Exam #1 Review
Fall 2017

1. Find all solutions to the linear system of equations

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 4x_3 & & & = & 5 \\ -x_1 & & & + & 2x_3 & - & 2x_4 & = & 7 \\ & - & x_2 & - & 3x_3 & + & x_4 & = & -6. \end{array}$$

   Express the solution set in vector form.

2. Is the vector $\begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix}$ a linear combination of the vectors

$$\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix}?$$

   Either find all such linear combinations, or explain why there are none.

3. V is a set on which two operations, $\oplus$ and $\odot$, are defined. The set is $\mathbb{R}^2$, and $\odot$ is the usual scalar multiplication, but $\oplus$ is defined by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_2 + v_2 \\ u_1 + v_1 \end{bmatrix}$$

   (a) Show that vector space axiom *7* is satisfied.

   (b) Show that vector space axiom *8* is not satisfied.

   (c) Is vector space axiom *3* satisfied? (Naturally, you should provide either a proof or a counterexample.)

4. Let P represent the statement

$$\forall\, x \in \mathbb{R} \quad (\exists\, y \in \mathbb{R} \quad (x = y^2 \text{ or } x = -y^2))$$

   (a) Find an expression for the negation of this statement, $\neg P$. Simplify your expression as much as possible.

   (b) One of P or $\neg P$ is true. Give a proof for whichever one it is.

5. Consider the system of equations

$$\begin{array}{rcrcrcrcrcr} & & & - & x_3 & + & 3x_4 & + & 5x_5 & = & 12 \\ x_1 & - & 2x_2 & + & x_3 & - & 2x_4 & & & = & -8 \\ 2x_1 & - & 4x_2 & & & + & 3x_4 & + & 12x_5 & = & 11. \end{array}$$

(a) What is the augmented matrix for this system?

(b) Use elementary row operations to put the matrix into reduced echelon form. Perform one operation at at time and indicate clearly which row operations are used.

(c) Write the solution to the system in vector form.

6. Is the vector $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$ a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}?$$

(a) Write down a system of linear equations in standard form whose solution will provide an answer to this question.

(b) Solve the system from part (a). Either give a description of all such linear combinations, or explain how you know there are none.

7. Consider the system

$$\begin{array}{ccccccc} x & & & + & 4z & = & a \\ & & y & - & 3z & = & b. \end{array}$$

Note that the augmented matrix of this system is in reduced echelon form.

(a) Suppose that $a = b = 0$. Find the vector form of the solution set in this case.

(b) When $a = b = 0$, is the solution set closed under addition? (i.e. does it satisfy the first axiom in the definition of a vector space?)

(c) Suppose that $a = 2$ and $b = -1$. Find the vector form of the solution set in this case.

(d) When $a = 2$ and $b = -1$, is the solution set closed under addition? (i.e. does it satisfy the first axiom in the definition of a vector space?)

8. Let P represent the statement

$$\forall\, m \in \mathbb{Z} \quad (x = 7m + 3 \implies \exists\, n \in \mathbb{Z} \quad (x^2 = 7n + 2))$$

(a) Find an expression for the negation of this statement, $\neg P$. Simplify your expression as much as possible.

(b) One of P or $\neg P$ is true. Give a proof for whichever one it is.

Definition 1 *Let V be a set with two operations, addition and scalar multiplication. If $\mathbf{u}, \mathbf{v} \in V$, we denote their sum as $\mathbf{u}+\mathbf{v}$. If $\mathbf{u} \in V$ and c is a scalar, we denote their scalar product as $c\mathbf{u}$.*

We say that V is a vector space *if the following axioms hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars c, d:*

1. $\mathbf{u}+\mathbf{v} \in V$
2. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. *There exists a vector $\mathbf{0} \in V$ such that $\mathbf{u}+\mathbf{0} = \mathbf{u}$ for all vectors $\mathbf{u} \in V$*
5. *For each vector $\mathbf{u} \in V$ there is a vector $(-\mathbf{u}) \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$*
6. $c\mathbf{u} \in V$
7. $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$
8. $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$