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Chapter 8

Black-Scholes

8.1 Introduction

In this chapter we derive the *Black-Scholes formulas* for the price of a call option and the price of a put option as the limit of the option prices in an N-period binomial model as the number of steps N goes to infinity. We also derive the *Black-Scholes partial differential equation*, and we verify that the Black-Scholes formulas are solutions of the Black-Scholes partial differential equation. We discuss the "Greeks," the partial derivatives of the function given by the Black-Scholes formulas.

To take the limit in an N-period binomial model, we need two major theorems from probability theory, the *Law of Large Numbers* and the *Central Limit Theorem*. We present these in the next two sections, and in Section 8.4, we use them to obtain the Black-Scholes formulas.

8.2 The Law of Large Numbers

 $\{\$8.1a\}1\}$

Theorem 8.2.1 Let X_1, X_2, \ldots be a sequence of independent random variables, all with the same distribution. Assume each random variable has expected value m and a finite variance. (All the random variables must have the same expected value and variance because they all have the same distribution.) Then the probability is one that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X_n = m.$$
(8.2.1) {8.1a.1}

{C8}

{S8.1}

For the limit of an *N*-period binomial model, we will use the Law of Large Numbers in the context of coin tossing. Suppose we toss a fair coin repeatedly and count the number of heads. Indeed, let

$$\{8.1a.2\} X_n = \begin{cases} +1 & \text{if the } n\text{-th coin toss results in } H, \\ -1 & \text{if the } n\text{-th coin toss results in } T. \end{cases} (8.2.2)$$

These random variables X_n are independent and all have the same distribution. The expected value of each X_n is 1 and the variance of each X_n is also 1. We define N

$$\{8.1a.2a\} M_N = \sum_{n=1} X_n, (8.2.3)$$

and call the process M_n, M_1, M_2, \ldots a symmetric random walk. The Strong Law of Large numbers says that the probability is one that

{8.1a.3}
$$\lim_{N \to \infty} \frac{M_N}{N} = 0.$$
(8.2.4)

We also consider the random variables X_n^+ , which are given by

$$X_n^+ = \begin{cases} 1 & \text{if the } n\text{-th coin toss results in } H, \\ 0 & \text{if the } n\text{-th coin toss results in } T. \end{cases}$$

The random variables X_1^+, X_2^+, \ldots are independent and indentically distributed. They all have expected value $\frac{1}{2}$, and their common variance, $\frac{1}{4}$, is finite. We define

$$H_N = \sum_{n=1}^{N} X_n^+, \qquad (8.2.5)$$

which is the number of heads in the first N tosses. The Law of Large Numbers applied to the sequence X_1^+, X_2^+, \ldots , implies that with probability one,

$$\{8.1a.5\} \qquad \qquad \lim_{N \to \infty} \frac{H_N}{N} = \frac{1}{2}. \tag{8.2.6}$$

The ratio of the number of heads to the number of tosses converges to $\frac{1}{2}$.

Finally, we consider the random variables $X_n^+ - X_n$, which are given by

$$X_n^+ - X_n = \begin{cases} 0 & \text{if the } n\text{-th coin toss results in } H, \\ 1 & \text{if the } n\text{-th coin toss results in } T. \end{cases}$$

 $\mathbf{2}$

{8.1a.4}

8.2. THE LAW OF LARGE NUMBERS

{8.1a.6}

The random variables $X_1^+ - X_1, X_2^+ - X_2, \ldots$ are indpendent and identically distributed. They all have expected value $\frac{1}{2}$, and their common variance, $\frac{1}{4}$, is finite. We define

$$T_N = \sum_{n=1}^{N} (X_n^+ - X_n), \qquad (8.2.7)$$

which is the number of tails in the first N tosses. The Law of Large Numbers applied to the sequence $X_1^+ - X_1, X_2^+ - X_2, \ldots$ implies that with probability one,

$$\lim_{N \to \infty} \frac{T_N}{N} = \frac{1}{2}.$$
 (8.2.8) {8.2.8a}

The ratio of the number of tails to the number of tosses converges to $\frac{1}{2}$.

Finally, note that $X_n^+ + (X_n^+ - X_n)$ is equal to 1, regardless of whether the *n*-th toss results in *H* or *T*. Therefore,

$$H_N + T_N = \sum_{n=1}^{N} \left[X_n^+ + (X_n^+ - X_n) \right] = N.$$
 (8.2.9) {8.1a.8}

This just says that in N tosses, the number of heads plus the number of tails is equal to the number of tosses. On the other hand,

$$H_N - T_N = \sum_{n=1}^N \left[X_n^+ - (X_n^+ - X_n) \right] = \sum_{n=1}^N X_n = M_N.$$
 (8.2.10) {8.1a.9]

We said we were tossing a "fair" coin, by which we mean that the probability \tilde{p} of a H on each toss is $\frac{1}{2}$ and hence the probability \tilde{q} of a T is also $\frac{1}{2}$. We use the notation of risk-neutral probabilities \tilde{p} , \tilde{q} , \mathbb{P} and \mathbb{E} here because in the derivation of the Black-Scholes formulas in this chapter, we shall use the Law of Large Numbers under the risk-neutral probability measure.

Equation (8.2.6) says we should expect about half the coin tosses to result in H. This does not mean that if H gets ahead in the first several coin tosses, there is a need for T to catch up. For example, if we toss the coin 10 times and get a H on every toss, there is no need to then get more tails than heads on subsquent tosses. In particular, it is not true that the probability of a T on the eleventh toss is more than $\frac{1}{2}$. The tosses are independent, and the outcome of the first ten tosses has no bearing on the eleventh toss. The probability of a T on the eleventh toss is still $\frac{1}{2}$. Consider the case that there are 10 heads in the first ten tosses. If from that point on there are as many tails as heads, so that T never catches up, then after 100 tosses there are 55 heads and $\frac{H_{100}}{100} = 0.55$, after 1000 tosses there are 505 heads and $\frac{H_{1000}}{1000} = 0.505$, and after 10,000 tosses there are 5005 heads and $\frac{H_{10,000}}{10,000} = 0.5005$. The ratio $\frac{H_N}{N}$ is converging to $\frac{1}{2}$, even though the number of heads is always ahead of the number of tails by the same amount.

In fact, if we toss the coin N times, it is normal to have the number of heads differ from $\frac{N}{2}$ by about \sqrt{N} . In 100 tosses, we should not be surprised if the number of heads differs from 50 by about $\sqrt{100} = 10$. In 10,000 tosses, we should not be surprised if the number of heads differs from 5,000 by about $\sqrt{10,000} = 100$. Note that if we have 50 + 10 = 60 heads in the first 100 tosses, then $\frac{H_{100}}{100} = 0.60$. If we have 5000 + 100 = 5100 heads in the first 10,000 tosses, then $\frac{H_{10,000}}{10,000} = 0.51$. The ratio $\frac{H_N}{N}$ is converging to $\frac{1}{2}$, even though H_N is getting farther from $\frac{N}{2}$ as N gets bigger. The Law of Large Numbers does not guarantee that the number of heads in the first N tosses is close to $\frac{N}{2}$, and in fact, these two quantities typically get farther apart rather than closer as $N \to \infty$.

8.3 The Central Limit Theorem

 $\{S8.2\}$

In the discussion of fair coin tossing in Section 8.2, we saw that

$$\lim_{N \to \infty} \frac{H_N}{N} = \frac{1}{2},\tag{8.2.6}$$

but that this does not guarantee that H_N and $\frac{N}{2}$ are close. However, there are some conclusions we can draw from (8.2.6). One of them is that if we divide by N to a power larger than 1 in (8.2.6), we will get a limit of zero. For example,

$$\{8.3.1\} \qquad \qquad \lim_{N \to \infty} \frac{H_N}{N\sqrt{N}} = \lim_{N \to \infty} \frac{H_N}{N} \cdot \lim_{N \to \infty} \frac{1}{\sqrt{N}} = \frac{1}{2} \cdot 0 = 0.$$
(8.3.1)

If we divide by N to a power between 0 and 1, we will get ∞ . For example,

$$\{8.3.2\} \qquad \lim_{N \to \infty} \frac{H_N}{\sqrt{N}} = \lim_{N \to \infty} \frac{H_N}{N} \cdot \lim_{N \to \infty} \sqrt{N} = \frac{1}{2} \cdot \lim_{N \to \infty} \sqrt{N} = \infty.$$
(8.3.2)

If we begin instead with (8.2.4),

$$\lim_{N \to \infty} \frac{M_N}{N} = 0, \tag{8.2.4}$$

8.3. THE CENTRAL LIMIT THEOREM

and replace N in the denominator by different powers of N, the situation is more complicated. Analogous to (8.3.1), we have

$$\lim_{N \to \infty} \frac{M_N}{N\sqrt{N}} = \lim_{N \to \infty} \frac{M_N}{N} \cdot \lim_{N \to \infty} \frac{1}{\sqrt{N}} = 0 \cdot 0 = 0.$$
(8.3.3)

However,

{8.3.3}

$$\lim_{N \to \infty} \frac{M_N}{\sqrt{N}} = \lim_{N \to \infty} \frac{M_N}{N} \cdot \lim_{N \to \infty} \sqrt{N}, \qquad (8.3.4)$$

and this leads to the indeterminant form $0 \cdot \infty$. In such a situation, the limit could be anything, and could even fail to exist. In fact, if we toss a coin repeatedly and write down the resulting sequence

$$M_1, \frac{M_2}{\sqrt{2}}, \frac{M_3}{\sqrt{3}}, \frac{M_4}{\sqrt{4}}, \dots,$$
 (8.3.5) {8.3.5}

this sequence of numbers will never settle down and have a limit.

Despite that fact that the sequence in (8.3.5) does not have limit, we can say something about what happens to $\frac{M_N}{\sqrt{N}}$ as N gets large. We can plot the histogram of the distribution of the random variable $\frac{M_N}{\sqrt{N}}$, and we discover that as N gets large, this histogram takes a particular shape, namely, the "bell-shaped curve."

We work out the details for the case N = 25. In this case,

$$\frac{M_{25}}{\sqrt{25}} = \frac{1}{5}M_{25}.$$

In 25 tosses, the number of heads that can occur is any integer between 0 and 25. If $H_{25} = 0$, then $T_{25} = 25$ and $M_{25} = H_{25} - T_{25} = -25$, so $\frac{1}{5}M_{25} = -5$. This is the smallest possible value for $\frac{1}{5}M_{25}$. If $H_{25} = 1$, then $T_{25} = 24$, $M_{25} = -23$, and $\frac{1}{5}M_{25} = -\frac{23}{5} = -4.6$. With each increase in H_{25} of one head, there is a decrease in T_{25} of 1 and hence an increase in M_{25} of 2 and an increase in $\frac{1}{5}M_{25}$ of $\frac{2}{5} = 0.4$. At the upper extreme, if $H_{25} = 25$, then $T_{25} = 0$, $M_{25} = 25$ and $\frac{1}{5}M_{25} = +5$. This is the largest possible value for $\frac{1}{5}M_{25}$. The set of possible values for $\frac{1}{5}M_{25}$ is thus

$$x_0 = -5, x_1 = -4.6, x_2 = -4.2, \dots, x_{12} = -0.2, x_{13} = 0.2, \dots, x_{25} = 5.$$

The probabilities that $\frac{1}{5}M_{25}$ takes these values are given by the formula

$$\widetilde{\mathbb{P}}\left\{\frac{1}{5}M_{25} = x_k\right\} = \frac{25!}{k!(25-k)!} \cdot \frac{1}{2^{25}}, \quad k = 0, 1, \dots, 25.$$
(8.3.6) {8.3.6}

k	x_k	$\widetilde{\mathbb{P}}\left\{\frac{1}{5}M_{25} = x_k\right\}$	h_k	$\varphi(x_k)$
0	-5.0	2.980×10^{-8}	7.451×10^{-8}	1.487×10^{-6}
1	-4.6	7.451×10^{-7}	1.863×10^{-6}	1.014×10^{-5}
2	-4.2	8.941×10^{-6}	2.235×10^{-5}	5.894×10^{-5}
3	-3.8	6.855×10^{-5}	1.714×10^{-4}	2.919×10^{-4}
4	-3.4	3.770×10^{-4}	9.425×10^{-4}	1.232×10^{-3}
5	-3.0	0.001583	0.003958	0.0044
6	-2.6	0.005278	0.01319	0.0136
7	-2.2	0.01433	0.03581	0.0355
8	-1.8	0.03223	0.08058	0.0790
9	-1.4	0.06089	0.1522	0.1497
10	-1.0	0.09742	0.2435	0.2420
11	-0.6	0.1328	0.3321	0.3332
12	-0.2	0.1550	0.3875	0.3910
13	0.2	0.1550	0.3875	0.3910
14	0.6	0.1328	0.3321	0.3332
15	1.0	0.09742	0.2435	0.2420
16	1.4	0.06089	0.1522	0.1497
17	1.8	0.03223	0.08058	0.0790
18	2.2	0.01433	0.03581	0.0355
19	2.6	0.005278	0.01319	0.0136
20	3.0	0.001583	0.003958	0.0044
21	3.4	3.770×10^{-4}	9.425×10^{-4}	1.232×10^{-3}
22	3.8	6.855×10^{-5}	1.714×10^{-4}	2.919×10^{-4}
23	4.2	8.941×10^{-6}	2.235×10^{-5}	5.894×10^{-5}
24	4.6	7.451×10^{-7}	1.863×10^{-6}	1.014×10^{-5}
25	5.0	2.980×10^{-8}	7.451×10^{-8}	1.487×10^{-6}

We record these probabilities in Table 8.3.1 below.

Table 8.3.1

We can use this table to construct the histogram in Figure 8.3.2. Above each of the points x_k we construct a bar. The width of each bar is 0.4. For example, the bar constructed above $x_{13} = 0.2$ has its left side at 0.0 and its right side at 0.4. The adjacent bar, the one constructed above $x_{14} = 0.6$, shares the side at 0.4 with the bar constructed above $x_{13} = 0.2$ and has its right side at 0.8. The width of the bar constructed above $x_{14} = 0.6$ is 0.8 - 0.4 = 0.4.

We construct the bars in the histogram so that the area in the bar

8.3. THE CENTRAL LIMIT THEOREM

above each x_k is the $\widetilde{\mathbb{P}}\{\frac{1}{5}M_{25} = x_k\}$. This means that the height of the bar above x_k is

$$h_k = \frac{1}{0.4} \widetilde{\mathbb{P}} \left\{ \frac{1}{5} M_{25} = x_k \right\}.$$
 (8.3.7) {8.3.7}

In Table 8.3.1, we record the values of h_k as well as the probabilities $\widetilde{\mathbb{P}}\left\{\frac{1}{5}M_{25}=x_k\right\}$ given by (8.3.6).



The standard normal density is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$
(8.3.8) {8.3.8}

The values of this function are reported in the last column of Table 8.3.1, and the graph of this function is superimposed on the histogram in Figure 8.3.2. We see that this function is a good approximation to the heights of the bars in the histogram. In particular

$$h_k \approx \varphi(x_k), \quad k = 0, 1, \dots, 25,$$
 (8.3.9) {8.3.9}

where \approx means "is approximately equal to."

Suppose that for some continuous function f(x), we want to evaluate

$$\widetilde{\mathbb{E}}\left[f\left(\frac{1}{5}M_{25}\right)\right].$$

This would require that we compute the probabilities of $\mathbb{P}\left\{\frac{1}{5}M_{25} = x_k\right\}$, as we have done, and then evaluate the sum

$$\{8.3.10\} \quad \widetilde{\mathbb{E}}\left[f\left(\frac{1}{5}M_{25}\right)\right] = \sum_{k=0}^{25} f(x_k)\widetilde{\mathbb{P}}\left\{\frac{1}{5}M_{25} = k\right\} = \sum_{k=0}^{25} f(x_k)(h_k \times 0.4). \quad (8.3.10)$$

This is already a long computation when the number of tosses is 25, as in (8.3.10), and it becomes extremely time consuming when the number of tosses is larger, say N = 100 or N = 1000. Fortunately, we can avoid this computation because we can get a good approximation to the expected value in (8.3.10) by using (8.3.9) to replace h_k in (8.3.10) and by $\varphi(x_k)$:

$$\{8.3.11\} \qquad \qquad \widetilde{\mathbb{E}}\left[f\left(\frac{1}{5}M_{25}\right)\right] \approx \sum_{k=0}^{25} f(x_k)\varphi(x_k) \times 0.4. \qquad (8.3.11)$$

The right-hand side of (8.3.11) is an approximating sum for a Riemann integral. In fact, if we wanted to approximate the Riemann integral

$$\{8.3.12\} \qquad \qquad \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx, \qquad (8.3.12)$$

as a sum, we would choose some points on the real line, say $x_0 = -5.0$, $x_1 = -4.6, \ldots, x_{25} = 5.0$, and above each of these points we would build a rectangle. Since the distance between the points is 0.4, we could build the rectangles to be centered at these points and each with width 0.4. Since $f(x)\varphi(x)$ is the function we want to integrate, we would make the height of the rectangle at x_k equal to $f(x_k)\varphi(x_k)$. This would result in the sum on the right-hand side of (8.3.11), which is therefore an approximation to the Riemann integral

$$\{8.3.13\} \qquad \qquad \int_{-5}^{5} f(x)\varphi(x) \, dx. \qquad (8.3.13)$$

But when |x| > 5, $\varphi(x)$ is very small (see the last column in Table 8.3.1), and so, provided f(x) does not grow too rapidly as $x \to \infty$ or $x \to -\infty$, the difference between the integral in (8.3.13) and the integral in (8.3.12)

8.3. THE CENTRAL LIMIT THEOREM

 $\{8.3.14\}$

is small. In general, we will use this approximation only for functions f(x) that satisfy

$$|f(x)| \le C \text{ for all } x \in \mathbb{R}$$
(8.3.14)

for some constant C. Such a function is said to be *bounded*, and for such a function, the difference between (8.3.13) and (8.3.12) can be ignored when the number of coin tosses is large. For a bounded function, we have the approximation

$$\widetilde{\mathbb{E}}\left[f\left(\frac{1}{5}M_{25}\right)\right] \approx \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx. \tag{8.3.15} \quad \{8.3.15\}$$

Saying that two things are approximately equal, as we just did in (8.3.15), is not a precise mathematical statement. We make precise the idea we are trying to capture in (8.3.15) using limits. The precise statement for the situation we have been discussing is the Central Limit Theorem, which we now state.

{T8.3.1}

Theorem 8.3.1 (Central Limit) Let X_1, X_2, \cdots be a sequence of independent, identically distributed random variables under a probability measure $\widetilde{\mathbb{P}}$. Assume that $\widetilde{\mathbb{E}}X_n = 0$ and $Var(X_n) = 1$. (The expected value and the variance is the same for each X_n because all these random variables have the same distribution.) Let f(x) be a bounded function defined on the real line that is continuous except possibly at finitely many points. Then

$$\lim_{N \to \infty} \widetilde{\mathbb{E}}\left[f\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n\right) \right] = \int_{-\infty}^{\infty} f(x)\varphi(x) \, dx. \tag{8.3.16}$$

It can happen that the random variables we wish to study are of the form

$$\frac{\gamma}{\sqrt{N}}\sum_{n=1}^{N}X_n + Y_N,$$

where the sequence X_1, X_2, \ldots is as in the Central Limit Theorem, γ is a real number, and Y_1, Y_2, \ldots is a sequence of random variables converging to a real number y. For such a case, we have the following generalization of the Central Limit Theorem.

{T8.3.2}

Theorem 8.3.2 (Generalized Central Limit) Let X_1, X_2, \cdots be a sequence of independent, identically distributed random variables under a probability measure $\widetilde{\mathbb{P}}$. Assume that $\widetilde{\mathbb{E}}X_n = 0$ and $Var(X_n) = 1$. Let γ be a real

number and let Y_1, Y_2, \ldots be a sequence of random variables such that, with probability 1,

$$\lim_{N \to \infty} Y_N = y,$$

where y is a real number. Let f(x) be a bounded function defined on the real line that is continuous except possibly at finitely many points. then

$$\{8.3.17\} \qquad \lim_{N \to \infty} \widetilde{\mathbb{E}}\left[f\left(\frac{\gamma}{\sqrt{N}} \sum_{n=1}^{N} X_n + Y_N\right) \right] = \int_{-\infty}^{\infty} f(\gamma x + y)\varphi(x) \, dx. \quad (8.3.17)$$

We close this section with a few observations about the standard normal density $\varphi(x)$. This function is positive for every $x \in \mathbb{R}$, and φ integrates to one:

$$\{8.3.18\} \qquad \qquad \int_{-\infty} \varphi(x) \, dx = 1. \tag{8.3.18}$$

We shall use (8.3.18) without deriving it. (To derive (8.3.18), write

$$\left(\int_{-\infty}^{\infty}\varphi(x)\,dx\right)^2 = \int_{-\infty}^{\infty}\varphi(x)\,dx \cdot \int_{-\infty}^{\infty}\varphi(y)\,dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)/2}\,dx\,dy,$$

and change to polar coordinates to compute the right-hand side, which turns out to be 1.)

The cumulative standard normal distribution is

$$\{8.3.19\} N(x) = \int_{-\infty}^{x} \varphi(y) \, dy. (8.3.19)$$

For $x \in \mathbb{R}$, N(x) is the area under the graph of φ to the left of the point x. Because is symmetric, (i.e., $\varphi(y) = \varphi(-y)$ for all $y \in \mathbb{R}$), the area under the graph of φ to the left of x, which is N(x), is the same as the area under the graph of φ to the right of -x. But the total area under the graph of φ is 1 and the area under the graph to the left of -x is N(-x), so the area under the graph to the right of -x is 1 - N(-x). In other words,

$$\{8.3.23\} N(x) = 1 - N(-x) \text{ for all } x \in \mathbb{R}. (8.3.20)$$

From the definition of N(x), we see that

$$\{8.3.20\} N'(x) = \varphi(x). (8.3.21)$$

8.4. THE BLACK-SCHOLES FORMULAS

In particular, N'(x) is strictly positive, so N is strictly increasing. From the definition of N(x), it is apparent that

$$\{8.3.21\} \qquad \qquad \lim_{x \to -\infty} N(x) = 0. \tag{8.3.22}$$

From (8.3.18), we have

{8.3.22}

$$\lim_{x \to \infty} N(x) = \int_{-\infty}^{\infty} \varphi(y) \, dy = 1. \tag{8.3.23}$$

8.4 The Black-Scholes Formulas

 $\{$ S8.4 $\}$

In this section we develop the Black-Scholes formula for European puts and calls.

8.4.1 Scaling of interest rate and volatility

We consider a stock with initial price per share S_0 . On this stock we have a put option expiring at a positive time τ (measured in years) and having a positive strike price K. We divide the time between 0 and τ into N steps, so that each step corresponds to a period of time of length $\frac{\tau}{N}$.¹

We want to build a binomial model which has N steps between time zero and time τ . To simplify the computations, we will design this model so that the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. If the up factor per period is u and the down factor is d, where 0 < d < 1 + r < u, then in order to have

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1}{2}, \quad \tilde{q} = \frac{u-1-r}{u-d} = \frac{1}{2},$$

we must have

$$2(1+r-d) = u - d = 2(u - 1 - r),$$

and so u - (1 + r) = (1 + r) - d. We call this common value σ , which is positive. In other words,

$$\sigma = u - (1+r) = (1+r) - d, \qquad (8.4.1) \quad \{8.4.1a\}$$

¹A stock price can change only when it is possible to trade it, which generally means only when the exchange on which it is traded is open. In practice, one needs to account for this fact when dividing the time interval between zero and τ into steps. For example, suppose $\tau = \frac{1}{4}$ years, so we have a "three-month option." In three months there are approximately 66 trading days. Thus, If we take N = 66, we are dividing time into steps of one trading day each and are excluding non-trading days from consideration. In these notes we ignore these so-called *day count* issues.

or equivalently,

$$u = 1 + r + \sigma, \quad d = 1 + r - \sigma.$$
 (8.4.2) {8.4.2a}

The risk-neutral expected return between time n and time n+1 is defined to be

$$\widetilde{\mathbb{E}}\left[\frac{S_{n+1} - S_n}{S_n}\right] = \widetilde{p}\frac{uS_n - S_n}{S_n} + \widetilde{q}\frac{dS_n - S_n}{S_n} \\ = \frac{1}{2}(u-1) + \frac{1}{2}(d-1) \\ = \frac{1}{2}(r+\sigma) + \frac{1}{2}(r-\sigma) \\ = r, \qquad (8.4.3)$$

which is what the expected return must be under the risk-neutral measure. To determine the risk-neutral variance of the return between time n and time n + 1, we first compute

$$\widetilde{\mathbb{E}}\left[\left(\frac{S_{n+1}-S_n}{S_n}\right)^2\right] = \widetilde{p}\left(\frac{uS_n-S_n}{S_n}\right)^2 + \widetilde{q}\left(\frac{dS_n-S_n}{S_n}\right)^2 \\ = \frac{1}{2}(u-1)^2 + \frac{1}{2}(d-1)^2 \\ = \frac{1}{2}(r+\sigma)^2 + \frac{1}{2}(r-\sigma)^2 \\ = r^2 + \sigma^2.$$
(8.4.4)

The risk-neutral variance of the return is

$$\operatorname{Var}\left[\frac{S_{n+1}-S_n}{S_n}\right] = \widetilde{\mathbb{E}}\left[\left(\frac{S_{n+1}-S_n}{S_n}\right)^2\right] - \left(\widetilde{\mathbb{E}}\left[\frac{S_{n+1}-S_n}{S_n}\right]\right)^2$$
$$= (r^2 + \sigma^2) - r^2$$
$$= \sigma^2. \tag{8.4.5}$$

Having thus computed the one-period expected return and variance of return, we next compute the expected return and variance of return for N periods. To do this, we need the formulas

$$\{8.4.6a\} \qquad \mathbb{E}\left[\frac{S_{n+1}}{S_n}\right] = \tilde{p}u + \tilde{q}d = \frac{1}{2}(1+r+\sigma) + \frac{1}{2}(1+r-\sigma) = 1+r \qquad (8.4.6)$$

and

$$\{\mathbf{8.4.7a}\} \qquad \widetilde{\mathbb{E}}\left[\frac{S_{n+1}^2}{S_n^2}\right] = \tilde{p}u^2 + \tilde{q}d^2 = \frac{1}{2}(1+r+\sigma)^2 + \frac{1}{2}(1+r-\sigma)^2 = (1+r)^2 + \sigma^2. \tag{8.4.7}$$

The actual return over the N periods beginning at time zero is $\frac{S_N-S_0}{S_0}$. Using the independence of the random variables $\frac{S_1}{S_0}$, $\frac{S_2}{S_1}$, ..., $\frac{S_{N-1}}{S_{N-2}}$, $\frac{S_{N-1}}{S_N}$ and using (8.4.6), we compute

$$\widetilde{\mathbb{E}}\left[\frac{S_N}{S_0}\right] = \widetilde{\mathbb{E}}\left[\frac{S_1}{S_0} \cdot \frac{S_2}{S_1} \cdot \dots \cdot \frac{S_{N-1}}{S_{N-2}} \cdot \frac{S_N}{S_{N-1}}\right] \\
= \widetilde{\mathbb{E}}\left[\frac{S_1}{S_0}\right] \cdot \widetilde{\mathbb{E}}\left[\frac{S_2}{S_1}\right] \cdot \dots \cdot \widetilde{\mathbb{E}}\left[\frac{S_{N-1}}{S_{N-2}}\right] \cdot \widetilde{\mathbb{E}}\left[\frac{S_N}{S_{N-1}}\right] \\
= (1+r)^N.$$
(8.4.8)

The expected return over the N periods beginning at time zero is

$$\widetilde{\mathbb{E}}\left[\frac{S_N - S_0}{S_0}\right] = \widetilde{\mathbb{E}}\left[\frac{S_N}{S_0}\right] - 1 = (1+r)^N - 1.$$
(8.4.9) {8.4.9a}

To compute the variance of the expected return, we first use (8.4.8), independence of $\frac{S_1^2}{S_0^2}, \frac{S_2^2}{S_1^2}, \ldots, \frac{S_{N-1}^2}{S_{N-2}^2}, \frac{S_N^2}{S_{N-1}^2}$, and (8.4.7) to compute

$$\begin{split} \widetilde{\mathbb{E}} \left[\left(\frac{S_N - S_0}{S_0} \right)^2 \right] \\ &= \widetilde{\mathbb{E}} \left[\frac{S_N^2 - 2S_N S_0 + S_0^2}{S_0^2} \right] \\ &= \widetilde{\mathbb{E}} \left[\frac{S_N^2}{S_0^2} \right] - 2\widetilde{\mathbb{E}} \left[\frac{S_N}{S_0} \right] + 1 \\ &= \widetilde{\mathbb{E}} \left[\frac{S_1^2}{S_0^2} \cdot \frac{S_2^2}{S_1^2} \cdots \cdot \frac{S_{N-1}^2}{S_{N-2}^2} \cdot \frac{S_N^2}{S_{N-1}^2} \right] - 2(1+r)^N + 1 \\ &= \widetilde{\mathbb{E}} \left[\frac{S_1^2}{S_0^2} \right] \cdot \widetilde{\mathbb{E}} \left[\frac{S_2^2}{S_1^2} \right] \cdots \cdots \widetilde{\mathbb{E}} \left[\frac{S_{N-1}^2}{S_{N-2}^2} \right] \cdot \widetilde{\mathbb{E}} \left[\frac{S_N^2}{S_{N-1}^2} \right] - 2(1+r)^N + 1 \\ &= ((1+r)^2 + \sigma^2)^N - 2(1+r)^N + 1. \end{split}$$
(8.4.10)

The variance of the N-period return is

$$\operatorname{Var}\left[\frac{S_N - S_0}{S_0}\right] = \widetilde{\mathbb{E}}\left[\left(\frac{S_N - S_0}{S_0}\right)^2\right] - \left(\widetilde{\mathbb{E}}\left[\frac{S_N - S_0}{S_0}\right]\right)^2 \\ = \left((1+r)^2 + \sigma^2\right)^N - 2(1+r)^N + 1 - \left((1+r)^N - 1\right)^2 \\ = \left((1+r)^2 + \sigma^2\right)^N - 2(1+r)^N + 1 - \left((1+r)^{2N} - 2(1+r)^N + 1\right) \\ = \left((1+r)^2 + \sigma^2\right)^N - (1+r)^{2N}.$$
(8.4.11)

We divide the time interval $[0, \tau]$ into N steps, each of which will have lenth $\frac{\tau}{N}$, and we ultimately let $N \to \infty$. When we do that, the average growth rate of the stock will be the continuously compounding rate. We would like the interest rate for this continuously compounding to be r per year. In particular, we want to have $\widetilde{\mathbb{E}}[S_N] = e^{r\tau}S_0$, or equivalently, we want to have the N-period risk-neutral expected return on the stock to be

$$\{8.4.12a\} \qquad \qquad \widetilde{\mathbb{E}}\left[\frac{S_N - S_0}{S_0}\right] = e^{r\tau} - 1. \qquad (8.4.12)$$

But in order to have interest rate r per year, we cannot also have interest rate r per period, which is a fraction of year. Indeed, comparing (8.4.12) to (8.4.9), we see that the interest accrued over each period should be computed using $\frac{r\tau}{N}$ rather than r. This makes sense because when r is the annual interest rate, the interest accrued on \$1 over a time period of length $\frac{\tau}{N}$ should be $\frac{r\tau}{N}$. If we substitute $\frac{r\tau}{N}$ for r in (8.4.9) and then let $N \to \infty$, we obtain the desired expected return,

$$\lim_{N \to \infty} \widetilde{\mathbb{E}} \left[\frac{S_N - S_0}{S_0} \right] = \lim_{N \to \infty} \left(1 + \frac{r\tau}{N} \right)^N - 1 = e^{r\tau} - 1, \qquad (8.4.13)$$

where the limit is justified by substituting $a = r\tau$ and b = 0 into Lemma 8.4.1, which follows.

{L8.4.1}

Lemma 8.4.1 Let a and b be real numbers. Then

{8.4.14a}
$$\lim_{N \to \infty} \left(1 + \frac{a}{N} + \frac{b}{N^2} \right)^N = e^a.$$
(8.4.14)

8.4. THE BLACK-SCHOLES FORMULAS

PROOF: We make the change of variable $x = \frac{1}{N}$ in the following calulation:

$$\lim_{N \to \infty} \ln\left(1 + \frac{a}{N} + \frac{b}{N^2}\right)^N = \lim_{x \to 0} \ln\left(1 + ax + bx^2\right)^{\frac{1}{a}} = \lim_{x \to 0} \frac{\ln(1 + ax + bx^2)}{x}.$$

This leads to the indeterminant form $\frac{0}{0}$, and so we use L'Hopital's rule to compute

$$\lim_{x \to 0} \frac{\ln(1 + ax + bx^2)}{x} = \lim_{x \to 0} \frac{a + 2bx}{1 + ax + bx^2} = a.$$

We have computed the logarithm of the limit in (8.4.14) and gotten a, so the limit in (8.4.14) is e^a .

We also want to have a limiting variance for the stock return as $N \to \infty$. The variance is given by (8.4.11). We have already seen that we should replace r by $\frac{r\tau}{N}$. To guarantee that there is a meaningful limit as $N \to \infty$ in (8.4.11), we similarly replace σ^2 by $\frac{\sigma^2 \tau}{N}$, or equivalently, replace σ by $\frac{\sigma \sqrt{\tau}}{\sqrt{N}}$. With this substitution and again using Lemma 8.4.1, we obtain the limiting variance of return

$$\lim_{N \to \infty} \operatorname{Var} \left[\frac{S_N - S_0}{S_0} \right]$$

$$= \lim_{N \to \infty} \left(\left(1 + \frac{r\tau}{N} \right)^2 + \frac{\sigma^2 \tau}{N} \right)^N - \lim_{N \to \infty} \left(1 + \frac{2r\tau}{2N} \right)^{2N}$$

$$= \lim_{N \to \infty} \left(1 + \frac{(2r + \sigma^2)\tau}{N} + \frac{r^2 \tau^2}{N^2} \right)^N - e^{2r\tau}$$

$$= e^{(2r + \sigma^2)\tau} - e^{2r\tau}.$$
(8.4.15)

Note that if we had instead replaced σ by $\frac{\sigma\tau}{N}$, then the limiting variance of $\frac{S_N-S_0}{S_0}$ would be zero, so that there would be no randomness left as $N \to \infty$. For that reason, we replace σ by $\frac{\sigma\sqrt{\tau}}{\sqrt{N}}$ rather than by $\frac{\sigma\tau}{N}$.

We now return to the formula (8.4.2) for the up and down factors in the N period model. When we divide the time interval $[0, \tau]$ into N periods, we take these to be

$$u = 1 + \frac{r\tau}{N} + \frac{\sigma\sqrt{\tau}}{\sqrt{N}}, \quad d = 1 + \frac{r\tau}{N} - \frac{\sigma\sqrt{\tau}}{\sqrt{N}}.$$
 (8.4.16) {8.4.17a}

With these choices of u and d, the risk-neutral probabilities are still $\tilde{p} = \tilde{q} = \frac{1}{2}$.

The parameter σ in (8.4.16) is called the *volatility* of the stock. This parameter describes how much the stock price moves over time and is thus a measure of the risk associated with investing in the stock.

When one builds a binomial model for a stock price on an interval of time from 0 to τ , dividing $[0, \tau]$ into N steps, one first estimates the volatility from price data and then takes the up and down factors to be either those given by (8.4.16) or by

$$\{8.4.1\} u = e^{\sigma\sqrt{\tau}/\sqrt{N}}, \quad d = e^{-\sigma\sqrt{\tau}/\sqrt{N}}. (8.4.17)$$

The formulas (8.4.16) and (8.4.17) are quite close, as we now show.

Recall from Taylor's Theorem that if a function f(x) has continuous first and second derivatives, then

$$\{8.4.3\} f(x) = f(0) + f'(0)x + \frac{1}{2}f''(\xi)x^2, (8.4.18)$$

where ξ is a point between 0 and x. So long as we restrict attention to $x \in [-1, 1]$, the term $f''(\xi)$ is bounded by a constant, and we may rewrite (8.4.18) as

$$\{8.4.4\} f(x) = f(0) + f'(0)x + O(x^2), (8.4.19)$$

where we use the notation $O(x^2)$ to denote any term that is bounded by a constant times x^2 so long as $x \in [-1, 1]$.

Applying Taylor's Theorem to $f(x) = e^x$, for which f'(0) = 1, we obtain from (8.4.19) that

$$\{8.4.5\} e^x = 1 + x + O(x^2). (8.4.20)$$

If $N > \sigma^2 \tau$, then we can replace x by $\pm \sigma \sqrt{\tau} / \sqrt{N}$, which is in [-1, 1], and (8.4.20) yields

$$\{8.4.6\} \qquad e^{\pm\sigma\sqrt{\tau}/N} = 1 \pm \frac{\sigma\sqrt{\tau}}{\sqrt{N}} + O\left(\frac{1}{N}\right), \qquad (8.4.21)$$

where we use the notation $O(\frac{1}{N})$ to denote any term that can be bounded by $\frac{1}{N}$ times a constant that does not depend on N. From (8.4.21) we see that the choice of u and d in (8.4.17) is close to the choice in (8.4.16), and in fact the difference is no larger than a constant times $\frac{1}{N}$. Because the choices in (8.4.16) and (8.4.17) are so close for large values of N, either choice will lead to the same Black-Scholes formulas; we make the choice (8.4.16) because it makes the derivation of the formulas simpler.

8.4. THE BLACK-SCHOLES FORMULAS

8.4.2 Black-Scholes price of a put

Consider an N-period binomial model with up and down factors u and d given by (8.4.16) and with per-period interest rate $\frac{r\tau}{N}$. The risk-neutral probabilities are

$$\tilde{p} = \frac{1 + \frac{r\tau}{N} - d}{u - d} = \frac{\sigma\sqrt{\tau}/\sqrt{N}}{2\sigma\sqrt{\tau}\sqrt{N}} = \frac{1}{2}, \quad \tilde{q} = 1 - \tilde{p} = \frac{1}{2}.$$
(8.4.22) (8.4.7)

This model is the result of dividing τ years into N steps, and so S_N is the stock price at time τ . This stock price is

$$S_N = S_0 u^{H_N} d^{T_N}, (8.4.23) {8.4.8}$$

where H_N is the number of heads obtained in N coin tosses, and T_N is the number of tails. In this subsection we price a put, so we are interested in computing

$$P_{0} = \frac{1}{(1 + \frac{r\tau}{N})^{N}} \widetilde{\mathbb{E}} \left[(K - S_{N})^{+} \right] = \frac{1}{(1 + \frac{r\tau}{N})^{N}} \widetilde{\mathbb{E}} \left[\left(K - S_{0} u^{H_{N}} d^{T_{N}} \right)^{+} \right]$$
(8.4.24) (8.4.9)

for some positive strike price K. In particular, we want to compute the limit in (8.4.24) as $N \to \infty$. We can use Lemma 8.4.1 to compute the limit of the discount term in (8.4.9). In fact, Lemma 8.4.1 implies

$$\lim_{N \to \infty} \frac{1}{(1 + \frac{r\tau}{N})^N} = e^{-r\tau}.$$
(8.4.25) {8.4.25}

We use the Law of Large Numbers and the Central Limit Theorem to compute the limit of the expected value in (8.4.24). In particular, we compute

$$\lim_{N \to \infty} \widetilde{\mathbb{E}} \left[\left(K - S_0 u^{H_N} d^{T_N} \right)^+ \right], \qquad (8.4.26) \quad \{8.4.10\}$$

and this will result in the Black-Scholes formula for the price of a put. For large values of N, the put price in the N-period binomial model will be close to the price given by the Black-Scholes formula.

We first work out the Taylor series expansion for the function $f(x) = \ln(1+x)$. We need the first two derivatives and their values at zero, which are as follows:

$$f(x) = \ln(1+x), \qquad f(0) = 0,$$

$$f'(x) = \frac{1}{1+x}, \qquad f'(0) = 1,$$

$$f''(x) = -\frac{1}{(1+x)^2}, \qquad f''(0) = -1.$$

According to Taylor's Theorem,

$$\ln(1+x) = f(0) + xf'(0) + \frac{1}{2}x^2f''(x) + O(x^3) = x - \frac{1}{2}x^2 + O(x^3). \quad (8.4.27) \quad \{8.4.11\}$$

Now

{8.4.

$$\ln S_{N}$$

$$= \ln \left(S_{0}u^{H_{N}}d^{T_{N}}\right)$$

$$= \ln S_{0} + H_{N}\ln u + T_{n}\ln d$$

$$= \ln S_{0} + H_{N}\ln \left(1 + \frac{r\tau}{N} + \frac{\sigma\sqrt{\tau}}{\sqrt{N}}\right) + T_{N}\ln \left(1 + \frac{r\tau}{N} - \frac{\sigma\sqrt{\tau}}{\sqrt{N}}\right)$$

$$= \ln S_{0} + H_{N}\left(\frac{r\tau}{N} + \frac{\sigma\sqrt{\tau}}{\sqrt{N}} - \frac{\sigma^{2}\tau}{2N} + O\left(\frac{1}{N\sqrt{N}}\right)\right)$$

$$+ T_{N}\left(\frac{r\tau}{N} - \frac{\sigma\sqrt{\tau}}{\sqrt{N}} - \frac{\sigma^{2}\tau}{2N} + O\left(\frac{1}{N\sqrt{N}}\right)\right)$$

$$= \sigma\sqrt{\tau}\frac{H_{N} - T_{N}}{\sqrt{N}} + \ln S_{0} + \left(r - \frac{1}{2}\sigma^{2}\right)\tau\frac{H_{N} + T_{N}}{N}$$

$$+ \frac{H_{N}}{N}O\left(\frac{1}{\sqrt{N}}\right) + \frac{T_{N}}{N}O\left(\frac{1}{\sqrt{N}}\right)$$

$$= \sigma\sqrt{\tau}\frac{M_{N}}{\sqrt{N}} + \ln S_{0} + \left(r - \frac{1}{2}\sigma^{2}\right)\tau + \frac{H_{N}}{N}O\left(\frac{1}{\sqrt{N}}\right) + \frac{T_{N}}{N}O\left(\frac{1}{\sqrt{N}}\right),$$

$$(8.4.28)$$

where we have used (8.2.10) and (8.2.9) in the last step. We define

$$Y_N = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\tau + \frac{H_N}{N}O\left(\frac{1}{\sqrt{N}}\right) + \frac{T_N}{N}O\left(\frac{1}{\sqrt{N}}\right).$$

According to (8.3.1) and the analogous equation for T_N , with probability one,

$$\{8.4.14\} \qquad \qquad \lim_{N \to \infty} Y_N = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\tau. \qquad (8.4.29)$$

Furthermore, M_N is given by (8.2.3), where the random variables X_1, X_2, \ldots are independent and identically distributed with expected value 0 and variance 1. The Generalized Central Limit Theorem, Theorem 8.3.2, implies that for any bounded, continuous function f(x) defined on \mathbb{R} ,

$$\lim_{N \to \infty} \widetilde{\mathbb{E}} \left[f(\ln S_N) \right] = \int_{-\infty}^{\infty} f\left(x \sigma \sqrt{\tau} + \ln S_0 + \left(r - \frac{1}{2} \sigma^2 \right) \tau \right) \varphi(x) \, dx.$$
(8.4.30)

8.4. THE BLACK-SCHOLES FORMULAS

To get the put payoff, we take

$$f(x) = (K - e^x)^+$$
(8.4.31)

so that

{8.4.21}

$$f(\ln S_N) = (K - S_N)^+.$$
 (8.4.32) {8.4.22}

This function is continuous and bounded between 0 and K, and thus satisfies the conditions of the Generalized Central Limit Theorem. Using (8.4.25) and (8.4.30), we see that the limit as $N \to \infty$ of the put price P_0 in (8.4.24) is

$$\lim_{N \to \infty} \frac{1}{(1 + \frac{r\tau}{N})^N} \widetilde{\mathbb{E}}\left[\left(K - S_N\right)^+\right]$$

$$= e^{-r\tau} \int_{-\infty}^{\infty} \left(K - \exp\left\{x\sigma\tau + \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\}\right)^+ \varphi(x) \, dx$$

$$= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(K - S_0 \exp\left\{x\sigma\tau + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\}\right)^+ \exp\left\{-\frac{x^2}{2}\right\} \, dx.$$
(8.4.34)

It remains to compute the right-hand side of (8.4.34). We must first determine the values of x for which

$$K - S_0 \exp\left\{x\sigma\tau + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\} > 0, \qquad (8.4.35) \quad \{8.4.36\}$$

so that the integrand on the right-hand side of (8.4.34) is not zero. Inequality (8.4.35) is equivalent to each of the following inequalities:

$$K > S_0 \exp\left\{x\sigma\sqrt{\tau} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\},$$
$$\frac{K}{S_0} > \exp\left\{x\sigma\sqrt{\tau} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\},$$
$$\ln\frac{K}{S_0} > x\sigma\sqrt{\tau} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\},$$
$$-\ln\frac{S_0}{K} - \left(r - \frac{1}{2}\sigma^2\right)\tau > x\sigma\sqrt{\tau},$$
$$-\frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right] > x.$$

We define

$$d_2 = \frac{1}{\sigma\sqrt{\tau}} \left[\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2 \right) \tau \right].$$
 (8.4.36) {8.4.37}

Then (8.4.35) is equivalent to

$$x < -d_2.$$
 (8.4.37) {8.4.38}

We only need to integrate the right-hand side of (8.4.34) over values of x satisfying (8.4.37); for other values of x, the integrand in (8.4.34) is zero. Thus, the right-hand side of (8.4.34) is

$$\frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} \left(K - S_0 \exp\left\{ x\sigma\sqrt{\tau} + \left(r - \frac{1}{2}\sigma^2\right)\tau \right\} \right) \exp\left\{-\frac{x^2}{2}\right\} dx = \frac{e^{-r\tau}K}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-x^2/2} dx - \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} \exp\left\{-\frac{1}{2}(x - \sigma\sqrt{\tau})^2\right\} dx.$$
(8.4.38)

The first term on the right-hand side of (8.4.38) is $e^{-r\tau}KN(-d_2)$. In the second term, we make the change of variable $y = x - \sigma\sqrt{\tau}$. To find the upper limit of integration, we note that when $x = -d_2$,

$$y = -d_2 - \sigma \sqrt{\tau}$$

= $-\frac{1}{\sigma \sqrt{\tau}} \left[\ln \frac{S_0}{K} + \left(r - \frac{1}{2} \sigma^2 \right) \tau \right] - \sigma \sqrt{\tau}$
= $-\frac{1}{\sigma \sqrt{\tau}} \left[\ln \frac{S_0}{K} + \left(r - \frac{1}{2} \sigma^2 \right) \tau + \sigma^2 \tau \right]$
= $-\frac{1}{\sigma \sqrt{\tau}} \left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2} \sigma^2 \right) \tau \right] = -d_1,$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left[\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2 \right) \tau \right].$$
(8.4.39)

The second term on the right-hand side of (8.4.38) is

$$-\frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{-d_1} e^{-y^2/2} \, dy = -S_0 N(-d_1).$$

We have thus determined that the right-hand side of (8.4.38) is

$$e^{-r\tau}KN(-d_2) - S_0N(-d_1).$$

This is the Black-Scholes price of a put. We summarize with a theorem.

{8.4.40}

Theorem 8.4.2 The Black-Scholes price of a put with strike price K and expiration time τ on a stock with volatility σ , obtained as the limit of the put price in a binomial model, is

$$\{8.4.41\} \qquad \lim_{N \to \infty} \frac{1}{(1 + \frac{r\tau}{N})^N} \widetilde{\mathbb{E}} \left[\left(K - S_N \right)^+ \right] = e^{-r\tau} K N(-d_2) - S_0 N(-d_1), \quad (8.4.40)$$

where d_1 and d_2 are given by (8.4.39) and (8.4.36).

8.4.3 Black-Scholes price of a call

We derived the Black-Scholes price of a put rather than a call because the put pay-off function is bounded (see (8.4.31)), and this is required in order to use the Generalized Central Limit Theorem. The call payoff function is unbounded (in place of (8.4.31), for the call we would have $f(x) = (e^x - K)^+$, and this has limit ∞ as $x \to \infty$), and so we cannot directly apply the Generalized Central Limit Theorem to the call pricing problem. However, we can use put-call parity to derive the call price from the put price, and we do that now.

 $\{T8.4.3\}$

Theorem 8.4.3 The Black-Scholes price of a call with strike price K and expiration time τ on a stock with volatility σ , obtained as the limit of the call price in a binomial model, is

$$\lim_{N \to \infty} \frac{1}{(1 + \frac{r\tau}{N})^N} \widetilde{\mathbb{E}} \left[\left(S_N - K \right)^+ \right] = S_0 N(d_1) - e^{-r\tau} K N(d_2), \qquad (8.4.41) \quad \{8.4.42\}$$

where d_1 and d_2 are given by (8.4.39) and (8.4.36).

PROOF: Let

$$C_0 = \frac{1}{(1 + \frac{r\tau}{N})^N} \widetilde{\mathbb{E}}\left[\left(S_N - K\right)^+\right)\right],$$

and recall from (8.4.24) the price of the put

$$P_0 = \frac{1}{(1 + \frac{r\tau}{N})^N} \widetilde{\mathbb{E}}\left[\left(K - S_N\right)^+\right)\right].$$

Because

$$(S_N - K)^+ - (K - S_N)^+ = S_N - K,$$

we have

$$C_0 - P_0 = \frac{1}{(1 + \frac{r\tau}{N})^N} \widetilde{\mathbb{E}} \left[S_N - K \right]$$

But $\frac{S_n}{(1+\frac{r_n}{r_n})^n}$, n = 0, 1, ..., N, is a martingale under the risk-neutral probability measure, so

$$C_0 - P_0 = S_0 - \frac{K}{(1 + \frac{r\tau}{N})^N}$$

From this equation, Theorem 8.4.2, and equation (8.4.25), we have

$$\lim_{N \to \infty} C_0 = \lim_{N \to \infty} P_0 + S_0 - \lim_{N \to \infty} \frac{K}{(1 + \frac{r\tau}{N})^N}$$

= $e^{-r\tau} K N(-d_2) - S_0 N(-d_1) + S_0 - e^{-r\tau} K$
= $S_0 (1 - N(-d_1)) - e^{-r\tau} K (1 - N(-d_2)).$ (8.4.42)

Finally, we use (8.3.20) to obtain (8.4.41) from (8.4.42).

8.4.4 Summary of formulas

For future reference, we record here the Black-Scholes formulas for the price of a call and a put. In these formulas, s is the price of the underlying stock at the time of pricing and τ is the time until expiration of the option. If the option expires at T and the time of pricing is t, where $0 \le t < T$, then

$$\{8.4.44\} \qquad \tau = T - t, \qquad (8.4.43)$$

and s is the stock price at time t. The variables d_1 and d_2 depend on both τ and s, and we indicate that explicitly in (8.4.44)–(8.4.47) below.

The Black-Scholes price of a call is

$$\{8.4.45\} \qquad c(t,s) = sN(d_1(\tau,s)) - e^{-r\tau}KN(d_2(\tau,s)), \qquad (8.4.44)$$

and the Black-Scholes price of a put is

$$\{8.4.46\} \qquad p(t,s) = e^{-r\tau} KN(-d_2(\tau,s)) - sN(-d_1(\tau,s)), \qquad (8.4.45)$$

where

$$d_1(\tau, s) = \frac{1}{\sigma\sqrt{\tau}} \left[\ln \frac{s}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau \right], \qquad (8.4.46)$$

$$d_2(\tau, s) = \frac{1}{\sigma\sqrt{\tau}} \left[\ln \frac{s}{K} + \left(r - \frac{1}{2}\sigma^2 \right) \tau \right].$$
 (8.4.47)

Using (8.3.20), we derive from (8.4.44) and (8.4.45) the *put-call parity* relationship

$$c(t,s) - p(t,s) = s \left[N(d_1(\tau,s)) + N(-d_1(\tau,s)) \right] -e^{-r\tau} K \left(N(d_2(\tau,s)) + N(-d_2(\tau,s)) \right] = s - e^{-r\tau} K.$$
(8.4.48)

8.5 Black-Scholes Partial Differential Equation

{S8.5}

The Black-Scholes functions c(t, s) and p(t, s) satisfy a partial differential equation that plays an important role in the construction of replicating portfolios. In this section we begin with the backward recursion developed for the binomial model and pass to the limit to derive the partial differential equation satisfied by both the call and the put formulas.

8.5.1 Fixing a time t

Consider a time interval $[0, \tau]$, which we divide into N periods, each of length $\frac{\tau}{N}$. The annual interest rate is r and the stock volatility is σ . The parameters in this N-period model depend on N, and we use a subscript N to remind us of this fact. The up and down factors are

$$u_N = 1 + \frac{r\tau}{N} + \frac{\sigma\sqrt{\tau}}{\sqrt{N}}, \quad d_N = 1 + \frac{r\tau}{N} - \frac{\sigma\sqrt{\tau}}{\sqrt{N}}.$$

We set

$$r_N = \frac{r\tau}{N}.$$

The risk-neutral probabilities are

$$\tilde{p} = \frac{1+r_N - d_N}{u_N - d_N} = \frac{1}{2}, \quad \tilde{q} = \frac{u_N - 1 - r_N}{u_N - d_N} = \frac{1}{2}.$$

When we divide the time interval $[0, \tau]$ into N periods, we need to be clear about what we mean by "time." If $\tau = \frac{1}{4}$, then in a 50-period model, the beginning of period zero will be time 0 and the beginning of period 25 will at time $\frac{1}{8}$. In a 100-period model with $\tau = \frac{1}{4}$, time $\frac{1}{8}$ is at the beginning of period 50. We shall speak of "the beginning of period 25" rather than "time 25." When we say "time," we mean a number between 0 and τ , not a value of n in the binomial model. The backward recursion for the N-period model is

$$v_{n,N}(s) = \frac{1}{1+r_N} \left[\frac{1}{2} v_{n+1,N}(u_N s) + \frac{1}{2} v_{n+1,N}(d_N s) \right].$$
(8.5.1) {8.5.1}

Here we denote by $v_{n,N}$ the value of an option in the N-period model at the beginning of period n. We denote by $v_{N,N}(s)$ the payoff of the option at time N. For a call, $v_{N,N}(s) = (s - K)^+$ if the stock price at time τ is s.

We fix a time t and consider the value of an option at this time in a sequence of binomial models. We do that by letting both n and N go to infinity in such a way that $t = \frac{n\tau}{N}$ remains fixed. For example, if $\tau = \frac{1}{4}$ and we want to consider the time $t = \frac{1}{8}$, in the 50 period model we choose n = 25 and in the 100 period model we choose n = 50. In general, we would always take $n = \frac{N}{2}$ and let both n and N goes to infinity. With $\tau = \frac{1}{4}$ and $t = \frac{1}{8}$ we would not consider odd values of N, since there is no period beginning at time $t = \frac{1}{8}$ when we divide $[0, \frac{1}{4}]$ into an odd number of periods.

We denote by v(t,s) the limit of $v_{n,N}(s)$ as $N \to \infty$ where we also let $n \to \infty$ so that the ratio $\frac{n\tau}{N}$ is always equal to t. If we are considering a call, then v(t,s) will be the Black-Scholes call price c(t,s) given by (8.4.44); if we are considering a put, then v(t,s) will be the p(t,s) given by (8.4.45).

When $t = \frac{n\tau}{N}$, then period n + 1 begins at time $t + \frac{\tau}{N}$. We replace $v_{n,N}(s)$ in (8.5.1) by the limit v(t,s), replace $v_{n+1,N}(u_N s)$ by $v(t + \frac{\tau}{N}, u_N s)$, and replace $v_{n+1,N}(d_N s)$ by $v(t + \frac{\tau}{N}, d_N s)$. We are replacing one set of terms by other terms that are only approximately the same. This introduces an error, and the size of that error depends on the value of N. In fact, the error is not larger than a constant times $\frac{1}{N\sqrt{N}}$, so we say the error is $O(\frac{1}{N\sqrt{N}})$. It is difficult to show that this is the size of the error, and we do not do so.

8.5.2 Expansion of backward recursion formula

We write (8.5.1) as

$$v_{n+1,N}(u_N s) + v_{n+1,N}(d_N s) - 2(1+r_N)v_{n,N}(s) = 0$$

and make the replacements just described. Because of the error introduced, we no longer have 0 on the right-hand side of the equation, but instead have $O(\frac{1}{N\sqrt{N}})$. Indeed, we have

$$\{8.5.2\} \quad v\left(t + \frac{\tau}{N}, u_N s\right) + v\left(t + \frac{\tau}{N}, d_N s\right) - 2(1 + r_N)v(t, s) = O\left(\frac{1}{N\sqrt{N}}\right). \quad (8.5.2)$$

8.5. BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION 25

We rearrange the left-hand side of (8.5.2) to obtain

$$\left[v\left(t+\frac{\tau}{N},u_Ns\right)-v(t,u_Ns)\right] + \left[v(t,u_Ns)-v(t,s)\right] + \left[v\left(t+\frac{\tau}{N},d_Ns\right)-v(t,d_Ns)\right] + \left[v(t,d_Ns)-v(t,s)\right]$$

$$-2r_Nv(t,s) = O\left(\frac{1}{N\sqrt{N}}\right).$$
(8.5.3)

We now apply Taylor's Theorem. In (8.5.4) and (8.5.5) below, we hold the second argument of v fixed and apply Taylor's Theorem to the variable t. To simplify notation, we denote $\frac{\partial v}{\partial t}$ by v_t . In (8.5.6) and (8.5.7), we hold the first argument of v fixed and apply Taylor's Theorem to the variable s. In this case we denote $\frac{\partial v}{\partial s}$ by v_s and $\frac{\partial^2 v}{\partial s^2}$ by v_{ss} . We have the four expansions

$$v\left(t+\frac{\tau}{N},u_Ns\right) = v(t,u_Ns) + v_t(t,u_Ns)\frac{\tau}{N} + O\left(\frac{1}{N^2}\right), \qquad (8.5.4)$$

$$v\left(t+\frac{\tau}{N},d_Ns\right) = v(t,d_Ns) + v_t(t,d_Ns)\frac{\tau}{N} + O\left(\frac{1}{N^2}\right), \qquad (8.5.5)$$

$$v(t, u_N s) = v(t, s) + v_s(t, s)(u_N - 1)s + \frac{1}{2}v_{ss}(t, s)(u_N - 1)^2 s^2 + O((u_N - 1)^3 s^3), \qquad (8.5.6)$$

$$v(t, d_N s) = v(t, s) + v_s(t, s)(d_N - 1)s + \frac{1}{2}v_{ss}(t, s)(d_N - 1)^2 s^2 + O((d_N - 1)^3 s^3).$$
(8.5.7)

Because $u_N - 1 = O(\frac{1}{\sqrt{N}})$ and $d_N - 1 = O(\frac{1}{\sqrt{N}})$, the $O(\cdot)$ terms appearing on the right-hand side of (8.5.6) and (8.5.7) are $O(\frac{1}{N\sqrt{N}})$. Furthermore,

$$(u_N - 1)^2 = \frac{\sigma^2 \tau}{N} + O\left(\frac{1}{N\sqrt{N}}\right) + O\left(\frac{1}{N^2}\right).$$

But here is no need to write $O(\frac{1}{N^2})$ in an equation containing $O\left(\frac{1}{N\sqrt{N}}\right)$, because anything that is bounded by a constant times $\frac{1}{N^2}$ is also bounded by a constant times $O(\frac{1}{N\sqrt{N}})$. Thus,

$$(u_N - 1)^2 = \frac{\sigma^2 \tau}{N} + O\left(\frac{1}{N\sqrt{N}}\right), \qquad (8.5.8) \quad \{8.5.8\}$$

and similary

$$(d_N - 1)^2 = \frac{\sigma^2 \tau}{N} + O\left(\frac{1}{N\sqrt{N}}\right)$$
(8.5.9) {8.5.9}

Using (8.5.4), (8.5.6), (8.5.5) and (8.5.7), in that order, we see that the first four terms on the left-hand side of (8.5.3) are

$$v\left(t+\frac{\tau}{N},u_Ns\right) - v(t,u_Ns) = \frac{\tau}{N}v_t(t,u_Ns) + O\left(\frac{1}{N^2},\right)$$

$$(8.5.10)$$

$$v(t, u_N s) - v(t, s) = \left(\frac{r\tau}{N} + \frac{\sigma\sqrt{\tau}}{\sqrt{N}}\right) sv_s(t, s) + \frac{\sigma^2 \tau s^2}{2N} v_{ss}(t, s) + O\left(\frac{1}{N\sqrt{N}}\right), \qquad (8.5.11)$$

$$v\left(t+\frac{\tau}{N},d_Ns\right) - v(t,d_Ns) = \frac{\tau}{N}v_t(t,d_Ns) + O\left(\frac{1}{N^2}\right), \qquad (8.5.12)$$

$$v(t, d_N s) - v(t, s) = \left(\frac{r\tau}{N} - \frac{\sigma\sqrt{\tau}}{\sqrt{N}}\right) sv_s(t, s) + \frac{\sigma^2 \tau s^2}{2N} v_{ss}(t, s)$$
$$O\left(\frac{1}{N\sqrt{N}}\right).$$
(8.5.13)

We substitute these expressions into (8.5.3), and again do not write any $O(\frac{1}{N^2})$ terms because an $O(\frac{1}{N\sqrt{N}})$ term is present. This yields

$$\frac{\tau}{N}v_t(t,u_Ns) + \frac{\tau}{N}v_t(t,d_ns) + \frac{2r\tau}{N}sv_s(t,s) + \frac{\sigma^2\tau s^2}{N}v_{ss}(t,s) - \frac{2r\tau}{N}v(t,s)$$
$$= O\left(\frac{1}{N\sqrt{N}}\right). \quad (8.5.14)$$

We multiply (8.5.14) by $\frac{N}{2\tau}$ to obtain the expanded backward recursion formula

$$\frac{v_t(t, u_N s) + v_t(t, d_N s)}{2} + rsv_s(t, s) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t, s) - rv(t, s) = O\left(\frac{1}{\sqrt{N}}\right).$$
(8.5.15)

$\{8.5.11\}$

8.5.3 Black-Scholes partial differential equation

Equation (8.5.15) is a Taylor expansion of the binomial backward recursion formula (8.5.1). As a last step in this discussion, we let $N \to \infty$. We observe that $\lim_{N\to\infty} u_N = 1$ and $\lim_{N\to\infty} d_N = 1$, so

$$\lim_{N \to \infty} \frac{v_t(t, u_N s) + v_t(t, d_N s)}{2} = \frac{v_t(t, s) + v_t(t, s)}{2} = v_t(t, s).$$

As $N \to \infty$, the right-hand side of (8.5.15) converges to zero. Thus, taking the limit in (8.5.15), we obtain the *Black-Scholes partial differential equation*

$$\{8.5.12\} rv(t,s) = v_t(t,s) + rsv_s(t,s) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t,s). (8.5.16)$$

Equation (8.5.16) is satisfied by both c(t,s) given by (8.4.44) and p(t,s) given by (8.4.45).

8.5.4 Delta hedging

In the binomial model, we can replicate an option by trading in the underlying stock and borrowing or investing in the money market account as necessary to finance this. If the price of a share of stock at the beginning of period n is s, then the number of shares of the stock held by the replicating portfolio from the beginning to the end of period n is

$$\Delta_n = \frac{v_{n+1,N}(u_N s) - v_{n+1,N}(d_N s)}{(u_N - d_N)s}.$$
(8.5.17) {8.5.13}

For large values of N, it is possible to replace (8.5.17) by a simpler formula by determining the limit in (8.5.17) as $N \to \infty$.

Again we fix t and let N and n both approach infinity so that $t = \frac{n\tau}{N}$ remains constant. We first observe that

$$(u_N - d_N)s = \frac{2\sigma s\sqrt{\tau}}{\sqrt{N}}.$$
 (8.5.18) {8.5.14}

In (8.5.17) we substitute (8.5.18), we replace $v_{n+1,N}(u_N s)$ by $v(t + \frac{\tau}{N}, u_N s)$, and we replace $v_{n,N}(d_N s)$ by $v(t + \frac{\tau}{N}, d_N s)$. This introduces an error $O(\frac{1}{N\sqrt{N}})$, so that (8.5.17) becomes

$$\Delta(t,s) = \frac{\sqrt{N}}{2\sigma s\sqrt{\tau}} \left[v \left(t + \frac{\tau}{N}, u_N s \right) - v \left(t + \frac{\tau}{N}, d_N s \right) + O \left(\frac{1}{N\sqrt{N}} \right) \right].$$
(8.5.19) (8.5.15)

We call the expression in (8.5.19) $\Delta(t, s)$ rather than Δ_n because as $N \to \infty$ (and along with it, $n \to \infty$), this expression will depends on the time t and the underlying stock price s at that time but not on the period n.

Using (8.5.10) and (8.5.12) in the second equality below, we have

$$\Delta(t,s) = \frac{\sqrt{N}}{2\sigma s\sqrt{\tau}} \left[\left(v \left(t + \frac{\tau}{N}, u_N s \right) - v \left(t, u_N (s) \right) \right) + \left(v (t, u_N s) - v (t, d_N s) \right) \right] - \left(v \left(t + \frac{\tau}{N}, d_N s \right) - v (t, d_N s) \right) \right] + O \left(\frac{1}{N} \right) = \frac{\sqrt{N}}{2\sigma s\sqrt{\tau}} \left[\frac{\tau}{N} v_t (t, u_N s) + v (t, u_N s) - v (t, d_N s) - \frac{\tau}{N} v_t (t, d_N s) \right] + O \left(\frac{1}{N^2} \right) \right] + O \left(\frac{1}{N} \right) = \frac{\sqrt{N}}{2\sigma s\sqrt{\tau}} \left[v (t, u_N s) - v (t, d_N s) \right] + O \left(\frac{1}{\sqrt{N}} \right).$$
(8.5.20)

From (8.5.6), (8.5.7), (8.5.8), (8.5.9), and (8.5.18), we have

$$v(t, u_N s) - v(t, d_N s) = v_s(t, s)(u_N - d_N)s + O\left(\frac{1}{N}\right)$$
$$= \frac{2\sigma s\sqrt{\tau}}{\sqrt{N}}v_s(t, s) + O\left(\frac{1}{N}\right). \quad (8.5.21)$$

Substitution of (8.5.21) into (8.5.20) yiels

$$\{8.5.22\} \qquad \Delta(t,s) = v_s(t,s) + O\left(\frac{1}{\sqrt{N}}\right). \qquad (8.5.22)$$

Letting $N \to \infty$, we see that at each time t, if the stock price is s, the replicating portfolio should hold

$$\{8.5.23\} \qquad \Delta(t,s) = v_s(t,s). \qquad (8.5.23)$$

This is called the *delta* of the option. When we hold $\Delta(t, s)$ shares of the underlying stock at time t, we are said to be "delta hedging."

For a call option, whose price is given by (8.4.44), the delta of the option is

$$\{8.5.24\} \qquad c_s(t,s) = N(d_1(\tau,s)). \qquad (8.5.24)$$

For the put option, whose price is given by (8.4.45), the delta of the option is

$$\{8.5.25\} \qquad p_s(t,s) = N(d_1(\tau,s)) - 1 = -N(-d_1(\tau,s)). \qquad (8.5.25)$$