21-370 **Discrete Time Finance** **Dr. Handron Fall 2010**

Exam #2 Review

1. Consider a binomial model with $N = 4$, $u = \frac{5}{3}$, $d = \frac{2}{3}$, $r = \frac{1}{3}$, and $S_0 = 9$. Let V be an *up-and-in American put option* on the stock with upper barrier $U = 13$ and strike price $K = 10$, expiring at time 4. This option becomes an American put the first time that the stock price is greater than or equal to 7. If the stock price is always strictly less than 7 then the option expires worthless at time 4. Find the arbitrage-free price V_0.

2. Let $\Omega = \{H, T\}^N$, with $N \geq 3$, and let $\mathbb{P}$ be a binomial product measure on Ω with probability of heads equal to p and probability of tails equal to q. For each $j \in \{1, 2, \dots, N\}$ define the random variable X_j on Ω by

$$X_j(w) = \begin{cases} j-1 & \text{if } w_j = H \\ \\ -j+1 & \text{if } w_j = T. \end{cases}$$

Define the process $(M_n)_{0 \leq n \leq N}$ by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} X_j, \ n = 1, 2, \dots, N.$$

   (a) Show that if $p = q = \frac{1}{2}$, then $(M_n)_{0 \leq n \leq N}$ is a martingale under $\mathbb{P}$.

   (b) Assume that $p = q = \frac{1}{2}$. Fix $n \in \{0, 1, \dots, N-1\}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a given function. Show that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}_n \left[f(M_{n+1}^2) \right] = g(M_n^2)$$

   (c) If $p \neq q$ is it still true that $(M_n)_{0 \leq n \leq N}$ is a martingale?

   (d) Fix $n \in \{2, \dots, N-1\}$ and assume that $p \neq q$. It is true that for every function $f : \mathbb{R} \to \mathbb{R}$ there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}_n \left[f(M_{n+1}^2) \right] = g(M_n^2)?$$

3. Let $\Omega = \{H, T\}^N$ and let $\mathbb{P}$ be a binomial product measure on Ω with probability of heads equal to p and probability of tails equal to q. Let $(Y_n)_{0 \leq n \leq N}$ be a process such that the following hold.

   - Each Y_n depends only on the n-th coin toss, not on the previous tosses.
   - For each $n, \mathbb{E}[Y_n]$ is a positive number μ_n.
   - For each $n, \mathbb{E}[Y_n^2]$ is positive number ν_n.

Define a process $(X_n)_{0 \le n \le N}$ recursively by $X_0 = 1$ and

$$X_{n+1} = X_n Y_{n+1} \quad \text{for all } n = 0, 1, \cdots, N-1.$$

Put $\alpha_0 = 1$. Find constants $\alpha_1, \cdots, \alpha_N$ such that the process $(\alpha_n X_n^2)_{0 \le n \le N}$ is a martingale under $\mathbb{P}$.

4. Consider an N-period binomial. Determine whether or not each random variable τ is a stopping rule.

   (a) Let
   $$S^*(\omega) = \max\{S_n(\omega) : n = 0, 1, \cdots, N\},$$
   and define $\tau : \Omega \to \mathbb{R}$ by
   $$\tau(\omega) = \min\{n = 0, 1, \cdots, N : S_n(\omega) = S^*(\omega)\} \quad \text{for all } \omega \in \Omega.$$

   (b) Let $K > 0$ be given and define and define $\tau : \Omega \to \mathbb{R}$ by
   $$\tau(\omega) = \min\{n = 0, 1, \cdots, N : S_n(\omega) = K\} \quad \text{for all } \omega \in \Omega.$$

   (c) Let $K > 0$ be given and define and define $\tau : \Omega \to \mathbb{R}$ by
   $$\tau(\omega) = \max\{n = 0, 1, \cdots, N : S_n(\omega) = K\} \quad \text{for all } \omega \in \Omega.$$