## 21 - 260

## **Differential Equations**

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## Problems

1. Consider the first order differential equation

$$\frac{dy}{dt} = \frac{t}{2y+1}$$

- (a) Find the general solution.
- (b) Find the particular solution with  $y(1) = \frac{3}{2}$ .
- (c) Is this the unique solution with  $y(1) = \frac{3}{2}$ ? How can you tell?
- 2. (a) Solve the initial value problem

$$xy' + 3y = x^3, \qquad y(1) = 10$$

- (b) What can you say about the existence and uniqueness of a solution to the above differential equation with initial condition y(3) = 2? What about y(0) = 1?
- 3. (a) Use isoclines to draw the slope field for the differential equation

$$y' = \frac{1}{4}x^2 + y^2 - 1.$$

Sketch the solution curve passing through the point (0,0).

- (b) Sketch the phase line for the differential equation  $y' = (y 1)(y + 1)(y + 2)^2$  Label each critical point as stable, semi-stable or unstable.
- 4. The diagram below shows a system of three tanks containing salt solutions. The first tank holds  $100\ell$  of solution, the second  $100\ell$  and the third initially holds  $50\ell$  of solution. Tank 1 initially contains 25g of salt in solution, Tank 2 contains 10g initially, and Tank 3 begins with 50g. Pure water is added to the first tank at a rate of  $5\ell$  per minute. The rates of flow through each pipe are shown on the diagram. Let  $x_j(t)$  be the ammount of salt in tank j at time t.
  - (a) Write a system of equations that describes  $x_1$ ,  $x_2$  and  $x_3$ .
  - (b) Verify that  $x_1(t) = 25e^{-t/20}$ .
  - (c) Find the solution for  $x_2(t)$ .
- 5. Consider the differential equation

$$\frac{dy}{dx} = (y-1)(y+2)^2(y^2+1).$$

(a) Draw the phase line (phase diagram) for this differential equation.

- (b) Let y(t) be the solution satisfying the initial condition y(0) = 0. Can the value of y(t) ever be less than -2? Why or why not?
- 6. Find the general solution for the differential equation

$$\frac{dy}{dx} = \frac{x(y^2+1)}{2y}$$

7. (a) Consider the second order, linear differential equation

$$y'' - 2y' - 8y = 0.$$

Find the general solution, and the particular solution satisfying y(0) = 0 and y'(0) = 0.

(b) The equation

$$x'' + 4x = F\sin(\omega t)$$

models a forced, undamped mass-spring system. Find the general solution for this differential equation. (Assume that  $\omega \neq 2$ .)

- (c) What can you say about the behavior of the solutions to the differential equation in (7b) when  $\omega = 2$ ? (You do *not* need to find any solutions, merely describe their behavior.)
- 8. Compute the Laplace transform of the function

$$g(t) = \begin{cases} e^{2t} & 0 \le t \le 3\\ e^{-3t} & 3 < t < 5\\ 0 & t \ge 5 \end{cases}$$

9. Consider the differential equation

$$\frac{dy}{dt} = y + f(t),$$

where f(t) is a positive, increasing function.

(a) We can find an approximate solution with initial condition y(1) = 1 using either Euler's Method or the Improved Euler's Method (a.k.a. the second order Runge-Kutta method).

Let  $t_0 = 1$  and  $y_0 = \tilde{y}_0 = 1$ . Let  $t_1 = t_0 + \Delta t$ . Let  $y_1$  be the Euler's Method approximation to  $y(t_1)$  and  $\tilde{y}_1$  the Improved Euler's Method approximation.

Which is greater  $y_1$  or  $\tilde{y}_1$ ? How did you determine the order?

- (b) This is a linear differential equation. If  $f(t) = e^t$ , find the exact solution satisfying the initial condition y(1) = 1.
- 10. (a) Find the general solution to the differential equation

$$\frac{dy}{dt} = y^2 \sin(t)$$

(b) Find an integrating factor for the linear equation

$$\frac{dy}{dt} + 2ty = 2t$$

and use it to find the general solution.

11. Solve the initial value problem

$$y'' + 7y' + 12y = 0$$
  
 $y(0) = 1$   
 $y'(0) = -1$ 

12. A vat initially (at time t = 0) contains 100  $\ell$  of pure water. An iodine solution with a concentration of  $e^{-t/20}g/\ell$  is added to the vat at a rate of 3  $\ell/\min$ . At the same time the resulting solution is drained from the vat, also at a rate of 3  $\ell/\min$ . Throughout this process the solution in the vat is kept well mixed.

Find an expression for the amount of iodine in the vat at time t.

13. Consider the differential equation

$$\frac{dy}{dt} = (y+1)(y-1)^2(y-4) = f(y).$$

- (a) Draw the Phase Line for this equation. (It may help to first sketch the graph of f(y) vs. y.) Be sure to label and classify each equilibrium point.
- (b) Let  $y_1(t)$  be a solution satisfying  $y_1(0) = 2$ . Can the value of  $y_1(t)$  ever be less then 0? Why or why not?
- 14. Consider the initial value problem

$$\frac{dy}{dt} = 2y$$
$$y(0) = 1$$

- (a) Verify that  $y(t) = e^{2t}$  is a solution to the initial value problem.
- (b) Use Euler's method with  $\Delta t = \frac{1}{n}$  to compute an approximate solution satisfying the initial condition  $(t_0, y_0) = (0, 1)$ : What are the values of  $y_1, y_2$  and  $y_3$ ? (The result will be in terms of n.) Find an expression for  $y_k$ .
- (c) The approximation to y(1) is given by  $y_n$  (since  $t_n = n\frac{1}{n} = 1$ ). What is this approximation. How does the approximation compare to the exact value  $y(1) = e^2$  as  $\Delta t \to 0$  (i.e. as  $n \to \infty$ )? [Note: It may help to know that  $\lim_{n\to\infty} (1 + \frac{a}{n})^n = e^a$ .]
- 15. Consider the first order differential equation

$$\frac{dy}{dt} + \sin(t)y = \sin(t)$$

(a) Find an integrating factor for the equation.

- (b) Find the general solution.
- (c) Find the particular solution with  $y(\frac{\pi}{2}) = 0$ .
- 16. Solve the initial value problem

$$\frac{dy}{dt} = 3 - 2y$$
$$y(0) = 0.$$

- 17. An impatient coffee drinker has placed his scalding hot (160°) coffee into a 40° refrigerator. After three minutes, the temperature has fallen to 140°. How long until the temperature of the coffee has fallen to a temperature that is safe to drink (120°). [You may want to use Newton's Law of Cooling:  $\frac{dT}{dt} = -k(T - T^*)$ . You may express your result in terms of natural logarithms of various numbers.]
- 18. A vat initially (at t = 0) holds 100gal of pure water. A salt solution is added to the vat at a rate of 2gal/min. The concentration of this solution (in oz/gal) varies and is given by  $c(t) = \frac{1}{2}t(50 t)$ . The (well mixed) solution is allowed to flow out of the vat at a rate of 4gal/min. Find an expression for the amount of salt in the vat as a function of t.
- 19. (a) Find the general solution to  $y' 2y = t^2 e^{2t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = y^2 e^x, \quad y(0) = 0.$$

$$t\frac{dy}{dt} = 2y + t^3\cos(t).$$

- (a) Find the general solution for this differential equation.
- (b) Find a value for b, such that the solution  $\phi(t)$  satisfying  $\phi(\frac{\pi}{2}) = b$  is positive when  $t \neq 0$ .
- (c) Is there a solution satisfying y(0) = 1? If so is it unique? Is there a solution satisfying y(0) = 0? If so is it unique? Should you expect these solutions to exist, or be unique? Explain why or why not.
- 21. Consider the following situation:
  - A vat initially holds 100  $\ell$  of pure water.
  - A salt solution is added to the vat at a rate of 2  $\ell$ /min. The concentration of salt is 10 g/ $\ell$ .
  - The vat is drained off at a rate of  $3 \ell/\min$ .
  - (a) What is the volume of water (salt solution) in the vat at time t. At waht time does the vat empty?

- (b) Find a differential equation for the ammount of salt in the vat at time t.
- (c) Solve the differential equation in (39b) and find the amount of salt in the vat at time t.
- (d) What is the *concentration* of salt in the vat at time t? What is the concentration of salt in the last drop of water to leave the vat?
- 22. The size of a certain population of gazelles is modeled by the differential equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right) - h$$

where the term h remresents a constant rate of hunting.

- (a) Draw the phase line for the case h = 0 (i.e. no hunting). Identify all the equilibrium points and determine whether they are stable, unstable or semi-stable.
- (b) Draw a phase line that corresponds to a *small* (but non-zero) rate of hunting. How is it different from the phase line in part (40a)?
- (c) Is there a level of hunting h that ensures the eventual demise of the gazell population? If so, find this level. If not, explain why not.
- 23. (a) Find the general solution to  $y' 2y = t^2 e^{2t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = y^2 e^x, \quad y(0) = 0.$$

24. Consider the differential equation

$$t\frac{dy}{dt} = 2y + t^3\cos(t).$$

- (a) Find the general solution for this differential equation.
- (b) Find a value for b, such that the solution  $\phi(t)$  satisfying  $\phi(\frac{\pi}{2}) = b$  is positive when  $t \neq 0$ .
- (c) Is there a solution satisfying y(0) = 1? If so is it unique? Is there a solution satisfying y(0) = 0? If so is it unique? Should you expect these solutions to exist, or be unique? Explain why or why not.
- 25. Consider the following situation:
  - A vat initially holds 100  $\ell$  of pure water.
  - A salt solution is added to the vat at a rate of 2 ℓ/min. The concentration of salt is 10 g/ℓ.
  - The vat is drained off at a rate of  $3 \ell/\min$ .

- (a) What is the volume of water (salt solution) in the vat at time t. At waht time does the vat empty?
- (b) Find a differential equation for the ammount of salt in the vat at time t.
- (c) Solve the differential equation in (39b) and find the amount of salt in the vat at time t.
- (d) What is the *concentration* of salt in the vat at time t? What is the concentration of salt in the last drop of water to leave the vat?
- 26. The size of a certain population of gazelles is modeled by the differential equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right) - h$$

where the term h remresents a constant rate of hunting.

- (a) Draw the phase line for the case h = 0 (i.e. no hunting). Identify all the equilibrium points and determine whether they are stable, unstable or semi-stable.
- (b) Draw a phase line that corresponds to a *small* (but non-zero) rate of hunting. How is it different from the phase line in part (40a)?
- (c) Is there a level of hunting h that ensures the eventual demise of the gazell population? If so, find this level. If not, explain why not.
- 27. (a) Find the general solution to  $y' + y = e^{-t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = \frac{-\sin(x)}{2y}, \quad y(0) = \frac{1}{\sqrt{2}}.$$

What is the domain of definition for this solution?

28. A certain population of animals can be modeled by the differential equation

$$\frac{dP}{dt} = kP\left(\frac{P}{m} - 1\right)\left(1 - \frac{P}{M}\right).$$

- (a) Find all equilibrium points for this differential equation and determine whether each is stable, unstable or semi-stable.
- (b) Draw the phase line for this (autonomous) differential equation and sketch the graphs of some solution curves.
- (c) Give an interpretation of m and M in terms of the behavior of the population.

29. Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2}.$$

- (a) Find an analytic solution y(t) for the initial value problem.
- (b) Using  $\Delta t = 1$ , compute the Euler's method (approximate) solution for  $0 \le t \le 2$ . (i.e. compute  $(t_1, y_1)$  and  $(t_2, y_2)$ .)
- (c) What is the error in the Euler's method approximation  $y_2$ , i.e. difference between  $y_2$  (corresponding to  $t_2 = 2$ ) and  $\lim_{t\to 2^-} y(t)$ .
- 30. Consider the differential equation

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1-t}{t^2}e^{-1/t}$$

- (a) Find the general solution. [Hint: It may help to know that  $\frac{d}{dt} \left(\frac{1}{t}e^{-1/t}\right) = \frac{1-t}{t^3}e^{-1/t}$ .]
- (b) Find the solution  $\phi(t)$  that satisfies the initial condition  $\phi(1) = \frac{1}{e}$ . What is  $\lim_{t\to\infty} \phi(t)$ ?
- (c) Find the solution satisfying y(1) = a
- (d) What is the largest value of a that results in a solution bounded above by 10 when t > 0?
- 31. Consider the differential equation

$$\frac{dy}{dt} = -\frac{1}{t}y + \frac{1}{t^3}$$

- (a) Show that  $y_1(t) = \frac{1}{t} \frac{1}{t^2}$  is a solution satisfying  $y_1(2) = 1/4$ .
- (b) Suppose  $t_0 > 1$  and that  $y_2(t)$  is a solution satisfying  $y_2(t_0) = 0$ . Show that  $y'(t_0) > 0$ .
- (c) Let  $y_3(t)$  be a solution satisfying  $y_3(2) = \frac{1}{8}$ . Why can't the graph of  $y_3(t)$  cross the *t*-axis at a point with t > 2? Why can't the graph of  $y_3(t)$  cross the graph of  $y_1(t)$  at a point with t > 2? What is  $\lim_{t\to\infty} y_3(t)$ ?
- 32. (a) Find the general solution to  $y' + 2y = te^{-2t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}, \quad y(0) = -\frac{1}{\sqrt{2}}.$$

33. A certain population of animals can be modeled by the differential equation

$$\frac{dP}{dt} = kP\left(\frac{P}{m} - 1\right)\left(1 - \frac{P}{M}\right).$$

- (a) Find all equilibrium points for this differential equation and determine whether each is stable, unstable or semi-stable.
- (b) Draw the phase line for this (autonomous) differential equation and sketch the graphs of some solution curves.
- (c) Give an interpretation of m and M in terms of the behavior of the population.
- 34. Consider the initial value problem

$$\frac{dy}{dt} = t - \frac{y}{t}, \quad y(1) = 0.$$

- (a) Find an analytic solution y(t) for the initial value problem.
- (b) Using  $\Delta t = 1$ , compute the Euler's method (approximate) solution for  $1 \le t \le 3$ . (i.e. compute  $(t_1, y_1)$  and  $(t_2, y_2)$ .)
- (c) What is the error in the Euler's method approximation  $y_2$ , i.e. difference between  $y_2$  (corresponding to  $t_2 = 3$ ) and y(3).
- 35. Consider the differential equation

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1-t}{t^2}e^{-1/t}$$

- (a) Find the general solution. [Hint: It may help to know that  $\frac{d}{dt} \left(\frac{1}{t}e^{-1/t}\right) = \frac{1-t}{t^3}e^{-1/t}$ .]
- (b) Find the solution  $\phi(t)$  that satisfies the initial condition  $\phi(1) = \frac{1}{e}$ . What is  $\lim_{t\to\infty} \phi(t)$ ?
- (c) Find the solution satisfying y(1) = a
- (d) What is the largest value of a that results in a solution bounded above by 10 when t > 0?
- 36. Consider the initial value problem

$$\frac{dy}{dt} = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1.$$

- (a) Show that  $y_1(t) = 1 t$  and  $y_2(t) = -t^2/4$  are solutions to the initial value problem.
- (b) What are the domains of definition for each of these solutions?
- (c) Does the existence of these two solutions contradict the theorems of existence and uniqueness? Why or why not?
- 37. (a) Find the general solution to  $y' 2y = t^2 e^{2t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = y^2 e^x, \quad y(0) = 0.$$

$$t\frac{dy}{dt} = 2y + t^3\cos(t).$$

- (a) Find the general solution for this differential equation.
- (b) Find a value for b, such that the solution  $\phi(t)$  satisfying  $\phi(\frac{\pi}{2}) = b$  is positive when  $t \neq 0$ .
- (c) Is there a solution satisfying y(0) = 1? If so is it unique? Is there a solution satisfying y(0) = 0? If so is it unique? Should you expect these solutions to exist, or be unique? Explain why or why not.
- 39. Consider the following situation:
  - A vat initially holds 100  $\ell$  of pure water.
  - A salt solution is added to the vat at a rate of 2 ℓ/min. The concentration of salt is 10 g/ℓ.
  - The vat is drained off at a rate of  $3 \ell/\min$ .
  - (a) What is the volume of water (salt solution) in the vat at time t. At waht time does the vat empty?
  - (b) Find a differential equation for the ammount of salt in the vat at time t.
  - (c) Solve the differential equation in (39b) and find the amount of salt in the vat at time t.
  - (d) What is the *concentration* of salt in the vat at time t? What is the concentration of salt in the last drop of water to leave the vat?
- 40. The size of a certain population of gazelles is modeled by the differential equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{4}\right) - h$$

where the term h remresents a constant rate of hunting.

- (a) Draw the phase line for the case h = 0 (i.e. no hunting). Identify all the equilibrium points and determine whether they are stable, unstable or semi-stable.
- (b) Draw a phase line that corresponds to a *small* (but non-zero) rate of hunting. How is it different from the phase line in part (40a)?
- (c) Is there a level of hunting *h* that ensures the eventual demise of the gazell population? If so, find this level. If not, explain why not.
- 41. (a) Suppose that  $y = \phi(t)$  is a solution to the initial value problem

$$\frac{dy}{dt} = 2t - y^2$$
$$y(0) = 0.$$

Use Euler's method with a step size of  $\Delta t = 1$  to find an approximate value for  $\phi(4)$ 

(b) Consider the linear system of algebraic equations

Find the augmented matrix for this linear system, and find the solution to the system using elementary row operations.

42. Consider the differential equation y' = f(y), where

$$f(y) = \frac{1}{6}y(y-4)(y^2 - 4y + 4 + \frac{1}{6}).$$

- (a) Sketch a graph of f(y) vs y, and draw the phase line for this differential equation. Identify the equilibrium points and determine their stability. [Hint:  $y^2 - 4y + 4 + \frac{1}{6}$  is always positive.]
- (b) A direction field for this equation is shown below. Use it to sketch a graph of the solution satisfying y(0) = 3.5.



- (c) What qualitative information about the solution in (b) is evident from the direction field, but not from the phase line?
- 43. For some reason, I put a brick in my freezer. When I closed the freezer door, the brick was 70° and the temperature in the freezer was 20°. Ten minutes later, the temperature of the brick had fallen to 60°. At what time will the temperature of the brick be 30°? [You may wish to recall Newton's Law of cooling,  $\frac{du}{dt} = -k(u-T)$ . You may leave your answer in terms of  $\ln(\frac{1}{5})$  and  $\ln(\frac{4}{5})$ .]
- 44. Consider the initial value problem

$$\frac{dy}{dx} = -2xy^2$$
$$y(-1) = a, \text{ where } a \neq 0.$$

- (a) Find an explicit solution to the initial value problem (you answer will involve the parameter a).
- (b) Determine the domain of this solution for each of the values  $a_1 = \frac{1}{2}$  and  $a_2 = \frac{4}{3}$  for the parameter a.
- (c) Find a value  $a_0$  such that for  $a < a_0$  the domain of the solution is  $(-\infty, \infty)$ , but for  $a > a_0$ , the solution becomes unbounded in some finite amount of time.
- 45. (a) Find the general solution to  $y' + y = e^{-t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = \frac{-\sin(x)}{2y}, \quad y(0) = \frac{1}{\sqrt{2}}.$$

What is the domain of definition for this solution?

46. A certain population of animals can be modeled by the differential equation

$$\frac{dP}{dt} = kP\left(\frac{P}{m} - 1\right)\left(1 - \frac{P}{M}\right).$$

- (a) Find all equilibrium points for this differential equation and determine whether each is stable, unstable or semi-stable.
- (b) Draw the phase line for this (autonomous) differential equation and sketch the graphs of some solution curves.
- (c) Give an interpretation of m and M in terms of the behavior of the population.
- 47. Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2}.$$

- (a) Find an analytic solution y(t) for the initial value problem.
- (b) Using  $\Delta t = 1$ , compute the Euler's method (approximate) solution for  $0 \le t \le 2$ . (i.e. compute  $(t_1, y_1)$  and  $(t_2, y_2)$ .)
- (c) What is the error in the Euler's method approximation  $y_2$ , i.e. difference between  $y_2$  (corresponding to  $t_2 = 2$ ) and  $\lim_{t\to 2^-} y(t)$ .
- 48. Consider the differential equation

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1-t}{t^2}e^{-1/t}$$

- (a) Find the general solution. [Hint: It may help to know that  $\frac{d}{dt}\left(\frac{1}{t}e^{-1/t}\right) = \frac{1-t}{t^3}e^{-1/t}$ .]
- (b) Find the solution  $\phi(t)$  that satisfies the initial condition  $\phi(1) = \frac{1}{e}$ . What is  $\lim_{t\to\infty} \phi(t)$ ?

- (c) Find the solution satisfying y(1) = a, for an arbitrary value of a.
- (d) What is the largest value of a that results in a solution bounded above by 10 when t > 0?
- 49. Consider the differential equation

$$y' + \frac{y}{t} = \frac{2}{t^2 - 4}$$

- (a) Find the general solution to this differential equation.
- (b) Why does the Existence and Uniqueness Theorem guarantee the existence of a solution satisfying the initial condition  $y(1) = \ln(3)$ ? On what interval is this solution guaranteed to be unique?
- (c) Find the solution satisfying  $y(1) = \ln(3)$ . What is the domain on which this solution is defined?
- 50. Consider the differential equation

$$\frac{dy}{dx} = 4x\sqrt{y}$$

- (a) Find all solutions to this differential equation.
- (b) Find the solution  $y = \phi(t)$  to this differential equation satisfying the initial condition  $\phi(-1) = 4$ . [Hint:  $\sqrt{y}$  means the positive square root.]
- (c) Use two iterations of Euler's method with a step size of  $\Delta x = \frac{1}{2}$  and initial condition of  $(x_0, y_0) = (-1, 4)$  to find  $(x_2, y_2)$ .
- 51. Consider the differential equation

$$\frac{dP}{dt} = \left(\frac{P}{300} - 1\right) \left(1 - \frac{P}{1000}\right)$$

which models the behavior of a certain population.

- (a) Draw the phase line for this differential equation. Indicate the stability of each equilibrium point. [Note: you must provide some justification for why your phase line looks as it does a graph or some calculations for example.]
- (b) Use information from the phase line to sketch some representative solution curves (i.e. curves in the tP-plane. Be sure to show each type of qualitative behavior.
- (c) What is the significance, in terms of the population dynamics, of the numbers 300 and 1000?
- 52. Consider the system of differential equations

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{\frac{te^{-t^2}}{2xy}}{-2ty + t}$$

- (a) Find the solution to the second equation satisfying y(0) = 1.
- (b) Given the solution in part (a), find the solution to the first equation satisfying  $x(0) = \sqrt{4 \ln(2)}$ .
- (c) If x(t) and y(t) are the solutions you found in parts (b) and (a) above, what is  $\lim_{t\to\infty} (x(t), y(t))$ ? Is this point an equilibrium point of the system?
- 53. (a) Find the general solution to  $y' + y = e^{-t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = \frac{-\sin(x)}{2y}, \quad y(0) = \frac{1}{\sqrt{2}}.$$

What is the domain of definition for this solution?

54. A certain population of animals can be modeled by the differential equation

$$\frac{dP}{dt} = kP\left(\frac{P}{m} - 1\right)\left(1 - \frac{P}{M}\right).$$

- (a) Find all equilibrium points for this differential equation and determine whether each is stable, unstable or semi-stable.
- (b) Draw the phase line for this (autonomous) differential equation and sketch the graphs of some solution curves.
- (c) Give an interpretation of m and M in terms of the behavior of the population.
- 55. Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2}.$$

- (a) Find an analytic solution y(t) for the initial value problem.
- (b) Using  $\Delta t = 1$ , compute the Euler's method (approximate) solution for  $0 \le t \le 2$ . (i.e. compute  $(t_1, y_1)$  and  $(t_2, y_2)$ .)
- (c) What is the error in the Euler's method approximation  $y_2$ , i.e. difference between  $y_2$  (corresponding to  $t_2 = 2$ ) and  $\lim_{t\to 2^-} y(t)$ .

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1-t}{t^2}e^{-1/t}$$

- (a) Find the general solution. [Hint: It may help to know that  $\frac{d}{dt} \left(\frac{1}{t}e^{-1/t}\right) = \frac{1-t}{t^3}e^{-1/t}$ .]
- (b) Find the solution  $\phi(t)$  that satisfies the initial condition  $\phi(1) = \frac{1}{e}$ . What is  $\lim_{t\to\infty} \phi(t)$ ?
- (c) Find the solution satisfying y(1) = a
- (d) What is the largest value of a that results in a solution bounded above by 10 when t > 0?
- 57. Consider the differential equation

$$\frac{dy}{dt} = -\frac{1}{t}y + \frac{1}{t^3}$$

- (a) Show that  $y_1(t) = \frac{1}{t} \frac{1}{t^2}$  is a solution satisfying  $y_1(2) = 1/4$ .
- (b) Suppose  $t_0 > 1$  and that  $y_2(t)$  is a solution satisfying  $y_2(t_0) = 0$ . Show that  $y'(t_0) > 0$ .
- (c) Let  $y_3(t)$  be a solution satisfying  $y_3(2) = \frac{1}{8}$ . Why can't the graph of  $y_3(t)$  cross the *t*-axis at a point with t > 2? Why can't the graph of  $y_3(t)$  cross the graph of  $y_1(t)$  at a point with t > 2? What is  $\lim_{t\to\infty} y_3(t)$ ?
- 58. Find all solutions to the following differential equations

(a) 
$$\frac{dy}{dt} = -\frac{4y}{t} + 8t^3 + 5$$
  
(b)  $\frac{dy}{dt} = t^3 \frac{y^2 - 1}{y}$ 

59. Solve the following initial value problems

(a) 
$$\frac{dy}{dx} = \frac{x-5}{y^2}, y(0) = 2.$$
  
(b)  $\frac{dy}{dx} = \frac{2y}{x} + x^2 \cos(x), y(\pi) = 1$ 

60. A certain population of muskrats can be modeled by the differential equation

$$\frac{dP}{dt} = \frac{5}{4}P\left(1 - \frac{P}{5}\right)$$

where P is the size of the population, measured in *thousands* of muskrats, and t is time, measured in months.

(a) Draw the phase line for this differential equation. Indicate the stability of each equilibrium point. [Note: you must provide some justification for why your phase line looks as it does — a graph or some calculations for example.]

- (b) A proposal has been made to "harvest" muskrats from the population at a rate of one thousand (1000) muskrats per month. How should the differential equation be modified ro reflect this change. Draw the phase line for the resulting differential equation.
- (c) Find a number  $h_0$  such that hunting at a rate greater than  $1000h_0$  muskrats per month causes populations of all sizes to decrease.
- 61. Consider the differential equation

$$\frac{dy}{dt} = 5y^{\frac{4}{5}}.$$

- (a) Find a solution to the equation satisfying the initial condition y(2) = 1.
- (b) Consider the following claim: There is an interval (a, b) with a < 2 < b on which the solution in part (a) is the unique solution satisfying the given initial condition. Is this claim justified? Why or why not?</li>
- (c) Describe all possible solutions to the equation satisfying the initial condition y(0) = 0.
- 62. (a) Find the general solution to  $y' + y = e^{-t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = \frac{-\sin(x)}{2y}, \quad y(0) = \frac{1}{\sqrt{2}}.$$

What is the domain of definition for this solution?

63. A certain population of animals can be modeled by the differential equation

$$\frac{dP}{dt} = kP\left(\frac{P}{m} - 1\right)\left(1 - \frac{P}{M}\right).$$

- (a) Find all equilibrium points for this differential equation and determine whether each is stable, unstable or semi-stable.
- (b) Draw the phase line for this (autonomous) differential equation and sketch the graphs of some solution curves.
- (c) Give an interpretation of m and M in terms of the behavior of the population.

64. Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2}.$$

- (a) Find an analytic solution y(t) for the initial value problem.
- (b) Using  $\Delta t = 1$ , compute the Euler's method (approximate) solution for  $0 \le t \le 2$ . (i.e. compute  $(t_1, y_1)$  and  $(t_2, y_2)$ .)
- 65. Consider the differential equation

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1-t}{t^2}e^{-1/t}$$

- (a) Find the general solution. [Hint: It may help to know that  $\frac{d}{dt} \left(\frac{1}{t}e^{-1/t}\right) = \frac{1-t}{t^3}e^{-1/t}$ .]
- (b) Find the solution  $\phi(t)$  that satisfies the initial condition  $\phi(1) = \frac{1}{e}$ . What is  $\lim_{t\to\infty} \phi(t)$ ?
- (c) Find the solution satisfying y(1) = a
- (d) What is the largest value of a that results in a solution bounded above by 10 when t > 0?
- 66. Consider the differential equation

$$\frac{dy}{dt} = -\frac{1}{t}y + \frac{1}{t^3}$$

- (a) Show that  $y_1(t) = \frac{1}{t} \frac{1}{t^2}$  is a solution satisfying  $y_1(2) = 1/4$ .
- (b) Suppose  $t_0 > 1$  and that  $y_2(t)$  is a solution satisfying  $y_2(t_0) = 0$ . Show that  $y'(t_0) > 0$ .
- (c) Let  $y_3(t)$  be a solution satisfying  $y_3(2) = \frac{1}{8}$ . Why can't the graph of  $y_3(t)$  cross the *t*-axis at a point with t > 2? Why can't the graph of  $y_3(t)$  cross the graph of  $y_1(t)$  at a point with t > 2? What is  $\lim_{t\to\infty} y_3(t)$ ?
- 67. Find all solutions to the following differential equations

(a) 
$$\frac{dy}{dt} = 3\sqrt{ty}$$
  
(b)  $2t\frac{dy}{dt} + y = 10\sqrt{t}$ 

68. Solve the initial value problem

- (a)  $\frac{dy}{dt} 2ty = t, y(0) = a.$
- (b) For what values of a do the solutions tend to  $\infty$  as  $t \to \infty$ ? For what values of a do the solutions tend to  $-\infty$  as  $t \to \infty$ ? Are there any solutions that do not tend to  $\pm \infty$  as  $t \to \infty$ ? If so what are they?

69. A fish hatchery has noticed that, left to itself, the size of the fish population in its pond is described by the differential equation

$$\frac{dP}{dt} = \frac{1}{5}P\left(1 - \frac{P}{10}\right)$$

where P is the size of the population, measured in hundreds of fish, and t is time, measured in months.

- (a) The hatchery would like to harvest fish from it's pond. If a fraction h of the population is to be harvested each month, i.e. hP fish are harvested, find a differential equation that models the resulting behavior of the population.
- (b) According to the model in part (a), what is the long term behavior of the fish population? Is there a maximum sustainable population? If so what is it? (You may assume h is a "small" number.) Is there a threshold size the population must exceed in order to survive? (Again, you may assume h is "small.")
- (c) Find the smallest number  $h_0$  such that hunting at a rate  $h > h_0$  causes populations of all sizes to decrease.
- 70. Consider the differential equation

$$\frac{dy}{dt} = f(y) = y(5-y).$$

- (a) Sketch a graph of f(y) vs y showing where f(y) is positive, negative and zero. Use information from this graph to draw the phase line for this differential equation. Indicate the stability of any equilibrium points.
- (b) Use Euler's method, with a step size of  $\Delta t = \frac{1}{2}$ , to compute a numerical solution satisfying the initial condition y(0) = 4. Compute  $y_1, y_2$ , and  $y_3$ .
- (c) In part (b), you should have found the approximation  $y(\frac{1}{2}) \approx y_1 = 6$ . I claim that  $y(\frac{1}{2}) \neq 6$ . Am I correct? Justify your answer.
- 71. Find all solutions to the differential equation

$$\frac{dy}{dx} = 3t^2(y-1)^2$$

72. Let  $\phi$  be a solution to the initial value problem

$$\frac{dy}{dt} = 2y - 3t^2, \quad y(0) = 1$$

Using Euler's method, with a step size of  $\Delta t = 1$ , find the approximation  $y_4 \approx \phi(4)$ .

73. Consider the initial value problem

$$y' - \frac{2}{(t+1)(t-1)}y = t - 1, y(0) = 0.$$

- (a) On what interval do the Existence and Uniqueness theorems guarantee the existence of a unique solution to the initial value problem?
- (b) Show that  $\mu(t) = \frac{t+1}{t-1}$  is an integrating factor for this equation.
- (c) Find an analytic expression for the solution to the initial value problem.
- 74. This problem concerns a population of animals where reproductive opportunities occur as a result of random encounters. This population is hunted with a constant effort. The assumptions for this model are
  - The population tends to grow at a rate proportional to the square of the size of the population.
  - Hunting tends to reduce the population at at rate proportional to its size.

Let P denote the size of the population, t denote time, k be a constant related to the rate of growth, and h be a constant related to the rate of hunting.

- (a) Write a differential equation to model the rate of growth of this population. How does your differential relate to the assumptions of the model?
- (b) What are the equilibrium points of your differential equations? For what sizes of the population does your model predict the population will increase? For what sizes does it predict a decreasing population? Draw the phase line for your differential equation.
- (c) Does your model predict a "carrying capacity," or maximum sustainable population? If so, what is it? Does it predict a "threshold" below which the population will not survive? If so, what is it?
- 75. Consider the differential equation

$$\frac{dy}{dt} = t^2 - y^2.$$

- (a) Sketch the direction field for this equation. You may use any technique we've discussed in class to do so. Your sketch should include the region  $-1 \le t \le 4$ ,  $-1 \le y \le 3$ .
- (b) I claim that the solution satisfying the initial condition  $y(1) = \frac{1}{2}$  also satisfies y(t) > 0 for t > 1. Am I right or wrong?
- (c) I claim that the solution satisfying the initial condition  $y(1) = \frac{1}{2}$  also satisfies y(t) < t for t > 1. Am I right or wrong?
- 76. Consider the differential equation

$$ty' + 2y = 1$$

- (a) Find the general solution to this equation.
- (b) Using Euler's method with  $\Delta t = 1$ , find a numerical approximation for the solution  $\phi$  satisfying  $\phi(1) = 1$ . What is  $y_3 \approx \phi(4)$ ?

77. Find all solutions to the differential equation

$$\frac{dy}{dt} = \frac{t(y^2 - 1)}{y}$$

78. A jug of milk is removed from a refrigerator and placed in a room where the thermostat is set at 70°. Initially the temperature of the milk is 38°. After on hour, the termserature of the milk is 54°. According to Newton's law of cooling,

$$\frac{du}{dt} = -k(u-T),$$

what will be the temperature of the milk after 4 hours?

79. Describe the pairs  $(t_0, y_0)$  for which the Existence and Uniqueness theorems do NOT guarantee the existence of a unique solution to the initial value problem

$$\frac{dy}{dt} = \frac{y}{y+t^2-1}, \quad y(t_0) = y_0.$$

80. Consider the differential equation

$$\frac{dy}{dt} = t^2 - y^2.$$

- (a) Sketch the direction field for this equation. You may use any technique we've discussed in class to do so. Your sketch should include the region  $-1 \le t \le 4$ ,  $-1 \le y \le 3$ .
- (b) I claim that the solution satisfying the initial condition  $y(1) = \frac{1}{2}$  also satisfies y(t) > 0 for t > 1. Am I right or wrong?
- (c) I claim that the solution satisfying the initial condition  $y(1) = \frac{1}{2}$  also satisfies y(t) < t for t > 1. Am I right or wrong?
- 81. (a) Find the general solution to  $y' + y = e^{-t}$ .
  - (b) Find a solution to the initial value problem

$$\frac{dy}{dx} = \frac{-\sin(x)}{2y}, \quad y(0) = \frac{1}{\sqrt{2}}.$$

What is the domain of definition for this solution?

82. A certain population of animals can be modeled by the differential equation

$$\frac{dP}{dt} = kP\left(\frac{P}{m} - 1\right)\left(1 - \frac{P}{M}\right).$$

- (a) Find all equilibrium points for this differential equation and determine whether each is stable, unstable or semi-stable.
- (b) Draw the phase line for this (autonomous) differential equation and sketch the graphs of some solution curves.
- (c) Give an interpretation of m and M in terms of the behavior of the population.
- 83. Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2}.$$

- (a) Find an analytic solution y(t) for the initial value problem.
- (b) Using  $\Delta t = 1$ , compute the Euler's method (approximate) solution for  $0 \le t \le 2$ . (i.e. compute  $(t_1, y_1)$  and  $(t_2, y_2)$ .)

$$\frac{dy}{dt} - \frac{1}{t}y = \frac{1-t}{t^2}e^{-1/t}$$

- (a) Find the general solution. [Hint: It may help to know that  $\frac{d}{dt} \left(\frac{1}{t}e^{-1/t}\right) = \frac{1-t}{t^3}e^{-1/t}$ .]
- (b) Find the solution  $\phi(t)$  that satisfies the initial condition  $\phi(1) = \frac{1}{e}$ . What is  $\lim_{t\to\infty} \phi(t)$ ?
- (c) Find the solution satisfying y(1) = a
- (d) What is the largest value of a that results in a solution bounded above by 10 when t > 0?
- 85. Consider the differential equation

$$\frac{dy}{dt} = -\frac{1}{t}y + \frac{1}{t^3}$$

- (a) Show that  $y_1(t) = \frac{1}{t} \frac{1}{t^2}$  is a solution satisfying  $y_1(2) = 1/4$ .
- (b) Suppose  $t_0 > 1$  and that  $y_2(t)$  is a solution satisfying  $y_2(t_0) = 0$ . Show that  $y'(t_0) > 0$ .
- (c) Let  $y_3(t)$  be a solution satisfying  $y_3(2) = \frac{1}{8}$ . Why can't the graph of  $y_3(t)$  cross the *t*-axis at a point with t > 2? Why can't the graph of  $y_3(t)$  cross the graph of  $y_1(t)$  at a point with t > 2? What is  $\lim_{t\to\infty} y_3(t)$ ?