21-124 MODELING WITH DIFFERENTIAL EQUATIONS

PROJECT 2

Project 2 requires you to use the techniques you have learned this semester to analyze a single modeling scenario in somewhat greater depth than we have in the past. You should complete one of the three problems. You may only discuss your progress with Ben or myself, and all questions regarding Project 2 should be addressed to Ben or myself. In particular, you should not discuss any aspect of the problems with your classmates.

Project 2 is due at 5:00 on Friday, May 3. If this deadline poses a problem, you should meet with me to discuss your special situation. Any time before then, you can show me a rough draft of your report, which I will review and discuss with you as promptly as possible. This is not required, but I strongly recommend it.

For each problem, there are several questions you are asked to answer. If you see another way to approach the problem that is more interesting to you, you may, in consultation with me, alter the questions somewhat. If you do that, you should definitely let me review a rough draft of your report.

1. A Wobbling Soda Bottle

Two liter plastic soda bottles and 20 oz. plastic soda bottles have a base with five "feet". If you cause one of these bottles to wobble slightly without falling over, it displays some interesting and unpredictable behavior. In this problem, we will develop a mathematical model for this system.

1. (An upside-down pendulum) We have seen that when the angle θ a pendulum makes with the vertical is small, it can be modeled by the same differential equation as a mass on a spring, $\frac{d^2\theta}{dt^2} = -k\theta$. Suppose that instead of measuring θ from the downward vertical position, we measure θ from the upward vertical position instead. Show that for small values of $|\theta|$ the behavior of this system can be modeled by the equation $\frac{d^2\theta}{dt^2} = +k\theta$. Show that the solutions to this system do not remain in the region where $|\theta|$ is small. We can think of this system as a "negative spring", which pushes on a mass, rather than pulling.
2. (A 2-dimensional spring) One end of a spring is fixed to a post in the middle of a smooth flat surface. The other end is attached to a mass that can move freely about the surface. This system can most easily be thought of in vector form. At any moment, the acceleration will be toward the post. If we make that the origin of our coordinate system, and let $Y = \begin{bmatrix} x \\ y \end{bmatrix}$, then:

$$\frac{d^2Y}{dt^2} = -kY.$$

   We can write this as a pair of second order equations: $x'' = -kx$, $y'' = -ky$. Change this to a system of first order equations, and write an m-file for the system for use with one of MATLAB's numerical solvers. Investigate the behavior of solutions to the system (use $k = 1$). Consider both situations where the initial velocity is zero, and where it is not zero. Note that this is a very idealized sort of set-up, where the mass is allowed to travel through the post!

3. (A 2-dimensional negative spring) The 2-dimensional spring above can be used to model the motion of a pendulum that is free to move in any direction (Like the ones on executives desks that trace patterns in a tray of sand) when it is near its stable equilibrium point. If such a pendulum were near its upward vertical position, it could be modeled by the the system
$$x'' = +kx$$
$$y'' = +ky.$$
What happens to solutions to this system? How can you interpret it when solutions become large?
4. (Wobbling bottle - a first try) The five feet of a soda bottle are arranged in a pentagon. If you choose a coordinate system so that the feet lie on the unit circle, their positions are described by the points $P_n = \begin{bmatrix} \cos(2\pi n/5) \\ \sin(2\pi n/5) \end{bmatrix}$, for $n = 1, \ldots, 5$. As it wobbles on one of its feet, it behaves like a pendulum pointing nearly straight up. Since the bottle never moves far from vertical (unless it falls over) we can try to model it as a mass attached to five negative springs, one at each of the five points P_n. Write down the second order equations for this system (remember, different forces just add together). Now write an m-file for the corresponding first order system. Use `ode45` or another numerical solver to comput a few solutions to the system. How do they behave? Why?
5. (Wobbling bottle - a second try) After the failure above, we need to look more carefully at the system. Notice that when the bottle is wobbling, only one (or sometimes two) feet are in contact with the table at any one time. Download the m-file `bottle.m` from the course website. How does this m-file try to solve the problem? Use `ode45` or another numerical solver to look at some solutions to this system. Do they behave like the bottle we are trying to model?
6. (Experimentation) The soda has two "modes of wobbling" that I find especially appealing. In one of them, the top of the bottle moves around in circles. In the other, the bottle rock back and forth, first balanced on two feet and then on the one foot opposite them. Each of these modes eventually degenerates into some sort of unpredictable motion. Can you find initial conditions for the system that produce these behaviors?
7. (Chaos?) Does the system in `bottle.m` seem to behaving chaotically? Why or why not?

2. The Human Immune System

In this problem, we will look at a model of the human immune system, and see how it reacts to invasion by a virus. As you complete the problem, you should think about diseases like the chicken pox, which once caught (usually by young children) confer immunity from the disease. (This problem is based on a project by John Polking, which is in turn based on an immune system model developed by Anderson and May.)

In this model, there are two different types of lymphocytes, whose populations are represented by $E_1(t)$ and $E_2(t)$. When no virus is present, the populations behave as follows:

- New lymphocites of type i are produced by bone marrow at a constant rate λ_i.
- Lymphocites of type i die at a rate proportional to E_i.
- The lymphocites of each type reproduce in response to contact with lymphocites of the other type, but the rate of reproduction never goes above a certain rate.

These assumptions lead to the differential equations

$$E_1' = \lambda_1 - \mu_1 E_1 + \frac{a_1 E_1 E_2}{1 + b_1 E_1 E_2}$$

$$E_2' = \lambda_2 - \mu_2 E_2 + \frac{a_2 E_1 E_2}{1 + b_2 E_1 E_2}.$$

The last term in each equation represents the reproduction that results from contact between the two types. Notice that when the populations are small, it is nearly proportional to the product $E_1 E_2$, but it never goes above the value a_i/b_i.

Use the values $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 1.25$, $a_1 = a_2 = 0.252$ and $b_1 = b_2 = 0.008$.

1. Find all the equilibrium points for the system where the populations are non-negative. You may use algebra to do this (by hand) or you may use the tools available in Pplane.
2. Which equilibrium points are asymptotically stable, and which are not? For each asymptotically stable equilibrium point, determine the "basin of attraction" (i.e. the set of initial conditions for which solutions will converge to the equilibrium point).
3. If a baby is born with a very small number of lymphocytes, which equilibrium point will the babies immune system converge to? (We will call this the "virgin state".

Now lets consider what happens when a virus is introduced into the system. Here are our assumptions:

- The virus reproduces at a rate proportional to the size of its population, V.
- Lymphocytes of type 1 kill the virus at a rate proportional to the number of contacts between them, and they themselves reproduce as a result of these contacts.
- Lymphocytes of type 2 do not interact directly with the virus, but continue to interact with the lymphocytes of type 1 as before.

These assumptions result in the system

$$E_1' = \lambda_1 - \mu_1 E_1 + \frac{a_1 E_1 E_2}{1 + b_1 E_1 E_2} + K E_1 V$$

$$E_2' = \lambda_2 - \mu_2 E_2 + \frac{a_2 E_1 E_2}{1 + b_2 E_1 E_2}$$

$$V' = rV - kE_1 V.$$

Use the parameters $r = 0.1$, $k = 0.01$ and $K = 0.05$ for the rest of the questions.

1. Show that
$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ 20 \\ 0 \end{bmatrix},$$
are equilibrium points for the system.
2. What happens to a person who is infected with a virus for the first time? That persons immune system will be in the virgin state when the virus is introduced. Compute the solution with initial condition $E_1 = E_2 = V = 1$. Turn in a graph showing all three components. To what equilibrium point does the solution converge? It may help to graph the solutions in the E_1E_2-plane.
3. What happens if the person is infected with the same virus a second time? Compute the solution with initial condition $E_1 = E_2 = 20$ and $V = 1$. What if the person is exposed to a massive dose of the virus (compute the solution with initial condition $E_1 = E_2 = 20$ and $V = 30$)?
4. Does this model seem to explain why people gain immunity to some diseases after getting sick a first time?

3. Things That Bob

The motion of an object floating in a pool of water can by modeled by a second order differential eqaution. There are three forces that act on the object: the force of gravity, the force of friction between the object and the water, and the force of the objects bouyancy bouyancy. (This project is due to Richard Bernatz, Luther College, as presented in Differential Equations and Boundary Value Problems, by Nagle Saff and Snyder.)

The assumptions on which this model is based are:

- Gravity exerts a force on the object equal to its weight in pounds. (Remember, pounds are a unit of force.)
- Friction with the water exerts a force proportional to the velocity of the object.
- The object is acted on by an upward force equal to the weight of the water displaced by the object. (This is Archimedes' principle.)

1. The density of water is $\rho = 62.57\text{lb/ft}^3$, and the coefficient of friction is $\gamma = 3\text{lb-s/ft}$. Let z measure distance from the surface of the water (positive is up, negative is down), and let $V(z)$ denote the volume of the object when its lowermost point is at z. Write down a differential equation that models the motion of the object.
2. Supose the bobbing object is a cube with side length 1ft and weight 32lbs. What is $V(z)$? What is the equilibrium position of the cube? If the oscillations are small (so that the cube is never completely submerged or completely out of the water) what can you say about solutions to the system. Are they similar to anything you have seen before?
3. Write an m-file and use a numerical solver to compute the solution for the case where the cube is placed on the surface of the surface of the water with no initial velocity. Does this solution agree with your observations above?
4. Now suppose the object is a sphere weighing 32lbs with radius 6in. What happens if this sphere is released from 5ft below the surface of the water? Will it ever leave the water? How high will it go? What if it is released from more or less than 5ft below the surface?

5. Investigate the behavior of a few other objects of your own design. Above we have compared two objects with the same weight and height, but different shapes. It might be interesting to compare objects with different shapes, but with constant density, or constant volume.