MATHEMATICAL STUDIES 1: ANALYSIS HW 7

SOLUTIONS

0.1. **Rudin, Chapter 5, Exercise 1.** Let f be defined for all real x and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.
If $x > y$, then we have

$$\begin{aligned} \frac{|f(x) - f(y)|}{x - y} &\leq x - y \\ \lim_{y \to x^-} \frac{|f(x) - f(y)|}{x - y} &\leq 0. \end{aligned}$$

But this limit must be non-negative because it is the quotient of two non-negative numbers, so

$$\lim_{y \to x^-} \frac{|f(x) - f(y)|}{x - y} = 0,$$

and we can remove the absolute value.
However, if $x < y$, then we get

$$\begin{aligned} \frac{|f(x) - f(y)|}{y - x} = \frac{|f(y) - f(x)|}{y - x} &\leq y - x \\ \lim_{y \to x^+} \frac{|f(y) - f(x)|}{y - x} &\leq 0. \end{aligned}$$

Again, this limit is non-negative, so

$$\lim_{y \to x^+} \frac{|f(y) - f(x)|}{y - x} = 0$$

and we can again remove the absolute value to get

$$\lim_{y \to x^+} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x^+} \frac{f(x) - f(y)}{x - y} = 0.$$

Therefore we have limits from both sides, giving

$$\lim_{y \to x} \frac{f(x) - f(y)}{x - y} = 0.$$

This limit is of course the derivative, so the derivative $f'(x) = 0$. Because this works for all real x, we have that the derivative of the function at every point is 0. Thus the function is constant.

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0.2. **Rudin, Chapter 5, Exercise 2.** Suppose $f'(x) > 0$ in (a,b). Given any $x, y \in (a,b)$ with $x < y$, since f is differentiable on $[x,y]$, f is continuous on $[x,y]$, and by the mean value theorem there exists some $z \in (x,y)$ such that $f(y) - f(x) = (y-x)f'(z) > 0 \Leftrightarrow f(x) < f(y)$, so f is strictly increasing on (a,b). Hence, the inverse function for f exists on (a,b), and call it g. For any $x \in (a,b)$, I wish to prove that g is differentiable at x. Fix some $\alpha, \beta \in (a,b)$ such that $\alpha < x < \beta$. Since f is differentiable on (a,b), f is differentiable on $[\alpha,\beta] \subset (a,b)$, and thus f is continuous on $[\alpha,\beta]$. Since $[\alpha,\beta]$ is compact and f is continuous on $[\alpha,\beta]$, from Theorem 4.17, g is continuous on $[f(\alpha), f(\beta)]$. Thus, as $y \to x$, $g(y) \to g(x)$. Also, since f is differentiable at x, $\lim_{y\to x} \frac{f(y)-f(x)}{y-x}$ exists. Since f is strictly increasing, $f(y) \neq f(x)$ whenever $y \neq x$, so $\lim_{y\to x} \frac{y-x}{f(y)-f(x)}$ exists. Finally, from all of the above, we have that the following limit exists:

$$\lim_{y\to x} \frac{g(y)-g(x)}{y-x} = \lim_{y\to x} \frac{g(y)-g(x)}{f(g(y)) - f(g(x))} = \lim_{g(y)\to g(x)} \frac{g(y)-g(x)}{f(g(y)) - f(g(x))}$$

and we may conclude that g is differentiable at x. Thus, g is differentiable on (a,b). From the identity $g(f(x)) = x$ and the chain rule, we can easily verify that for all $x \in (a,b)$,

$$(g(f(x)))' = 1 = g'(f(x))f'(x) \Leftrightarrow g'(f(x)) = \frac{1}{f'(x)}$$

0.3. **Rudin, Chapter 5, Exercise 6.** Suppose that f is continuous for all $x \geq 0$, $f'(x)$ exists for all $x > 0$, $f(0) = 0$, and that f' is monotonically increasing. Define

$$g(x) = \frac{f(x)}{x}$$

for $x > 0$.

By theorem 5.3, because both $f(x)$ and x are differentiable, we see that for any $x > 0$

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

By the "mean value theorem", there exists a $y \in (0,x)$ such that

$$f(x) - f(0) = (x-0)f'(y) = xf'(y)$$

Combining this with the information given, we see that

$$f(x) = xf'(y)$$

for some $y < x$.

Therefore, we observe that for any $x > 0$ and some $0 < y < x$

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{xf'(x) - xf'(y)}{x^2} = \frac{f'(x) - f'(y)}{x}$$

Since f' monotonically increasing, we know for $x > y$ that $f'(x) \geq f'(y) \iff f'(x) - f'(y) \geq 0$ which implies that $\frac{f'(x)-f'(y)}{x} \geq 0$ and thus for all $x > 0$

$$g'(x) \geq 0$$

Then by theorem 5.11, we see that g is monotonically increasing. $\square$

0.4. **Rudin, Chapter 5, Exercise 26.**

Problem 1: Rudin, pg. 119, (26)

Suppose that f is differentiable on $[a,b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

The result is immediate if $A = 0$. From the inequality $|f'(x)| \leq A|f(x)|$ we know that $A \geq 0$, so let A be positive. Fix some arbitrary $x_0 \in [a,b]$ and let

$$M_0 = \sup|f(x)|, \text{ and } M_1 = \sup|f'(x)|,$$

where the suprema are taken over all $x \in [a, x_0]$. In particular, $M_1 \leq AM_0$. Again, we apply the mean value theorem in a clever way to show that for any $x \in [a, x_0]$, there is a $c \in (a,x)$ such that $f(x) - f(a) = f'(c)(x-a)$. Because $f(a) = 0$, this simply implies that $f(x) = f'(c)(x-a)$. Taking the absolute value of both sides, $|f(x)| = |f'(c)|(x-a)$, furthermore,

$$|f(x)| \leq M_1(x-a) \leq M_1(x_0 - a) \leq AM_0(x_0 - a).$$

Now, we fix a number $\epsilon > 0$ such that $\epsilon \leq 1/(2A)$ and such that $(b-a)/\epsilon$. We fix $x_0 = a + \epsilon \leq a + 1/(2A)$. So $(x_0 - a) \leq 1/(2A)$ and $AM_0(x_0 - a) \leq M_0/2$, so

$$|f(x)| \leq \frac{M_0}{2} = \frac{\sup|f(x)|}{2}$$

This is a contradiction unless $f(x) = 0$ for all $x \in [a, x_0]$. We iterate this process, defining $x_n = x_{n-1} + \epsilon$, and considering the interval $[x_{n-1}, x_n]$. Since $x_n - x_{n-1} = \epsilon$ and ϵ divides the interval $[a,b]$ into an finite (and integer) number of subintervals, the process will terminate after a finite number of iterations. We will have established at that point that $f(x) = 0$ for all $x \in [x_0, x_1] \cup \cdots \cup [x_{n-1}, x_n] \cup \cdots = [a,b]$. $\square$

0.5. **Rudin, Chapter 5, Exercise 27.**

0.6. **Rudin, Chapter 7, Exercise 2.** Suppose $\{f_n\}, \{g_n\}$ converge uniformly to f, g respectively.
Claim: $\{f_n + g_n\}$ converges uniformly.
Fix $\epsilon > 0$.
$\exists N_1$ s.t. $n \geq N_1 \Rightarrow \forall x \in E,\ |f_n(x) - f(x)| < \frac{\epsilon}{2}$.
$\exists N_2$ s.t. $n \geq N_2 \Rightarrow \forall x \in E,\ |g_n(x) - g(x)| < \frac{\epsilon}{2}$.
Thus, $|f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon$.
Well $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon \Rightarrow \{f_n + g_n\}$ converges uniformly to $f + g$.

Claim: If $\{f_n\}, \{g_n\}$ are bounded, then $\{f_n g_n\}$ converges uniformly.

$\exists N$ s.t. $n \geq N \Rightarrow \forall x \in E,\ |g_n(x) - g(x)| < 1$.
Choose any $k \geq N$. We know that g_k is bounded, i.e. $\forall x \in E,\ |g_k(x)| < M \in \mathbb{R}$.
But $|g_k(x) - g(x)| < 1$, so $\forall x \in E,\ |g(x)| < M + 1$.
Furthermore, $\forall n \geq N, \forall x \in E,\ |g_n(x) - g(x)| < 1 \Rightarrow |g_n(x)| < M + 2$.

There are only finitely many g_n not bounded by $M + 2$, so it is clear that there exists $M_1 \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N},\ x \in E,\ |g_n(x)| < M_1$ and $|g(x)| < M_1$.
By a similar argument, $\exists M_2 \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N},\ x \in E,\ |f_n(x)| < M_2$ and $|f(x)| < M_2$.

Fix $\epsilon > 0$.
Consider $|f_n(x)g_n(x) - f(x)g(x)|$.
This is equal to $|f_n(x)g_n(x) - f(x)g(x) + f_n(x)g(x) - f_n(x)g(x)|$
$\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)|$.
Factoring, this is the same as $|f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$.

Well, $\exists N_1$ s.t. $n \geq N_1 \Rightarrow \forall x \in E,\ |g_n(x) - g(x)| < \frac{\epsilon}{2M_2}$.
$\exists N_2$ s.t. $n \geq N_2 \Rightarrow \forall x \in E,\ |f_n(x) - f(x)| < \frac{\epsilon}{2M_1}$.

Thus, $n \geq \max(N_1, N_2) \Rightarrow |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$
$\leq M_2|g_n(x) - g(x)| + M_1|f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
$\Rightarrow \{f_n g_n\}$ converges uniformly to fg.

0.7. **Rudin, Chapter 7, Exercise 3.** Let $f_n(x) = x + 1/n$ and let $g_n(x) = f_n(x)$ for all $x \in \mathbb{R}$. Then it is trivial that $f_n(x)$ and $g_n(x)$ converge uniformly to $f(x) = x$ and $g(x) = x$, respectivley. However, $f_n(x)g_n(x) = x^2 + (2x)/n + 1/n^2$ converges pointwise to $f(x)g(x) = x^2$, but not uniformly. This is true because, given any $\epsilon > 0$ and N, choose $x > 0$ large enough so that $(2x)/N + 1/N^2 > \epsilon$. Then we have

$$\left| x^2 + \frac{2x}{N} + \frac{1}{N^2} - x^2 \right| = \left| \frac{2x}{N} + \frac{1}{N^2} \right| > \epsilon$$

This method can be used for any ϵ and N, so therefore $\{f_n g_n\}$ does not converge uniformly.