21-260 **Differential Equations** **D. Handron**

Final Exam Review

1. (a) Find a solution to the initial value problem

$$ty' + 3y = 2t - t^2 - t + 2; \quad y(1) = \frac{1}{2}.$$

What is the domain of this solution?

(b) Find all possible solutions to the differential equation

$$\frac{dy}{dx} = \frac{xy^2 - x}{y}$$

2. Consider the differential equation

$$\frac{dy}{dx} = xy + \frac{y}{x}$$

(a) Find the isoclines for this equation and use them to help sketch the direction field.

(b) On a separate set of axes (you can redraw the direction field) use the direction field to sketch several solutions to the differential equation.

(c) Discuss a connection between the existence and uniqueness theorems and the graphs of the solutions you have drawn.

3. A certain population of animals can be modeled by the differential equation

$$\frac{dP}{dt} = P(P-1)(10-P)$$

(a) Draw the phase line for this differential equation. Sketch a representative sample of solution curves in the ty-plane. Identify any equilibrium points and determine their stability.

(b) Suppose that hunting the animals at a rate of h per unit time is to be allowed. Assuming that the rate h is "small", how will the phase line change from that in part (a)? Sketch the modified phase line.

(c) What qualitative behaviors may be observed as the rate of hunting h changes continuously from "small" values to "large" values. [You may find it convenient to sketch some diagrams when answering this problem.]

4. Solve the initial value problem

$$y'' - 3y' + 2y = u(t-3) \cdot \cos(2t-6); \quad y(0) = 1, y'(0) = 2.$$

using the Laplace transform method

5. Consider the initial value problem

$$y'' + y = \sum_{k=1}^{15} \delta(t - (2k-1)\pi); \quad y(0) = 0, y'(0) = 0.$$

(a) Find the solution to the initial value problem.

(b) Sketch a graph of the solution on the interval $[0, 6\pi]$.

(c) What happens to the solution after the sequence of impulses ends at $t = 29\pi$?

6. Consider the system of differential equations

$$\begin{aligned} x' &= 7x - y \\ y' &= -2x + 8y \end{aligned}$$

(a) Find the solution to the system satisfying the initial conditions $x(0) = 1$, $y(0) = 0$.

(b) Find the x- and y-nullclines for the system. Use the nullclines and information from the analytic solution you found in (a) to sketch a phase portrait for the system.

(c) Make a mark next to each term that describes this system:

__ center	__ spiral	__ sink	__ node	__ saddle
__ stable	__ asymptotically stable	__ source	__ simplex	__ unstable

7. Consider the partial differential equation problem

$$\begin{gathered} u_x + u_{xt} + u_t = 0 \\ u(0,t) = 1 \\ u(x,0) = 1 \end{gathered}$$

(a) Using the technique of separation of variables, assume that $u(x,t) = X(x)T(t)$, and replace the partial differential equation $u_x + u_{xt} + u_t = 0$ with a pair of ordinary differential equations.

(b) Solve the equations in (a). What solutions to the original partial differential equation problem can be determined from these results?

8. A string with it's ends fixed at $x = 0$ and $x = 4$ is released with zero initial velocity and an initial displacement of

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & 2 \le x \le 4 \end{cases}$$

The motion of the string is described by the wave equation, $u_{tt} = a^2 u_{xx}$. Determine appropriate boundary conditions and initial conditions, and find a formula for the displacement of the string, $u(x,t)$ at position x along the string, and time t after the release of the string.

[It may help to know that $\int x \sin(\frac{n\pi x}{4})\, dx = \frac{16}{n^2\pi^2} \sin(\frac{n\pi x}{4}) - \frac{4x}{n\pi} \cos(\frac{n\pi x}{4})$.]

9. Find the solution to the wave equation problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\begin{gathered} u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = \sin(x) \\ \frac{\partial u}{\partial t}(x,0) = 1 \end{gathered}$$

It may be helpful to express your solution as a sum $u(x,t) = v(x,t) + w(x,t)$ as discussed in class/homework.

10. Consider the partial differential equation problem

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} = \frac{1}{t}\frac{\partial u}{\partial t} \\ u(0,t) = 0 \\ u(1,t) = 0 \end{gathered}$$

(a) Using the technique of separation of variables, assume that $u(x,t) = X(x)T(t)$, and replace the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{t}\frac{\partial u}{\partial t}$ with a pair of ordinary differential equations.

(b) Using the boundary conditions, $u(0,t) = 0$ and $u(1,t) = 0$, determine appropriate boundary conditions for one of the ordinary differential equations you found in part (a).

(c) Find solutions to the equations in (a) subject to the boundary conditions in (b). What solutions to the original partial differential equation problem can be determined from these results?