21-260 **Differential Equations** **D. Handron**

Exam #2 Review Questions

1. Solve the initial value problem

$$\begin{gathered} y'' + 2y' + 5y = 0 \\ y(0) = 2 \\ y'(0) = -1 \end{gathered}$$

using the Laplace transform method.

2. Consider the following differential equation, which models a damped mass-spring system:

$$y'' + 14y' + 58y = 0.$$

(a) Classify the system as overdamped, underdamped or critically damped.

(b) Suppose an external force $F(t) = e^{-3t}$ is applied to the system above. Find the solution to the resulting non-homogeneous differential equation that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 0$.

3. The motion of an undamped mass-spring system is described by the differential equation

$$2x'' + 36x = \sin(\omega t).$$

(a) For what values of ω will the system exhibit resonance?

(b) Use the Laplace transform method to find a particular solution in the case where ω is *not* the resonant frequency.

4. Let

$$g(t) = \begin{cases} t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & 2 \le t. \end{cases}$$

(a) Find $G(s) = \mathcal{L}\{g(t)\}$.

(b) Find $\mathcal{L}^{-1}\left\{\dfrac{1 - 2e^{-(s-5)} + e^{-2(s-5)}}{s-5}\right\}$.

5. Find the solution to the initial value problem

$$\begin{gathered} y'' + 16y = 4\mathcal{U}(t) - 4\mathcal{U}(t - \tfrac{\pi}{4}) \\ y(0) = 0 \\ y'(0) = 0. \end{gathered}$$

6. (a) Find the general solution to the second order linear equation

$$4y'' + 16y' + 16y = 0.$$

(b) Solve the initial value problem

$$y'' + y' - 6y = 0, \quad y(0) = 1, y'(0) = 1.$$

7. An undamped, forced harmonic harmonic oscillator may be modelled by the equation $mx'' + kx = f(t)$, where m is the mass of the weight, k is the spring constant, and $f(t)$ is the force applied to the mass. Consider the case where $m = 2$, $k = 8$ and $f(t) = 3\cos(\omega t)$.

   (a) Use the method of undetermined coefficients to find a particular solution, $x_p(t)$.

   (b) For what value of ω will resonance occur? How can this be determined from the particular solution found in part (a)?

8. Compute the Laplace transform of the function

$$f(t) = e^{at}g(t), \quad \text{where} \quad g(t) = \begin{cases} 2t & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

9. Use Laplace transforms to solve the initial value problem

$$x'' + 4x = 1, \quad \text{where} \quad x(0) = 1, \text{ and } x'(0) = 0.$$

10. A certain damped mass-spring system is modeled by the equation

$$y'' + 4y' + 5y = 4\cos(t) - 4\sin(t).$$

   (a) If the initial state of the system is $y(0) = 1$, $y'(0) = 1$, find the behavior of the system for $t > 0$.

   (b) What is the long term behavior of the system? Identify the "steady-state solution" and the "transient solution".

   (c) If this "experiment" is repeated with different initial conditions what, if any, difference be noticed in the long term behavior of the system?

11. Use the Laplace transform method to solve the initial value problem

$$y'' + 6y' + 13y = u_{4\pi}(t)[12\cos(t) + 27\sin(t)], \quad y(0) = 1, y'(0) = -1.$$

   [Hint: You may find it useful to know that $\frac{12s+27}{(s^2+1)(s^2+6s+13)} = \frac{1}{(s^2+1)} + \frac{2}{(s^2+6s+13)}$.]

12. Consider the differential equation

$$x'' + x' + 4x = 4\cos(t) + 2\sin(t).$$

   (a) Find the solution satisfying initial conditions $x(0) = 0$, $x'(0) = 0$.

   (b) Identify the steady-state solution and the transient solution.

13. A certain mass-spring system exhibits the following properties:

- The mass is 2kg.
- The spring exerts a force of 6N when the mass is displaced 2m from its equilibrium.
- A viscous force of 5N slows the system when the mass moves with a velocity of 1m/s.

(a) What is the spring constant, k, for this system?

(b) What is the damping coefficient, γ, for this system?

(c) If $x(0) = 1$ and $x'(0) = -1$, what is the position of the mass when $t = 1$?

(d) Is this system overdamped, underdamped or critically damped?

14. Use the method of Laplace transforms to find the solution to the initial value problem

$$x'' + 4x' + 8x = 11\cos(t) + s\sin(t), \quad x(0) = 1, x'(0) = -2$$

[Hint: It may help to know that $\frac{11s+3}{(s^2+1)(s^2+4s+8)} = \frac{s+1}{s^2+1} - \frac{-s-5}{s^2+4s+8}$.]

15. Suppose that $\mathcal{L}\{f(t)\} = F(s)$ is defined for $s > a$, and that $c > 0$.
Show that $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c})$, and that this transform is defined for $s > ac$.

16. (a) Find the general solution to the differential equation $y'' - 2y' - 8y = 0$.

17. Let $x(t)$ be a solution to the differential equation

$$x'' + \gamma x' + x = 0.$$

which models a certain (damped) mass-spring system. Also suppose that this solution satisfies the initial conditions

$$x(0) = 1, \quad x'(0) = -2.$$

(a) For what values of γ will this system be underdamped? Sketch a graph showing the behavior of a solution in this case (i.e. underdamped). How many times does this graph cross the t axis?

(b) Suppose that the system is critically damped. At what time or times will the mass pass through the equilibrium point? Sketch a graph of this solution.

18. Consider the differential equation

$$x'' + 4x = \cos(2t).$$

(a) Use the Laplace transform method to find a solution to this differential equation.

(b) Sketch a graph of the solution you have found. What is the behavior of this solution as $t \to \infty$?

(c) What phenomenon do you observe here?

19. Let $f(t)$ be a piecewise continuous function that is of exponential order as $t \to \infty$. Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $s > a$. Justify the following two statements:

(a) If $c > 0$, then $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c})$ for $s > ac$.

(b) $\mathcal{L}\{tf(t)\} = -F'(s)$. [Hint: It may help to know that with the above assumptions, $F'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{d}{ds}e^{-st}f(t)dt$.]

20. For $n > 0$, let

$$f_n(t) = \begin{cases} 1 - \frac{t}{n}, & 0 \le t < n \\ 0, & n \le t \end{cases}$$

(a) Compute the Laplace transform $F_n(s) = \mathcal{L}\{f_n(t)\}$.

(b) What is the limit $\lim_{n\to\infty} F_n(s)$?

21. Consider the initial value problem

$$x'' + 2x' + x = 0, \quad x(0) = 1, x'(0) = -r$$

where $r > 0$.

(a) Suppose $r = 2$. Find the solution to the initial value problem. How many times does the solution to the initial value problem cross the positive x-axis?

(b) Suppose $r = 0$. Find the solution to the initial value problem. How many times does this solution cross the positive x-axis?

(c) What is the largest value of r such that the solution does *not* cross the positive x-axis?

22. Consider the equation

$$x'' + x' + x = \cos(\omega t),$$

which represents a forced, damped harmonic oscillator. Find an expression for the amplitude of the steady state solution as a function of the forcing frequency, ω.

23. Consider the differential equation

$$y'' + 2y' + 5y = \cos(\omega t),$$

which models the motion of a damped mass-spring system with mass 1, damping coefficient 2, and spring constant 5 under the influence of the external force $F(t) = \cos(\omega t)$.

Show that if $\omega = 2$, then the amplitude of the steady-state solution is $\frac{1}{\sqrt{17}}$.

24. Let f be the function defined by

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ t - 1 & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$$

(a) Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$.

(b) Solve the initial value problem

$$x'' + (2\pi)^2 x = 0; \quad x(0) = 1, \quad x'(0) = 2$$

25. Consider a damped mass-spring system with an external forcing function, which can be modeled by

$$mx'' + \gamma x' + kx = \cos(\omega t),$$

where $m = 2$, $\gamma = 4$, $k = 4$ and $\omega = 1$.

(a) What is the natural frequency, ω_0, of this system? What is the pseudo-frequency, μ?

(b) Find the steady state solution.

(c) What is the amplitude of the steady state solution?

26. Use Laplace transform methods to solve the initial value problem

$$x'' - 2x' + 10x = 0, \quad x(0) = 3, x'(0) = 8$$

27. Solve the initial value problem

$$x'' - 6x' + 8x = g(t), \quad x(0) = 0, x'(0) = 0$$

where

$$g(t) = \begin{cases} 8 & t < 1 \\ 16 - 8t & 1 \leq t. \end{cases}$$

[Hint: $\frac{8}{s(s-2)(s-4)} = \frac{1}{s} - \frac{2}{s-2} + \frac{1}{s-4}$ and $\frac{8}{s^2(s-2)(s-4)} = \frac{3/4}{s} + \frac{1}{s^2} - \frac{1}{s-2} + \frac{1/4}{s-4}$.]

28. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.

(a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right), \quad s > ca$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right).$$