21-378 **Mathematics of Fixed Income Markets** **Fall 2016**

Solutions to Review Problems for Midterm

1. (a) $97.947 = 100d(.5)$, so $d(.5) = .97947$.
   $98.475 = 2d(.5) + 102d(1)$ and consequently
   $$d(1) = \frac{98.487 - 2d(.5)}{102} = .94635.$$
   $101.80 = 3d(.5) + 3d(1) + 103d(1.5)$ and consequently
   $$d(1.5) = \frac{101.80 - 3d(.5) - 3d(1)}{103} = .93226.$$
   (b) Since
   $$d(1.5) = \frac{1}{\left(1 + \frac{\hat{r}(1.5)}{2}\right)^3},$$
   we have
   $$\hat{r}(1.5) = 2\left[\left(\frac{1}{.93226}\right)^{\frac{1}{3}} - 1\right] = .047315.$$
   We also have
   $$f(1) = 2\left[\frac{d(.5)}{d(1)} - 1\right] = .06999.$$
   (c)
   $$y_{pc}(1.5) = \frac{2[1 - d(1.5)]}{d(.5) + d(1) + d(1.5)} = .047404$$

2. (a) The amount of each coupon payment is $10,000\frac{.08}{2} = 400$. The yield to maturity y satisfies the equation
   $$11,245.36 = \frac{10,000}{(1 + \frac{y}{2})^{40}} + 400\sum_{i=1}^{40}\frac{1}{(1 + \frac{y}{2})^i}.$$
   Other forms are acceptable as well.
   (b) Since the bond is trading above par, the value of y must be below the coupon rate, i.e. $y < .08$.

3. Let us put
   $$\lambda_a = \frac{1}{1 + \frac{.0468}{2}}.$$
   Then we have
   $$\sum_{i=1}^{20} d(\frac{i}{2}) = \frac{\lambda_a}{1 - \lambda_a}(1 - \lambda_a^{20}) = 15.82739.$$

Let us put

$$\lambda_b = \frac{1}{1.025}.$$

The price (per \$100 face) of the 6% coupon bond is

$$P = 100\lambda_b^{20} + 3\frac{\lambda_b}{1-\lambda_b}(1-\lambda_b^{20}) = 107.79458.$$

We also know that

$$P = 100d(10) + 3\sum_{i=1}^{20} d(\frac{i}{2}) = 100d(10) + 3(15.82739).$$

solving for $d(10)$ we find that

$$d(10) = .603124.$$

Finally, we have

$$y_{pc}(10) = \frac{2[1-d(10)]}{\sum_{i=1}^{20} d(\frac{i}{2})} = .05051.$$

4. We begin by computing the prices of the two bonds (per \$100 face). Put

$$\lambda_1 = \frac{1}{1+\frac{.05}{2}}, \quad \lambda_2 = \frac{1}{1+\frac{.054}{2}}.$$

Then we have

$$P_1 = 100\lambda_1^{20} + 2\frac{\lambda_1}{1-\lambda_1}(1-\lambda_1^{20}) = 92.2054,$$

$$P_2 = 100\lambda_2^{20} + 2\frac{\lambda_2}{1-\lambda_2}(1-\lambda_2^{20}) = 104.5896.$$

If we purchase 100α face of Bond #1 and $100(1-\alpha)$ face of Bond #2 we will produce a coupon bond with face value 100 and coupon rate $\alpha(.04)+(1-\alpha)(.06)$. Therefore we want to choose α such that

$$\alpha P_1 + (1-\alpha)P_2 = 100.$$

We find that

$$\alpha = \frac{100-P_2}{P_1-P_2} = .37060.$$

The par-coupon rate will be given by

$$y_{pc}(10) = \alpha(.04) + (1-\alpha)(.06) = .05259.$$

Remark: You can also solve this problem by using the bond prices to determine $d(10)$ and $\sum_{i=1}^{20} d(\frac{i}{2})$ and using the formula that expresses y_{pc} in terms of discount factors. This method is a bit longer, but still a good way to solve the problem.

5. Observe that
$$d(10) = \frac{1}{(1.02)^{20}} = .67297.$$
The time-0 price of the coupon stream is
$$100,000(1.25)(1 - d(10)) = 40,878.58.$$
The time-0 price of the coupon stream plus a payment of 100,000 at maturity is
$$40,878.58 + 100,000d(10) = 108,175.72.$$

6. Throughout this problem, we use the term *yield* to mean *yield to maturity.* Denote the yields of the four securities by y_1, y_2, y_3, and y_4, respectively. Recall that for coupon bonds and annuities, the yield is always between the lowest and the highest spot rate corresponding to payment times and that for a zero-coupon bond with maturity T, the yield to maturity is exactly equal to $\hat{r}(T)$. It follows that $y_1 = \hat{r}(10)$, which much be the largest of the 4 yields, and that $y_2 < \hat{r}(5)$. The yield of an annuity is independent of the payment size (it cancels from both sides of the equation). Recall that if a security can be expressed as the sum of two securities (both having nonnegative payments) then the yield of the composite security lies between the yields of the two individual securities. It follows that the yield of the 10-year annuity is between the yield of the 5-year annuity and an annuity that makes payments at times $t = 5.5, 6, 6.5, \cdots, 10$. This latter yield is between $\hat{r}(5.5)$ and $\hat{r}(10)$. The yield of a coupon bond depends on the coupon rate (not on the face value). However, the yield of a 10 year coupon bond must lie between the yield of a 10-year annuity and a 10-year zero. We conclude that
$$y_2 < y_3 < y_4 < y_1.$$
These conclusions regarding the ordering of the yields, are, of course, based on the fact that the spot-rate curve is upward sloping.

7. (a) When the spot-rate curve is flat at 4%, the Macaulay duration of a par-coupon bond with face value F and maturity T is given by
$$D_{Mac} = (1.02)\frac{1}{.04}\left[1 - \frac{1}{(1.02)^{2T}}\right].$$
Therefore we have
$$D_{Mac}^{(1)} = (1.02)\frac{1}{.04}\left[1 - \frac{1}{(1.02)^{12}}\right] = 5.39342.$$
The Macaulay duration of a ZCB is the maturity, so
$$D_{Mac}^{(2)} = 20.$$
The price of the par-coupon bond is
$$P^{(1)} = 10,000,000.$$

The price of the ZCB is

$$P^{(2)} = \frac{18,000,000}{(1.02)^{40}} = 8,152,027.$$

The Macaulay duration of a portfolio is a weighted average of the Macaulay durations of the components, with weights being the percentage of capital in the components. Therefore

$$D_{Mac}^{(A)} = \frac{10(5.39342) + 8.152027(20)}{18.152027} = 11.9532.$$

The duration of the portfolio is given by

$$D^{(A)} = \frac{D_{Mac}^{(A)}}{1.02} = 11.7188.$$

The DV01 of the portfolio is given by

$$DV01^{(A)} = \frac{P^{(A)}D^{(A)}}{10,000} = \frac{18,152,027(11.7188)}{10,000} = 21,272.03.$$

(b) If the spot-rate curve shifts upward by 14 basis points the price of portfolio A will decrease by approximately

$$(21,272.03) \times (14) = \$297,808.47$$

(c) The DV01 of portfolio B is given by

$$DV01^{(B)} = \frac{F_3}{10,000(.04)}\left[1 - \frac{1}{(1.02)^{20}}\right] = .000817572F_3.$$

Setting this equall to 21,272.03 and solving for F_3 gives

$$F_3 = 26,018,550.$$

Remark: There are several other reasonable ways to obtain these numbers.

8. (a) When the spot-rate curve is flat at 4%, the DV01 of a par-coupon bond with face value F and maturity T is given by

$$DV01 = \frac{F}{10,000(.04)}\left(1 - \frac{1}{(1.02)^{2T}}\right).$$

Therefore we have

$$DV01^{(1)} = \frac{15,000,000}{10,000(.04)}\left(1 - \frac{1}{(1.02)^{20}}\right) = 12,263.58,$$

$$DV01^{(2)} = \frac{10,000,000}{10,000(.04)}\left(1 - \frac{1}{(1.02)^{40}}\right) = 13,677.74,$$

$$DV01^{(3)} = \frac{25,000,000}{10,000(.04)}\left(1 - \frac{1}{(1.02)^{60}}\right) = 43,451.11.$$

The DV01 of a portfolio is the sum of the DV01's of the components, so we have

$$DV01^A = DV01^{(1)} + DV01^{(2)} + DV01^{(3)} = 69,392.43.$$

The price of portfolio A is 50,000,000. The duration of portfolio A is given by

$$D^A = \frac{10,000DV01^A}{50,000,000} = 13.87849.$$

The Macaulay duration of portfolio A is given by

$$D^A_{Mac} = 1.02D^A = 14.15606.$$

(b) If the spot-rate curve shifts downward by 21 basis points the price of portfolio A will increase by approximately

$$\$69,392.43 \times (21) = \$1,457,241.03.$$

(c) The DV01 of portfolio B is given by

$$DV01^B = \frac{F}{10,000}\left[\frac{12}{(1.02)^{25}} + \frac{18}{(1.02)^{37}}\right] = \frac{15.965367F}{10,000}.$$

Setting this equal to$DV01^A$ and solving for F we find

$$F = \frac{10,000DV01^A}{15.965367} = \$43,464,350.05.$$

Remark: There are several other reasonable ways to obtain these numbers.

9. Notice that Security #3 is a premium bond, so it has a larger coupon than Security #2.

   (a) The Macaulay duration of a ZCB is the time to maturity, and the Macaulay duration of a portfolio of ZCBs is a weighted average of the durations of the individual ZCBs with weights proportional to the percentage of the total price that is invested in each of the individual bonds. The Macaulay duration of an annuity is therefore smaller than the Macaulay duration of bonds of the same maturity. Also, the Macaulay duration of bonds having the same maturity decreases as the coupon increases. We conclude that

$$D^{(2)}_{Mac} > D^{(3)}_{Mac} > D^{(1)}_{Mac}.$$

(b) The convexity of a ZCB increases with increasing maturity, and the convexity of a portfolio of ZCBs is a weighted average of the convexities of the individual ZCBs with weights proportional to the percentage of the total price that is invested in each of the individual bonds. Therefore an annuity will have smaller convexity than bonds of the same maturity. Also, for bonds having the same maturity convexity decreases with increasing coupon. We conclude that

$$C^{(2)} > C^{(3)} > C^{(1)}.$$

(c) The DV01 of a portfolio is the sum of the DV01's of the components. Security #2 is equal to Security #1 plus a ZCB, so we know that $DV01^{(2)} > DV01^{(1)}$. Security #3 is equal to security #2 plus an annuity, so we know that $DV01^{(3)} > DV01^{(2)}$. It follows that

$$DV01^{(3)} > DV01^{(2)} > DV01^{(1)}.$$

10. (a) The Macaulay duration of a ZCB is the time to maturity, and the Macaulay duration of a portfolio of ZCBs is a weighted average of the durations of the individual ZCBs with weights proportional to the percentage of the total price that is invested in each of the individual bonds. The Macaulay duration of a par-coupon bond is therefore smaller than the Macaulay duration of a ZCB with the same maturity. We conclude that

$$D^{(3)}_{Mac} > D^{(1)}_{Mac} > D^{(2)}_{Mac}.$$

(b) The convexity of a ZCB increases with increasing maturity, and the convexity of a portfolio of ZCBs is a weighted average of the convexities of the individual ZCBs with weights proportional to the percentage of the total price that is invested in each of the individual bonds. Therefore a par-coupon bond will have smaller convexity than a ZCB with the same maturity. We conclude that

$$C^{(3)} > C^{(1)} > C^{(2)}.$$

(c) We can say that $DV01^{(2)} > DV01^{(1)}$, but we cannot compare $DV01^{(3)}$ to $DV01^{(1)}$ or $DV01^{(2)}$ without more information.

11. A's payments to B can be replicated by purchasing a floater with face value F and maturity 5, shorting a ZCB with face value F and maturity 5 and purchasing ZCB's with face value Fq and maturities $5.5, 6.6.5, \cdots, 10$ at time 0. The time-0 price of this portfolio is

$$F(1 - d(5)) + Fq \sum_{i=11}^{20} d(\frac{i}{2}).$$

B's payments to A can be replicated by purchasing ZCB's with face value $F\frac{q}{2}$ and maturities $.5, 1, 1.5, \cdots, 5$ at time 0, shorting a ZCB with face value $2F$ and maturity 10 at time 0, and purchasing a floater with face value $2F$ that matures 5 years later at time 5. The time-0 price of this portfolio is

$$2Fd(5) - 2Fd(10) + F\frac{q}{2}\sum_{i=1}^{10} d(\frac{i}{2}).$$

Since nothing is paid to enter the agreement we have

$$F(1 - d(5)) + Fq\sum_{i=11}^{20} d(\frac{i}{2}) = 2Fd(5) - 2Fd(10) + F\frac{q}{2}\sum_{i=1}^{10} d(\frac{i}{2}).$$

Assuming that

$$\frac{1}{2}\sum_{i=1}^{10} d(\frac{i}{2}) \neq \sum_{i=1}^{20} d(\frac{i}{2}),$$

we can solve for q to obtain

$$q = \frac{1 - 3d(5) + 2d(10)}{\frac{1}{2}\sum_{i=1}^{10} d(\frac{i}{2}) - \sum_{i=11}^{20} d(\frac{i}{2})}.$$

12. Let us denote by $y_{pc}(T)$ the par-coupon yield for maturity T. By Proposition 4.1, we know that

$$y_{pc}(5) > y_{pc}(3) > y_{pc}(2).$$

We shall refer to investor A's bond as $B^{(A)}$ and investor B's bond as $B^{(B)}$. Let us look at the cash flows for the two investors, per \$100 invested initially, under the assumption that the term structure remains unchanged.

At each of the times .5,1,1.5,2, investor A will receive coupon payments of amount $50y_{pc}(2)$ and investor B will receive coupon payments of amount $50y_{pc}(5)$. At time 2, investor A will also receive a payment of \$100 (face value of $B^{(A)}$). Let's look at the value of $B^{(B)}$ at time 2: This bond will have 3 years until it matures and it has coupon rate strictly greater than that for newly issued 3-year par bonds (because $y_{pc}(5) > y_{pc}(3)$). Therefore, at time 2, $B^{(B)}$ will be a premium bond and consequently will have price strictly greater than \$100. We see that investor B receives strictly greater coupon payments and (after the coupon payments at time 2) will be left with a bond that is worth more than investor A's bond (which is worth face value). Assuming that they treat coupon payments in the same way, investor B will be better off at time 2. (Of course, it is possible that investor A will invest the coupon payments in a better way and conceivably be better off.)

Notice that if the term structure remains unchanged, an investor who shorts the 2-year par bond and purchases the 5-year par bond at time 0 and then sells the 5-year bond at time 2 will pay nothing initially and will receive strictly positive payments at each of the times .5, 1, 1.5 and 2. However, this is not an arbitrage because there is no way to be certain in advance that the term structure will remain flat.