21-124 MODELING WITH DIFFERENTIAL EQUATIONS

LECTURE 2: BUILDING MATHEMATICAL MODELS EXPERIMENTS AND DISCUSSION

In the second lecture, we will experiment with some modeling scenarios using MATLAB. We will be using a MATLAB routine called `dfield`, written by John Polking at Rice University. This routine provides a convenient way to plot solutions to many differential equations.

1. Obtaining the `dfield` File

The file you need is called `dfield6.m`. It is available from a Rice University website: http://math.rice.edu/∼dfield. There are many different versions, but you want the one for MATLAB version 6. You can also follow the link from the course webpage.

In order to run `dfield`, find an available Xterm window, and enter `matlab` on the command line. Once MATLAB is up and running, enter `dfield6` on the MATLAB command line. This will produce a dialog box into which you can enter the differential equation and limits for the independent and dependent variables. You can also change the letters representing these variables, if you so desire. Once you have done this, click the "Proceed" button.

MATLAB will open a new window containing the direction field for the differential equation you entered. Clicking anywhere on the graph will cause MATLAB to plot an approximate solution through that point.

Go ahead and explore some of the menus to see what else you can do.

2. Population Models

We will add some additional assumptions to the Logistic Growth Model. In order to facilitate an intuitive understanding of the situation, assume the population we are discussing consists of rabbits on an isolated island. Assume that the population P is measured in number of rabbits, and that time t is measured in months.

In each case, modify the Logistic Growth Model to account for the new situation and use `dfield` to plot some sample solutions. Try to answer the questions posed. The questions might also give you ideas about what might be interesting to look at.

1. Assume that hunting is allowed on the island. Hunters find it easier to track down their prey when the population is larger. Because of this, you may assume that the rate of hunting is proportional to the size of the population.

   How does this assumption affect the Logistic Growth Model? Do any solutions increase without bound? Will the population ever go extinct? Does the result depend on the rate of hunting? On the growth coefficient? What effect does the hunting have on the size of the population that can be supported?

2. Now assume that the rate of hunting is restricted to a certain number of hunting permits which are issued each month. Hunting then takes place at a constant rate of h rabbits per month.

   How does this assumption affect the Logistic Growth Model? Do any solutions increase without bound? Will the population ever go extinct? Does the result depend on the rate of hunting? On the growth coefficient? What effect does the hunting have on the size of the population that can be supported?

3. Let's forget about hunting. But suppose instead that the resources on the island (food, water, etc.) are not being replenished as rapidly as they once were. Because of this, the carrying capacity of the island is decreasing exponentially from its initial level to one half that amount.

   Describe the effect this has on the solution curves, as compared to the original Logistic Model.

3. Toricelli's Law

Now let's take a look at some specific cases of Toricelli's Law. Recall that Toricelli's Law implies that

$$\frac{dy}{dt} = -a\sqrt{2g}\frac{\sqrt{y}}{A(y)},$$

where a is the area of the hole, g is the acceleration of gravity, y is the depth of the water in the tank, and $A(y)$ is the area of the water's surface when the depth of the water is y.

In each case, the equation of a curve is given. The "tank" is then obtained by revolving the curve around the y-axis. You may assume that distances are measured in meters, and time in seconds, so that $g = 9.8$.

1. Use the curve $x = y(5 - y) + .01$ for $y \in [0, 5]$. The hole at the bottom has a radius of 1 cm. What does the tank look like? How long does it take the tank to empty? At what time is the tank half empty? Which takes longer, draining the top half of the tank, or the bottom half? Why?

2. Use the curve $x = \frac{1}{(y-2.5)^2+1}$ for $y \in [0, 5]$. The hole in the bottom of the tank has radius $\frac{1}{7.25}$ m. What does the tank look like? Why do the solution curves look the way they do (with reference to the shape of the tank)?