21-370 **Discrete Time Finance** **D. Handron**

Final Exam
Pearl Harbor Day, 2010 **Name:**____________________________________

Closed book and notes. Only the following calculators will be permitted: TI-30Xa, TI-30X II, TI-30X IIS

Write your solutions in the space below each problem. Be sure to show all work and explain your reasoning as clearly as possible. You may use the back of the page if you need more space; please indicate clearly if you do so. Good Luck.

Problem	**Points**	**Score**
1	40	
2	30	
3	30	
4	30	
5	40	
6	30	
Total	200	

"On my honor, I have neither given nor received any aid on this exam."

Signature:____________________________________

1. **(40 points)** Consider a 2-period binomial pricing model with $u = \frac{5}{3}$, $d = \frac{2}{3}$, $r = \frac{1}{3}$ and $S_0 = 9$.

   (a) A European *up-and-in rebate option* with barrier U pays \$1 at $t = 2$ if $S_k > U$ for some $k \in \{1, 2\}$. Find the arbitrage-free price V_0 of an up-and-in rebate option V on the stock with $U = 12$.

   (b) A European *down-and-in rebate option* with barrier L pays \$1 at $t = 2$ if $S_k < L$ for some $k \in \{1, 2\}$. Find the arbitrage-free price W_0 of an down-and-in rebate option W on the stock with $L = 7$.

   (c) Let X be a European option that pays \$1 at $t = 2$ if *either* $S_k > 12$ or $S_k < 7$ for some $k \in \{1, 2\}$. How does the arbitrage-free price X_0 compare to $V_0 + W_0$? Give an explanation that does not involve computing X_0 via backward induction or the risk-neutral pricing formula.

2. **(30 points)** Consider a 2-period binomial pricing model with $u = \frac{6}{3}$, $d = \frac{2}{3}$, $r = \frac{1}{3}$ and $S_0 = 9$, and an American put on this stock with expiration $t = 2$ and strike price $K = 16$.

   (a) Find P_0, the arbitrage-free price of the put.

   (b) Describe an optimal exercise policy for this option. Is there more than one optimal exercise policy? Explain why or why not.

3. **(30 points)** Consider a 2-period binomial pricing model with $u = \frac{6}{3}$, $d = \frac{2}{3}$, $r = \frac{1}{3}$ and $S_0 = 9$. Let

$$M_n = \max_{k \in \{0,\ldots,n\}} S_k, \qquad L_n = \min_{k \in \{0,\ldots,n\}} S_k.$$

V is an American option with intrinsic value $G_n = M_n - L_n$.

(a) Find the arbitrage-free price V_0 of this option.

(b) Explalin in detail how to construct a replicating portfolio for the option, and how the replicating portfolio operates.

4. **(30 points)** Consider a 2-period binomial pricing model with $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$ and $S_0 = 4$.

   (a) V is a European option that pays $(S_2 - S_1)^+$ at time $t = 2$. (This is an example of a *forward starting call.*) Find the arbitrage-free price V_0 of this option at $t = 0$.

   (b) Find a value K such that a European call, C with strike price K has arbitrage-free price $C_0 = V_0$.

5. **(40 points)**

(a) Let $\Omega = \{\omega_1 \dots \omega_N : \omega_i \in \{H, T\}\}$ be a coin toss space, with $\mathbb{P}(H) = p$ and $\mathbb{P}(T) = q$. For $k \in \{1, \dots, N\}$ let

$$X_k = \begin{cases} 2k, & \omega_k = H \\ -k, & \omega_k = T \end{cases}$$

and define the process $\{M_n\}_{n=0}^N$ by $M_0 = 0$, $M_n = \sum_{k=1}^n X_k$. Find all the values $\{p, q\}$ (if any) for which $\{M_n\}$ is a martingale.

(b) Let Y_k be an adapted process generated by rolling a six-sided die; $Y_k \in \{1, \dots, 6\}$, each with probability $\frac{1}{6}$. Let $T_n = \sum_{k=1}^n Y_k$ and $L_n = \min_{k \in \{1,\dots,6\}} S_k$. Is the following claim true or false:

For $n = 1, ..., N-1$, there are functions g_n such that $\mathbb{E}_n[T_N - L_N] = g_n(M_n, L_n)$.

Explain your reasoning.

6. **(30 points)** Consider the one-period binomial model with interest rate $r = \frac{1}{5}$ and initial stock price $S_0 = 5$. The price of the stock at $t = 1$ is \$9 with probability p or \$2 with probability q. An investor with wealth $x > 0$ at $t = 0$ and utility function $U(x) = \ln(x)$ if $x > 0$ and $U(x) = -\infty$ if $x \leq 0$ chooses to invest all his capital in the stock. What does this investor believe the probabiities p and q to be?