and maturity i.

(a) Show that for any choice of $\sigma > 0$, and any observed values of $B_{0,i}^{market}$, $i = 1, 2$ it is possible to choose Λ_0, Λ_1 so that we have

$$B_{0,i}^{market} = B_{0,i}^{model}, \quad i = 1, 2.$$

(Your formula for Λ_1 should involve sigma.)

(b) Assume that $0 < B_{0,2}^{market} < B_{0,1}^{market} < 1$. Do all choices of σ lead to positive interest rates?

(c) Suppose that $B_{0,1}^{market} = .9445$ and $B_{0,2}^{market} = .8883$. Find R_0, $R_1(H)$, and $R_1(T)$ for each of the choices $\sigma = .01$ and $\sigma = .005$.

Solution:

12. (a) Matching the model and market prices for the maturity-1 bond, we find that

$$B_{0,1}^{market} = \frac{1}{1 + \Lambda_0},$$

$$\Lambda_0 = \frac{1}{B_{0,1}^{market}} - 1.$$

Matching the model and market prices for the maturity-2 bond we find that

$$B_{0,2}^{market} = \frac{1}{2(1 + \Lambda_0)} \left(\frac{1}{1 + \Lambda_1 + \sigma} + \frac{1}{1 + \Lambda_1 - \sigma} \right),$$

$$B_{0,2}^{market} = \frac{1 + \Lambda_1}{1 + \Lambda_0} \left(\frac{1}{(1 + \Lambda_1)^2 - \sigma^2} \right).$$

Solving for Λ_1, we find that

$$\Lambda_1 = \frac{1 + \sqrt{1 + (2B_{0,2}^{market}(1+\Lambda_0))^2\sigma^2}}{2B_{0,2}^{market}(1+\Lambda_0)} - 1.$$

(b) If σ is large enough, $\Lambda_1 - \sigma$ becomes negative.

(c) Substituting in $B_{0,1}^{market} = .9445$, $B_{0,2}^{market} = .8883$, and $\sigma = .01$, we find that

$$\Lambda_0 = .058761249, \quad \Lambda_1 = .063360956;$$

this gives

$$R_0 = .058761249, \quad R_1(H) = .073360956, \quad R_1(T) = .053360956.$$

Substituting in $B_{0,1}^{market} = .9445$, $B_{0,2}^{market} = .8883$, and $\sigma = .005$, we find that

$$\Lambda_0 = .058761249, \quad \Lambda_1 = .063290426;$$

this gives

$$R_0 = .058761249, \quad R_1(H) = .068290426, \quad R_1(T) = .058290427.$$

(6) Consider an N-period binomial model with interest rate process $(R_n)_{0\le n\le N-1}$, let $A > 0$ be a given constant and let V be a security that pays the amount

$$V_N(\omega) = A(1 - R_{N-1}(\omega))$$

at time N.

(a) Let $\mathrm{For}_{0,N}$ be the forward price at time 0 for delivery of V at time N. Find a formula that expresses $\mathrm{For}_{0,N}$ in terms of A and the forward interest rate $F_{0,N-1}$.

(b) Let $\mathrm{Fut}_{n,N}$ denote the futures prices at time n for delivery of V at time N. Assume that the interest rate process has the special form

$$R_n(\omega) = \Lambda_n + \sigma M_n(\omega),$$

where $\Lambda_0, \Lambda_1, \cdots \Lambda_{N-1}, \sigma > 0$ are constants and

$$M_n(\omega) = \#H(\omega_1, \omega_2, \cdots, \omega_n) - \#T(\omega_1, \omega_2, \cdots, \omega_n),$$

and that the risk-neutral measure is a binomial product measure with probability of heads equal to .5. Find formulas expressing $\mathrm{Fut}_{n,N}(\omega_1, \cdots, \omega_n)$ in terms of $A, \Lambda_0, \Lambda_1, \cdots, \Lambda_{N-1}, \sigma, \#H(\omega_1, \cdots, \omega_n), \#T(\omega_1, \cdots, \omega_n)$ for $n = 0, 1, 2, \cdots, N-1$. (Your formula for $\mathrm{Fut}_{n,N}$ might not involve all of those quantities.)

Solution:

(a)

$$\begin{aligned}\tilde{\mathbb{E}}[D_N R_{N-1}] &= \tilde{\mathbb{E}}[D_N(1+R_{N-1})] - \tilde{\mathbb{E}}[D_N] \\ &= \tilde{\mathbb{E}}[D_{N-1}] - \tilde{\mathbb{E}}[D_N] \\ &= B_{0,N-1} - B_{0,N}\end{aligned}$$

$$\begin{aligned}\frac{\text{For}_{0,N}}{A} &= \frac{\tilde{\mathbb{E}}[(1-R_{N-1})D_N]}{\tilde{\mathbb{E}}[D_N]} \\ &= \frac{2B_{0,N}-B_{0,N-1}}{B_{0,N}} = 2 - \frac{B_{0,N-1}}{B_{0,N}} \\ &= 2-(1+F_{0,N-1}) = 1 - F_{0,N-1}\end{aligned}$$

It follows that

$$\text{For}_{0,N} = A(1-F_{0,N-1}).$$

(b) Observe that

$$\tilde{\mathbb{E}}_n[R_{N-1}](\omega_1,\cdots,\omega_n) = \Lambda_{N-1} + \sigma M_n(\omega_1,\cdots,\omega_n).$$

It follows that

$$\begin{aligned}\text{Fut}_{n,N}(\omega_1,\cdots,\omega_n) &= \tilde{\mathbb{E}}_n[A(1-R_{N-1})](\omega_1,\cdots,\omega_n) \\ &= A(1-\Lambda_{N-1}-\sigma M_n(\omega_1,\cdots,\omega_n)) \\ &= A(1-\Lambda_{N-1}-\sigma(\#H(\omega_1,\cdots,\omega_n) - \#T(\omega_1,\cdots,\omega_n)))\end{aligned}$$

(7) Consider a pool of mortgages with maturity 2. Assume the one-period mortgage rate is $Y = 0.10$ and the pool size is $P = 10,000,000$.

(a) In the absence of prepayments, compute the level monthly payment, as well as the principal payments, interest payments and remaining balance for times $n = 1, 2$.

(b) Repeat the above calculation, but now assume prepayments at a one period CPR of 10%.

(c) For the interest rate model of Problem 1, price the IO MBS security assuming no prepayments and 10% CPR.

Solution:

(a)

$$10,000,000 = A\left(\frac{1}{1.1}+\frac{1}{(1.1)^2}\right) \Longrightarrow A = 5,761,905$$

We know $B(0) = P = 10,000,000$ so that

$$\begin{aligned} I(1) &= B(0)Y = 10,000,000 * .1 = 1,000,000 \\ P(1) &= A - I(1) = 5,761,905 - 1,000,000 = 4,761,905 \\ B(1) &= B(0) - P(1) = 5,238,095 \end{aligned}$$

and

$$\begin{aligned} I(2) &= 5,238,095 * .1 = 523,809.5 \\ P(2) &= 5,761,905 - 523,809.5 = 5,238,095 \\ B(2) &= 0 \end{aligned}$$

(b) Since the one period CPR is .1 this means that 10% of the mortgages pay off at time 1. We have the same $A = A(0)$ and $B(0)$ as above. To get the cash flows we have

$$\begin{aligned} I(1) &= B(0)Y = 10,000,000 * .1 = 1,000,000 \\ P_{sch}(1) &= A - I(1) = 5,761,905 - 1,000,000 = 4,761,905 \\ P_{pre}(1) &= .10 * (10,000,000 - 4,761,905) = 523,809.5 \\ P(1) &= P_{sch}(1) + P_{pre}(1) = 5,285,714,5 \\ B(1) &= .9 * (10,000,000 - 4761,905) = 4,714,285.5 \end{aligned}$$

and

$$\begin{aligned} A(1) &= .9A(0) = 5,185,714.5 \\ I(2) &= 471,428.55 \\ P_{sch}(2) &= 4,714,285 \\ P_{pre}(2) &= 0 \\ P(2) &= 4,714,285 \\ B(2) &= 0 \end{aligned}$$

(c) From problem 1 we have the bond prices

$$B_{0,1} = .9259; \qquad B_{0,2} = .85736.$$

We thus price the IO in the two scenarios as

(i) No prepayments:

$$\text{Px}^{IO} = I(1)B_{0,1} + I(2)B_{0,2} = 925,900 + 449,093 = 1,374,993.$$

(ii) Prepayments

$$\text{Px}^{IO} = 925,900 + 471,428.55 * .85736 = 1,330,083.$$