21-370 **Discrete Time Finance** **D. Handron**

Homework — Week #11

1. Consider a binomial asset pricing model in which $\mathbb{P}(\omega_1 \ldots \omega_N) = p^{\#H(\omega_1 \ldots \omega_N)} q^{\#T(\omega_1 \ldots \omega_N)}$ and a random variable $\tau : \Omega \to \mathbb{R}$. For $A \subseteq \Omega$ denote by 1_A the function

$$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

   Show that τ is a stopping time if and only if

   (a) $\sum_{n=0}^{N} 1_{\{\tau=n\}} = 1$ (i.e. the function $\Omega \to \mathbb{R}$ given by $\omega \mapsto 1$), and

   (b) The sequence $\{X_n = 1_{\{\tau=n\}}\}$ (each term in the sequence is a function) is an adapted stochastic process.

2. Consider a binomial asset pricing model with $u = 2$, $d = \frac{3}{4}$, $r = \frac{3}{8}$ and $S_0 = 8$. A derivative security with random stopping time, V, pays \$1 at $t = n$ if $S_n \geq 20$ or if $S_n \leq 5$. After making one payment, the option expires. At $t = 4$, if no payment has been made, the option expires worthless.

   (a) Use a backward induction process to determine the arbitrage-free value of the security, V_0, at $t = 0$.

   (b) Express the value V_0 as the discounted risk neutral expected value of a self-financing portfolio, i.e. describe the portfolio, X, and show that

$$\widetilde{\mathbb{E}}\left[\frac{X_4}{(1+r)^4}\right] = V_0.$$

3. Problem 4.3 in Shreve.