21-370 DISCRETE TIME FINANCE
WRITTEN HOMEWORK — WEEK #15

2016 FALL

The Black-Scholes formulas for the prices of put and call options with strike price K and maturity τ on a stock with spot price S_0 are

$$\begin{aligned} P_0 &= e^{-r\tau} K\, N(-d_2) - S_0\, N(-d_1) \\ C_0 &= S_0\, N(d_1) - e^{-r\tau} K\, N(d_2) \end{aligned}$$

where r is the annual interest rate under the continuous compounding convention, N is the standard normal cumulative distribution function, and

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right] \\ d_2 &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right] \end{aligned}$$

where σ is the *volatility* of the stock price.

It is easily shown that P_0 and C_0 are monotonically increasing functions of σ. Treating K, τ, S_0, ad r as constants, we can view P_0 and C_0 as invertible functions of σ. For each $\widehat{P_0} > e^{-r\tau}K - S_0$ there is a unique $\widehat{\sigma}$ such that using $\sigma = \widehat{\sigma}$ in the Black-Scholes formula formula for the price of a put results in $\widehat{P_0}$ as the computed option price. Similarly, for each $\widehat{C_0} > S_0 - e^{-r\tau}K$ there is a unique corresponding $\widehat{\sigma}$.

If $\widehat{P_0}$ or $\widehat{C_0}$ is the quoted price of an option, the $\widehat{\sigma}$ is said to be the *implied volatility* of the option.

(1) On December 2, 2016, shares of MMM were valued at $S_0 =$ \$172.43. Short term interest rates at that time were approximately $r = 0.0034$. Quoted prices for call options on MMM with maturity December 9, 2016 ($\tau = 7/360$) on December 2 were:

Strike Price (K)	Ask ($\widehat{C}_0$)
157.5	17.45
160	14.45
162.5	12.40
165	9.75
167.5	6.50
170	3.05
172.5	1.29
175	.35
177.5	.11
180	.08
182.5	.23
185	.09
187.5	.31

Compute the implied volatility corresponding to the options with strike prices 165, 172.5, and 180.

You will need to use some sort of numerical method to do these computations. I had good luck using the "solve" command in Maple, but you can use whatever technology you find most convenient. After computing these three implied volatilities, you may find that you have developed a workflow that makes computing additional implied volatilities relatively painless. If so, you should compute as many more as seems feasible.

(2) Plot the implied volatilities you computed in the previous problem as a function of the strike price. Place implied volatility on the vertical axis and strike price on the horizontal axis. What characteristic shape does the data display?