Final Exam — Review

1. Let W be a subspace of $\mathbb{R}^n$ and $W^\perp$ its orthogonal complement. Show that for any $\mathbf{v} \in \mathbb{R}^n$

$$\text{proj}_W(\text{proj}_{W^\perp}(\mathbf{v})) = \mathbf{0}$$

2. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$, and $P = [p_{ij}]_{i,j}$ an invertible $n \times n$ matrix.

   (a) For $i \in \{1, \ldots, n\}$ set $\mathbf{u}_i = p_{1i}\mathbf{v}_1 + \ldots + p_{ni}\mathbf{v}_n$. Show that $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$.

   (b) Show that P is the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

3. Let

$$A = \begin{bmatrix} -14 & -20 \\ 12 & 17 \end{bmatrix}$$

   (a) Find the eigenvalues of A.

   (b) Find a basis for each eigenspace of A.

   (c) Is A diagonalizable? Either find a diagonalization or explain why it is not.

4. (a) Use the Gram-Schmidt process (or "modified" Gram-Schmidt process) to find an orthonormal basis for

$$\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

5. **(25 points)** Let $\mathcal{B}$ be the basis $\{1, x, x^2\}$ for the vector space $\mathcal{P}_2$, and $\mathcal{C}$ be the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for the vector space $\mathbb{R}^2$.

   Let $T : \mathcal{P}_2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(p(x)) = \begin{bmatrix} \int_0^1 p(x)\, dx \\ p(0) \end{bmatrix}$$

   Find the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ for the linear transformation T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.