Exam #3 — Review

1. Consider a one-period model with $\Omega = \{\omega_1, \omega_1, \omega_1\}$ and $r = .25$. There is a single stock S that can be bought or sold at $t = 0$ for S_0 per share. There are two other traded securities in the model, a call option with strike price $K_c = 60$ and a put option with strike price $K_p = 40$. Both options have the same expiration date, $T = 1$. The options can be bought or sold at $t = 0$ for $C_0 = \$0.80$ and $P_0 = \$1.60$.

   The values of the securities at $t = 1$ are given in the following table:

Ω	$S_1(\omega)$	$C_1(\omega)$	$P_1(\omega)$
ω_1	65	5	0
ω_2	50	0	0
ω_3	35	0	5

   (a) Find a risk neutral probability measure for this model.

   (b) What is the initial value of the stock, S_0.

2. Consider the probability space $(\Omega, \mathbb{P})$ and the random variables X and Y described in the following table:

Ω	$\mathbb{P}$	X	Y
HH	5/12	1	1
HT	1/6	1	-1
TH	1/6	-1	1
TT	1/4	-1	-1

   (a) Are the random variables X and Y independent?

   (b) Recall that the conditional probability of event A given event B is $\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$. Compute the conditional probability $\mathbb{P}[X = 1|Y = 1]$.

   (c) Show that random variables V and W are independent if and only if

   $$\mathbb{P}[V = a|W = b] \;=\; \mathbb{P}[V = a]$$

   for any numbers a and a.

3. Consider a one-period binomial model in which the price of the stock at $t = 1$ may be $S_1(H) = 80$ or $S_1(T) = 20$. Let C denote a call option on the stock with strike price $K = 40$ and P a put option on the stock, also with strike price $K = 40$. The call and the put are trading at $t = 0$ at the initial prices $C_0 = 10$ and $P_0 = 10$. Given that there is no arbitrage, determine as much as you can about the values of S_0, r, $\tilde{p}$, and $\tilde{q}$.

4. Consider a single stock that pays no dividends and has current price S_0 per share. Let $T > 0$ and $K > 0$ be given and let P and C denote European put and call options on the stock with exercise date T and strike price K. There is also an ideal money market with constant effective rate R.

(a) Show, by constructing a portfolio and making a no-arbitrage argument that

$$P_0 + C_0 \leq S_0 + KD(T).$$

(b) Is this result still valid if P is an American option?

5. A coin with the characteristics ($\mathbb{P}(H) = \frac{2}{3}$, $\mathbb{P}(T) = \frac{1}{3}$) is tossed three times.

   (a) What is the probability of the event "At least two heads" occurring? What is the probability of the event "Exactly two heads" occurring?

   (b) Given that one of the three tosses comes up heads, what is the probability that exactly two of the tosses are heads?

   (c) Given that the first toss comes up heads, what is the probability that exactly two of the tosses are heads?

6. A certain family has decided to have children until there is one boy and one girl, or until there are three children.

   (a) Describe a sample space Ω and probability measure $\mathbb{P} : \Omega \to [0, 1]$ that is appropriate to this problem. Assume that any pregnancy results in a boy with probability 1/2 or a girl with probability 1/2.

   (b) Using the sample space and probability measure from (a), find the expected value and variance for the number of boys and the number of girls.

7. Suppose that $(\Omega, \mathbb{P})$ is a finite probability space, and X a random variable. Let $\mu = \mathbb{E}[X]$ and $\sigma^2 = \text{Var}(X)$. (Assume $\sigma > 0$).

   Let $Y = \frac{X-\mu}{\sigma}$. What are the expected value and variance of Y?

8. A stock, S, is available today (t=0) for $S_0 = \$80$ per share. In 6 months, the stock will pay a dividend equal to 1% of the share value at that time. A put option with strike price $K = \$75$ and expiration date $T = 1$ year can be bought or sold for $P_0 = \$3$. A bank is offering an effective interest rate of $R_*(1) = .02$ for borrowing or investment.

   What is price C_0 to buy or sell a call option on the stock with a strike price $K = \$75$ and expiration date $T = 1$ year?

9. Let $T > 0$, $K > 0$, and $\alpha \in (0, 1)$ be given and let S be a stock. Consider a (European-style) derivative security V that pays its holder the amount

$$V_T = \max(\alpha S_T, S_T - K)$$

   at time T. Determine constants β, γ, and $\hat{K}$ such that the arbitrage-free price V_0 of this security is given by

$$V_0 = \beta S_0 + \gamma \hat{C}_0$$

   where $\hat{C}$ is a European call option on S with expiration T and strike price $\hat{K}$. (The constants that you find may depend on α and K.)

10. Let S be a stock, and $\mathcal{F}$ the forward price for delivery of one share of this stock at time T. Let C be an American call option for one share of the stock, with strike price K and expiration date T.

    Show, by considering an appropriate portfolio, that

$$C_0 \geq (\mathcal{F} - K)D(T)$$

Exam #3 Formula Sheet

Forward exchange rate (forward price in units of A for a unit of B delivered at T)

$$F_A^B = E_A^B \frac{D^B(T)}{D^A(T)}$$

Forward price for delivery of a fixed-income security at time T_j

$$\mathcal{F} = \sum_{i=j+1}^{N} \frac{F_i D(T_i)}{D(T_j)}$$

Forward price for delivery of a share of stock at time T (no dividends)

$$\mathcal{F} = \frac{S_0}{D(T)} = S_0(1 + R_*(T))^T$$

Forward price for delivery of a share of stock at time T (known dividends)

$$\mathcal{F} = \left(S_0 - \sum_{i=1}^{N} D(\tau_i) d_{\tau_i} \right) (1 + R_*(T))^T$$

Forward price for delivery of a share of stock at time T (known dividend yield)

$$\mathcal{F} = (1 - \alpha)^N S_0(1 + R_*(T))^T$$

Put-call Parity

$$P_0 - C_0 = D(T)(K - \mathcal{F})$$

Formula used to derive put-call parity

$$(x - y)^+ - (y - x)^+ = x - y$$