

RESEARCH STATEMENT

GARRETT ERVIN

My research interests are in set theory, dynamics, order theory, graph theory, and geometric group theory. Much of my work has centered on structures satisfying *multiplicative invariance relations*. The basic techniques used to study such structures resemble the basic techniques used to study self-similar sets and iterated function systems, with the structures playing the role of attractors. The work also has connections to the theory of paradoxical decompositions and non-amenable groups.

In my thesis, I solved the *cube problem for linear orders*, originally posed by Sierpiński in 1958. The problem is to determine whether there exists a linear order that is isomorphic to its lexicographically ordered cartesian cube but is not isomorphic to its square. The corresponding question has been answered positively for many different kinds of structures, including groups, rings, graphs, Boolean algebras, and topological spaces of various kinds. However, the answer to Sierpiński's question is negative: every linear order isomorphic to its cube is already isomorphic to its square. Subsequently, I solved a related problem of Sierpiński's by constructing a pair of non-isomorphic linear orders that are both left-hand and right-hand divisors of one another.

More recently, I have been working on problems concerning locally finite graphs that are motivated by questions about Cayley graphs of finitely generated groups. I am especially interested in the dividing line between amenable and non-amenable groups. Non-amenability of a finitely generated group can be characterized in terms of an isoperimetric criterion on the Cayley graph of the group, which in turn can be used to decompose the Cayley graph into uniformly splitting trees.

One can ask more generally when isoperimetric conditions on a locally finite graph yield the existence of certain uniform subgraphs of the graph, or even a decomposition into such subgraphs. In this direction, I proved that every locally finite graph G contains a pruned tree T that, in a precise sense, splits as early and as often as possible. The proof uses a matroid on the vertices of G whose independent sets are precisely those collections of vertices that can serve as sources for pairwise disjoint infinite one-sided paths. I also proved that every locally finite graph G with at most countably many ends is bilipschitz equivalent to a graph G' that can be partitioned into a collection of infinite paths realizing each of its ends exactly once.

1. CUBE PROBLEMS

Suppose that (\mathfrak{C}, \times) is a class of structures with an associative product, like the class of groups with the direct product or the class of topological spaces with the topological product. It is often possible to find examples of infinite structures $X \in \mathfrak{C}$ that are isomorphic to their own squares. If $X \cong X^2$, then $X \cong X^3$ as well. The question of whether the converse holds for a given class \mathfrak{C} , that is, whether

$X^3 \cong X \implies X^2 \cong X$ for all $X \in \mathfrak{C}$, is called the *cube problem* for \mathfrak{C} . If it has a positive answer, then \mathfrak{C} is said to have the *cube property*.

For “large” or “general” classes of structures, the cube property typically fails. There exist groups, rings, modules, topological spaces, Boolean algebras, graphs, and partial orders that are isomorphic to their cubes but not their squares, to name a few examples. See [1] [2] [6] [9] [10] [13] [14] [19] [20] [21] [22], or my paper [3] for a detailed list of these results and further historical context.

In his 1958 book *Cardinal and Ordinal Numbers*, Sierpiński asked whether there exists a counterexample to the cube property for the class (LO, \times) of linear orders under the lexicographical product. Despite the wealth of counterexamples that have been constructed for other classes, Sierpiński’s question remained open until I showed that in fact the cube property holds for (LO, \times) .

Theorem 1. (E.) If X is a linear order such that $X^3 \cong X$, then $X^2 \cong X$. More generally, for any order X and $n > 1$ we have $X^n \cong X \implies X^2 \cong X$.

The proof is in my paper [3].

Also in *Cardinal and Ordinal Numbers*, Sierpiński asked if there exist non-isomorphic orders X and Y that are both left-handed and right-handed divisors of one another (that is, $X \cong AY \cong YB$ and $Y \cong CX \cong XD$ for some orders A, B, C, D). This problem is related to the cube problem: if there were an order X isomorphic to its cube but not its square, then X and $Y = X^2$ would give such orders. There is no such X , but the answer to Sierpiński’s question is still positive.

Theorem 2. (E.) There exist non-isomorphic orders X, Y of size 2^{\aleph_0} that divide one another on both the left and right.

See [4]. There are, however, no countable examples of such orders.

2. FUNCTION SYSTEMS AND SELF-SIMILAR STRUCTURES

My solutions to Sierpiński’s problems rely on a general theorem I proved about bijections between Cartesian products of an infinite set X and itself. The theorem can be ported into many contexts to characterize when a structure X is isomorphic to a product of itself, or power of itself.

Theorem 3. (E.) Given sets A and X , it is possible to characterize, without the axiom of choice, exactly when there is a bijection $F : A \times X \rightarrow X$. Specifically, there is such a bijection if and only if X can be partitioned into a family of subsets $X = \bigcup \{I_u : u \in A^\omega\}$, indexed by points in A^ω , such that any two subsets I_u, I_v that are indexed by tail-equivalent points u, v are of the same cardinality.

Here, $u, v \in A^\omega$ are *tail-equivalent* if there exist finite sequences $r, s \in A^{<\omega}$ and an infinite tail $u' \in A^\omega$ such that $u = ru'$ and $v = su'$. I use the notation $X = A^\omega(I_{[u]})$ to mean that X can be partitioned into such a family of subsets.

Given a class of structures (\mathfrak{C}, \times) and a fixed structure $A \in \mathfrak{C}$, it is often possible to adapt the proof of the theorem to characterize those structures $X \in \mathfrak{C}$ such that $A \times X \cong X$. One may think of such structures X in this way: they may only be obtained by “replacing” each point u in the direct product A^ω by structures $I_u \in \mathfrak{C}$ such that tail-equivalent points are replaced by isomorphic structures. How to turn this into a concrete result depends on context. Here are some examples.

Theorem 4. (E.)

- a. Fix a group G , and suppose X is a group such that $G \times X \cong X$. Then there is a subgroup $H \leq G^\omega$ that is closed under tail-equivalence, and a normal subgroup $N \trianglelefteq X$, such that X/N is isomorphic to H .
- b. Fix a topological space T . For any topological space X , we have $T \times X \cong X$ if and only if $X \cong T^\omega(I_{[u]})$, where the topology on T^ω can be the product topology, the box topology, or any intermediate topology that is “closed under multiplication by T .”
- c. Fix a linear order L and let \times denote the lexicographical product. Then for any order X , we have $L \times X \cong X$ if and only if $X \cong L^\omega(I_{[u]})$ for some collection of linear orders $I_{[u]}$.

See my thesis [5]. My solution to the cube problem for linear orders crucially relies on part (c.) of Theorem 4.

An *iterated function system* (IFS) is a finite collection of contraction mappings $\{f_1, \dots, f_n\}$ on some complete metric space. A fundamental result, due to Hutchinson [8], is that any such system has a unique attractor. That is, there is a unique compact set K such that $K = \bigcup f_i(K)$. Moreover, this attractor is naturally homeomorphic to a quotient of Cantor space (on n symbols), and under this homeomorphism each f_i becomes the shift map $u \mapsto iu$.

Theorem 4 can be viewed as an analogue to Hutchinson’s result. If A and X are structures such that $A \times X \cong X$, then X can be decomposed into “ A -many copies of itself.” Hence there is a collection of mappings $\{f_a : a \in A\}$ such that for each $a \in A$, the map f_a sends X onto the a th copy of itself within itself, and we have $X = \bigcup f_a(X)$. Moreover there is a natural isomorphism identifying X , not as a quotient of Cantor space, but as a replacement of A^ω . Under this isomorphism the f_a become shift maps on A^ω . Since there is no notion of metric, the f_a are not contractions. As a result, the iterated images of X under a sequence of these maps need not converge to a point, as they do in the case of an IFS. However, they do converge to a substructure (or, in certain instances, the “coset of a substructure”), and it is possible to show that substructures associated to tail-equivalent sequences are isomorphic.

An elaboration of this idea yields the following.

Theorem 5. (E.) Given a set X , it is possible to characterize, without the axiom of choice, when there is a bijection $F : X \times X \rightarrow X$. There is such an F if and only if there is a family of functions $\{f : 2^\omega \rightarrow X^\omega\}$ that is closed under dyadic dissection and concatenation.

It is generally not as easy to characterize those structures X from a given class \mathfrak{C} that are isomorphic to their own squares, as it is to characterize those satisfying $AX \cong X$. However, in many cases the proof of Theorem 5 can be adapted to get useful information about such X .

3. BUILDING COMBINATORIAL STRUCTURES USING ISOPERIMETRY CONDITIONS IN CAYLEY GRAPHS

Suppose that Γ is a finitely generated group. We say that Γ admits a *paradoxical decomposition* if there is a partition $\Gamma = A_1 \cup \dots \cup A_n \cup B_1 \cup \dots \cup B_m$ and a collection of group elements $g_1, \dots, g_n, h_1, \dots, h_m$ in Γ such that

$$\Gamma = \bigcup_i g_i A_i = \bigcup_j h_j B_j.$$

A group Γ is said to be *paradoxical* if it admits a paradoxical decomposition. Tarski showed that a group is paradoxical if and only if it is non-amenable.

If Γ is paradoxical, then the Cayley graph G of Γ with respect to the generators g_i, h_j contains a bijection $F : 2 \times \Gamma \rightarrow \Gamma$. Such bijections are characterized by Theorem 4, whose proof yields a kind of “representation theorem” for paradoxical groups Γ that can be used to recover the following result of Kevin Whyte.

Theorem. (Whyte [23]) (“Geometric Von Neumann Conjecture”) Any finitely generated non-amenable group Γ has a Cayley graph G that can be partitioned into subgraphs that are each isomorphic to a 4-regular tree.

Any group Γ containing a copy of the free group F_2 on two generators also has a Cayley graph that can be partitioned into 4-regular trees: the pieces of the partition are just the cosets of F_2 . Von Neumann conjectured that every non-amenable group contains a copy of F_2 . This turned out to be false in general, but Whyte’s theorem shows that the graph-theoretic version of the conjecture holds.

In the paper in which he proved his theorem, Whyte asked if the “Geometric Burnside Conjecture” holds: does every finitely generated infinite group Γ have a Cayley graph that can be partitioned into bi-infinite paths (i.e. copies of the Cayley graph of \mathbb{Z})? Whyte’s theorem above implies the answer is yes for groups with at least three ends. Seward later showed that the answer is yes for finitely generated infinite groups with at most two ends. In fact he showed something stronger: every finitely generated infinite group Γ with at most two ends has a Cayley graph G with a Hamiltonian path. That is, G contains a subgraph isomorphic to \mathbb{Z} that includes every vertex of G . This follows from a purely graph-theoretic result that Seward proved.

Theorem. (Seward [15]) Any infinite, locally finite graph with at most two ends is bilipshitz equivalent to a bi-infinite path.

I proved the following generalization of this result.

Theorem 6. (E.) Suppose G is an infinite, locally finite graph, and E is a countable, dense subset of its space of ends. There is a bilipshitz equivalent graph G' that can be partitioned into a family of subgraphs P , each isomorphic to a one-sided path, such that for each $e \in E$, there is exactly one $p \in P$ converging to e . In particular, if G has only countably many ends, it is bilipshitz equivalent to a graph that can be partitioned into a set of paths realizing each of its ends exactly once.

I am in the process of writing up this result.

The proof of Whyte’s theorem depends on the so-called “Gromov doubling condition” that characterizes non-amenable groups. It says that a finitely generated group Γ is non-amenable if and only if it has a Cayley graph G in which the vertex boundary $\partial(A)$ of any finite set of vertices A is of size at least $|A|$. Whyte uses Hall’s matching theorem to translate this condition into a decomposition of the Cayley graph by regular trees. In turn, one may view these trees as concretely witnessing the doubling condition, since any subset of vertices in a regular tree satisfies it.

Given an arbitrary finite set of vertices A in a locally finite graph G , let us say that an *isoperimetry condition* is any condition that gives a lower bound on the size of the vertex boundary $|\partial(A)|$ in terms of $|A|$. One can ask if there is a general way

of witnessing isoperimetry conditions in a graph G by a combinatorial structure (like a regular tree), or a collection of such structures.

I proved that there is a general way of translating between isoperimetry conditions satisfied by a graph G and combinatorial structures that can be decomposed into infinite paths.

Theorem 7. (E.) Let $G = (V, E)$ be a locally finite graph. Define a (finite or infinite) set of vertices $I = \{x_1, x_2, \dots\}$ to be *independent* iff there are pairwise vertex-disjoint infinite paths p_1, p_2, \dots whose sources are x_1, x_2, \dots respectively. Let \mathcal{I} denote the collection of independent sets. Then (V, \mathcal{I}) is a matroid.

Using this result, I proved that, given a vertex x in a locally finite graph G , there is a pruned tree T rooted at x that, in a precise sense, splits as early and as often as possible. This T can be viewed as a witness to *any* isoperimetry condition satisfied by sets of vertices X containing the root x .

I am also in the process of writing up this result.

4. EVERYWHERE ISOMORPHIC LINEAR ORDERS

I have also studied partitions of linear orders into suborders that are isomorphic on every open interval. I proved that complete orders never admit decompositions into two such suborders.

Theorem 8. (E.) It is not possible to decompose $(\mathbb{R}, <)$ into two suborders that are isomorphic to one another on every open interval. That is, if $\mathbb{R} = A \cup B$ is a partition of \mathbb{R} , there is an open interval $I = (a, b)$ such that $A \cap I$ is not isomorphic to $B \cap I$ (as orders). More generally, it is impossible to decompose *any* dense, complete linear order L into two everywhere isomorphic pieces.

The proof depends crucially on the completeness of \mathbb{R} . And if we delete a countable dense subset, in fact we get can get such a decomposition.

Theorem 9. (E.) There is a partition of the irrationals into two everywhere isomorphic suborders. That is, there is a partition $\mathbb{R} \setminus \mathbb{Q} = A \cup B$ such that for every open interval I , we have $A \cap I \cong B \cap I$.

I am also in the process of writing up these results.

5. DIRECTIONS FOR FURTHER WORK

5.1. Sierpiński's other problems. Two questions from *Cardinal and Ordinal Numbers* remain unresolved.

Q1. Do there exist linear orders X, Y such that $X^3 \cong Y^3$ but $X^2 \not\cong Y^2$?

Q2. Do there exist linear orders X, Y such that $X^2 \cong Y^2$ but $X^3 \not\cong Y^3$?

These questions, as well as those discussed in the first section, are instances of a much more general problem. Given a class of structures (\mathfrak{C}, \times) and a semigroup (S, \cdot) , we say that S can be *represented* in \mathfrak{C} if there is a map $i : S \rightarrow \mathfrak{C}$ such that for all $a, b \in S$, we have $i(a \cdot b) \cong i(a) \times i(b)$ and $a \neq b$ implies $i(a) \not\cong i(b)$. The statement that there is an $X \in \mathfrak{C}$ isomorphic to its cube but not its square is equivalent to the statement that \mathbb{Z}_2 can be represented in \mathfrak{C} . It is typical that when the cube property fails for \mathfrak{C} that it is possible to prove much more general representation results. For example, Ketonen showed that every countable commutative semigroup

can be represented in the class (BA, \times) of countable Boolean algebras under the cartesian product.

Theorem 1 is equivalent to the statement that \mathbb{Z}_n cannot be represented in (LO, \times) for any $n > 1$.

Q3. Which semigroups can be represented in (LO, \times) ?

Q4. Can any non-trivial group be represented in (LO, \times) ?

A complete answer to Question 3 would yield answers to Question 1, 2, and 4, but may be difficult to find. I am interested in working on all of these problems.

5.2. Cantor Algebras and Thompson's Group. Given an infinite set X and bijection $F : X \times X \rightarrow X$, one may view (X, F) as an algebraic structure with a binary operation given by F . Such structures are sometimes called *Cantor algebras*. Theorem 5 can be viewed as a representation theorem for Cantor algebras, and it can be used to give simpler proofs of many of the fundamental results concerning such algebras, proved by Tarski and Jónsson [11] and Smirnov [17] [18].

Automorphism groups of Cantor algebras are sometimes of interest. The well-known Thompson group F is such a group. It may be far-fetched, but perhaps Theorem 5 can give information about Thompson's group and the question of its amenability, which is notoriously open.

Q5. Is Thompson's group amenable?

Q6. For which classes of structures \mathfrak{C} can Theorem 5 be adapted to characterize those $X \in \mathfrak{C}$ such that $X \cong X^2$?

5.3. Binary Trees in Groups of Exponential Growth. Another reasonable sounding but false conjecture concerning groups is the following: every group of exponential growth contains a subsemigroup isomorphic to the free semigroup on two generators. However, like the geometric versions of the Von Neumann conjecture and Burnside conjecture, it may be that the geometric version of this conjecture has a positive answer. The following question is open.

Q7. Does every group Γ of exponential growth have a Cayley graph G containing a complete binary tree?

Using Theorem 7 above, I can prove the following partial result.

Theorem 10. (E.) Suppose Γ is a group of exponential growth, and B_n is the ball of radius n around the identity. Let G be a Cayley graph of Γ satisfying $|B_n| > 2^n$ for all n . Then there is a partition of G (up to some finite pockets) into one-sided paths, such that the number of path sources in B_n is at least 2^n for every n .

Such a partition is implied by the existence of a binary tree as a subgraph, but I do not know if the converse is true.

REFERENCES

- [1] A. L. S. Corner, *On a conjecture of Pierce concerning direct decompositions of abelian groups*, Proc. Colloq. Abelian Groups (1964): 43-48.
- [2] P. Eklof and S. Shelah, *The Kaplansky test problems for \aleph_1 -separable groups*, Proceedings of the American Mathematical Society 126.7 (1998): 1901-1907.
- [3] G. Ervin, *Every linear order isomorphic to its cube is isomorphic to its square*, Advances in Mathematics 313 (2017): 237-281.
- [4] G. Ervin, *Distinct orders dividing each other on both sides*, Proc. Amer. Math. Soc. 147 (2019), 3729-3741.

- [5] G. Ervin, *The Cube Problem for Linear Orders*, ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)?University of California, Irvine.
- [6] T. Gowers, *A solution to the Schroeder-Bernstein problem for Banach Spaces*, Bull. London Math. Soc. (1996) 28(3): 297-304.
- [7] W. Hanf, *On some fundamental problems concerning isomorphism of Boolean algebras*, Math. Scand. 5 (1957): 205-217.
- [8] J.E. Hutchinson, *Fractals and self similarity*, University of Melbourne, Department of Mathematics, 1979.
- [9] J.M. Tyrer Jones, *On isomorphisms of direct powers*, Studies in Logic and the Foundations of Mathematics 95 (1980): 215-245.
- [10] B. Jónsson, *On the isomorphism types of groups and other algebraic systems*, Math. Scand. (1958), 224-229.
- [11] B. Jónsson, A. Tarski, *On two properties of free algebras*, Math. Scand. (1961): 95-101.
- [12] I. Kaplansky, *Infinite abelian groups*, University of Michigan Publications in Mathematics 2, Ann Arbor, 1954.
- [13] J. Ketonen, *The structure of countable Boolean algebras*, Annals of Mathematics, Second Series, Vol. 108, No. 1 (July 1978): 41-89.
- [14] V. Koubek, J. Nešetřil, V. Rödl, *Representing groups and semigroups by products in categories of relations*, Algebra Universalis 4.1 (1974): 336-341.
- [15] B. Seward, *Burnside's problem, spanning trees and tilings*, Geometry & Topology 18.1 (2014): 179-210.
- [16] W. Sierpiński, *Cardinal and ordinal numbers*, Vol. 34. Państwowe Wydawn. Naukowe, 1958.
- [17] D.M. Smirnov, *Cantor algebras with single generator I*, Algebra and Logic 10.1 (1971): 40-49.
- [18] D.M. Smirnov, *Bases and automorphisms of free Cantor algebras of finite rank*, Algebra and Logic 13.1 (1974): 17-33.
- [19] A. Tarski, *Remarks on direct products of commutative semigroups*, Math. Scand. (1958), 218-223.
- [20] V. Trnková, *Isomorphisms of sums of countable Boolean algebras*, Proc. Amer. Math. Soc. 261 (1980): 463-482.
- [21] V. Trnková, *Categorical aspects are useful for topology—after 30 years*, Topology and its Applications 155.4 (2008): 362-373.
- [22] V. Trnková, V. Koubek, *Isomorphisms of products of infinite graphs*, Commentationes Mathematicae Universitatis Carolinae, 19.4 (1978): 639-652.
- [23] K. Whyte, *Amenability, Bilipschitz Equivalence, and the Von Neumann Conjecture*, Duke mathematical journal 99.1 (1999): 93-112