Week #15 Written Assignment: Due on *Friday*, December 6.

1. **Linearization of Non-Linear Systems.** In first semester calculus, you learned that the linear approximation to a differentiable function $f(x)$ at the point $x = a$ is $f(x) \approx f(a) + f'(a)(x - a)$. A differentiable function of two variables, $F(x, y)$, also has a linearization at a point (a, b). It is given by

$$F(x,y) \approx F(a,b) + \left.\frac{\partial F}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial F}{\partial y}\right|_{(a,b)} (y-b)$$

The non-linear system

$$x' = F(x,y), \quad y' = G(x,y)$$

Has an equilibrium point at (a, b) if both $F(a, b) = 0$ and $G(a, b) = 0$. At the equilibrium point of a non-linear system we can consider the *linearized* system

$$\begin{aligned} x' &= \left.\frac{\partial F}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial F}{\partial y}\right|_{(a,b)} (y-b) \\ y' &= \left.\frac{\partial G}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial G}{\partial y}\right|_{(a,b)} (y-b) \end{aligned}$$

or

$$\begin{aligned} u' &= \left.\frac{\partial F}{\partial x}\right|_{(a,b)} u + \left.\frac{\partial F}{\partial y}\right|_{(a,b)} v \\ v' &= \left.\frac{\partial G}{\partial x}\right|_{(a,b)} u + \left.\frac{\partial G}{\partial y}\right|_{(a,b)} v \end{aligned}$$

where $u = x - a$ and $v = y - b$.

(a) Consider the non-linear system

$$\begin{aligned} x' &= y - x(x-1) \\ y' &= -y + x(x+1) \end{aligned}$$

Find the nullclines for the system and use them to sketch a phase portrait for the system.

(b) The system in part (a) has an equilibrium point at $(0, 0)$. Find the linearization of this system at $(0, 0)$. Find the general solution to the linearized system.

(c) Sketch a phase portrait for the linearized system in part (b). Note that the qualitative behavior of solutions near the equilibrium point in part (a) is very close to the behavior of solutions in (b).

(d) Now consider the system

$$\begin{aligned} x' &= y - x(x-1) \\ y' &= y - x(x+1) \end{aligned}$$

This system also has an equilibrium point at $(0, 0$. Sketch the nullclines, and indicate the directions that solution curves must cross the nullclines. Note that $(0, 0)$ appears to be a spiral. (Or a center. Or maybe an impoper node...) But do the solution curves move *toward* the equilibrium point or *away*? It's hard to tell, isn't it.

(e) The solutions near $(0, 0)$ will behave in a way similar to the linearized system at (0.0). Find the linearization of the system in (d) at the point $(0, 0)$. Determine the type and stability of the equilibrium point. Use that information to sketch a phase portrait for the non-linear system in part (d).

2. **The Heat Eqauation** Consider the differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad k > 0$$

This equation models the flow of head in a one-dimensional rod: $u(x, t)$ represents the temperature of the object at position x and time t. Using the technique of separation of variables, as we did in class for the wave equation, look for non-zero solutions of the form $u(x, t) = X(x)T(t)$.

(a) Substitute $u(x, t) = X(x)T(t)$ into the heat equation to get an expression involving $X(x)$, $X''(x)$, $T(t)$, and $T'(t)$.

(b) Divide by $kX(x)T(t)$ to separate the variables. Explain why the left and right sides must be constants. (Use $-\lambda$ as this "separation constant"). Show that X must satisfy a particular second order linear differential equation, and T must satisfy a first order equation.

(c) Assume the ends of the rod are kept at a constant temperature of zero: $u(0, t) = u(L, t) = 0$. What conditions does this place on the function $X(x)$?

(d) Explain why the allowable solutions for X are $X(x) = X_n(x) = c\sin(\frac{n\pi x}{L})$

(e) Show that the allowable solutions for T are $T(t) = T_n(t) = Ce^{-\frac{n^2\pi^2 kt}{L^2}}$

(f) Using the fact that the heat equation is a linear equation, argue that the solution to the heat equation must be of the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2 kt}{L^2}} \sin(\frac{n\pi x}{L})$$

3. Consider the wave equation problem

$$\begin{gathered} \tfrac{\partial^2 u}{\partial t^2} = \alpha^2 \tfrac{\partial^2 u}{\partial x^2} \\ u(0, t) = 0 \\ u(L, t) = 0 \\ u(x, 0) = 0 \\ \tfrac{\partial u}{\partial t}(x, 0) = g(x) \end{gathered}$$

In class we looked at problems where the initial velocity was zero, and the initial displacement was non-zero. In the above problem the situation is reversed.

Apply the technique of separation of variables to this problem:

(a) Assuming that u has the form $u(x,t) = X(x)T(t)$, rewrite the wave equation in terms of $X(x)$ and $T(t)$. Divide in order to "separate" the variables, and replace the partial differential equation with an appropriate pair of ordinary differential equations.

(b) Replace the conditions $u(0,t) = 0$ and $u(L,t) = 0$ with suitable conditions for the functions X and T. We are interested in conditions that allow for a non-zero $u(x,t)$.

(c) Replace the condition $u(x,0) = 0$ with suitable conditions for the functions X and T. We are interested in conditions that allow for a non-zero $u(x,t)$.

(d) Find the allowable solutions $u_n(x,t) = X_n(x)T_n(t)$.

(e) How can these solutions be combined to solve the above problem?

4. Using the linearity of the wave equation, solve the wave equation problem

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \\
u(0,t) = 0 \\
u(L,t) = 0 \\
u(x,0) = \sin\left(\frac{\pi x}{L}\right) \\
\frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{2\pi x}{L}\right)
\end{gathered}$$