21-124 MODELING WITH DIFFERENTIAL EQUATIONS

HOMEWORK 2: DUE IN CLASS ON FEBRUARY 26

1. The differential equation modeling the motion of a freely swinging, undamped pendulum of length 2m is
$$\frac{d^2\theta}{dt^2} + \frac{9.8}{2}\sin(\theta) = 0.$$
For small values of θ this can be approximated by
$$\frac{d^2\theta}{dt^2} + \frac{9.8}{2}\theta = 0.$$
   (a) Write the second order equations as systems of first order equations.
   (b) Use `pplane` to examine the solutions of these two systems. Do they appear similar for "small" values of θ? How are they different when θ is large?
   (c) Describe all the "different types" of solutions for the first system. You should be able to find at least three (five if you distinguish between clockwise and counterclockwise rotation). (Hint: One is tricky to find. Try the initial conditions $\theta(0) = 0$, $\theta'(0) = 4.4275$.)
2. We will explore the forced, damped equation
$$y'' + .1y' + 9y = f(t).$$
Our goal is to produce a grpah of the steady periodic solution without solving for it anylitically.
   (a) We will use the forcing function $f(t) = \mathrm{mod}(t, 10)$. Use MATLAB to produce a graph of this function on the range $0 \leq t \leq 30$. You need not write an M-file for this unless you want to. Note that this function is periodic with period 10. Since this fuction is not exponential, polynomial or sinusoidal the method of undetermined coefficients can not be used in this case.
   (b) Write the second order equation as a system of first order equations, and write a function M-file that can be used with `rk4` to compute approximate solutions. Also write an M-file for the associated homogeneous equation.
   (c) Compute a solution to the non-homogeneous equation for $0 \leq t \leq 100$, using initial conditions $y(0) = 0$, $y'(0) = 0$ and stepsize (0.1). Plot $y(t)$ as a function of t. Does the graph look like it has "settled down" to a periodic function? It should.
   (d) Since the forcing function has period 10, we expect the steady periodic solution to have period 10 as well. Thus if y_{sp} is the steady periodic solution, then $y_{sp}(100) = y_{sp}(0)$.
   (e) Have MATLAB display the matrix of y and v values. The last row represents $y(100)$ and $v(100)$. These should be very close to $y(0)$ and $v(0)$.

(f) Compute a new solution for $0 \leq t \leq 30$ using the values of $y(100)$ and $v(100)$ as initial conditions. Plot the graph of $y(t)$ vs. t. Does it look periodic? It should. This is the steady periodic solution.

(g) If $z(t)$ is the solution to the non-homogeneous equation with initial conditions $z(0) = 3$, $z'(0) = 2$ then $z(t) = y_{tr}(t) + y_{sp}(t)$. Compute a numerical approximation of $z(t)$, and then subtract to get the transient solution $y_{tr}(t) = z(t) - y_{sp}(t)$. Plot this transient solution. Does it look like a solution to the homogeneous equation?