Orthogonality

(insert 1)

- the <u>inner product</u> of vectors $\vec{u}, \vec{v}$ is $\vec{u}^* \vec{v}$, where $\vec{u}^*$ = transpose of conjugate of $\vec{u}$ ($= \vec{u}^T$ when $\vec{u}$ real)

a scalar

- commutative when $\vec{u}, \vec{v}$ real
- recall: $\|\vec{u}\|^2 = \vec{u}^* \vec{u} = \sum_{i=1}^{n} |u_i|^2$
- $\vec{u}, \vec{v}$ <u>orthogonal</u> if $\vec{u}^* \vec{v} = 0$

e.g.
$\vec{u} = \begin{bmatrix} 1+i \\ i \end{bmatrix}$
$\vec{v} = \begin{bmatrix} 1+i \\ -2i \end{bmatrix}$
orthog.
also:
$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

  - if $\vec{u}, \vec{v}$ real $\vec{u}^T \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

    angle between $\vec{u}, \vec{v}$

  - $\vec{u}, \vec{v}$ orthog $\Rightarrow \cos\theta = 0 \Rightarrow \theta$ between $\vec{u}, \vec{v}$ is $90°$

prove w/ law of cosines

- good to know: $(\vec{u} + \vec{v})^* = \vec{u}^* + \vec{v}^*$ $\left((\vec{u} + \vec{v})^T = \vec{u}^T + \vec{v}^T \right)$
  easy to check
  ↳ implies bilinearity of inner product
- also: $(AB)^* = B^* A^*$ $\left((AB)^T = B^T A^T \right)$

↳ Assume all matrices / vectors real for now, though most everything carries over with $*$ replacing T.

Claim: If $\{\vec{q}_1, \dots, \vec{q}_k\}$ a set of nonzero orthogonal vectors, then this set is independent

Pf: - if not, there is $j \le k$ s.t.

$$\vec{q}_j = c_1\vec{q}_1 + \dots + c_{j-1}\vec{q}_{j-1} + c_{j+1}\vec{q}_{j+1} + \dots + c_k\vec{q}_k$$

w/ not all c_i 0.

- then $\vec{q}_j^{\,T}\vec{q}_j = 0 + \dots + 0 = 0$

$$\underset{0 \neq \|\vec{q}_j\|^2}{\shortparallel} \quad \text{contradiction} \checkmark$$

- Given an <u>orthonormal</u> set $\{\vec{q}_1, \dots, \vec{q}_k\}$ (i.e. $\vec{q}_i^{\,T}\vec{q}_j = 0$ if $i \neq j$ and $= 1$ if $i = j$) of vectors in $\mathbb{R}^n$ and $\vec{u} \in \mathbb{R}^n$, can decompose $\vec{u}$ into its components along each $\vec{q}_i$ + a component orthog. to all $\vec{q}_i$

↳ consider, for fixed i, the vector $\overset{\text{a scalar}}{(\vec{q}_i^{\,T}\vec{u})}\vec{q}_i$

↳ in direction of $\vec{q}_i$ - will be $\vec{u}$'s component along $\vec{q}_i$

↳ define $\vec{r} = \vec{u} - \sum_{i=1}^{k}(\vec{q}_i^{\,T}\vec{u})\,\vec{q}_i$

↳ then $\vec{r}$ orthog. to each $\vec{q}_i$ since

$$\vec{q}_i^{\,T}\vec{r} = \vec{q}_i^{\,T}\vec{u} - \text{bunch of 0's} - (\vec{q}_i^{\,T}\vec{u})\,\underbrace{\vec{q}_i^{\,T}\vec{q}_i}_{1} = 0$$

↳ so writing

$$\vec{u} = \vec{r} + \sum_{i=1}^{k} (\vec{q}_i^T \vec{u}) \vec{q}_i$$

writes $\vec{u}$ as $\vec{r} + \vec{v}$ where $\vec{v}$ has coeffs $\vec{q}_i^T \vec{u}$ in basis $\{\vec{q}_1, \ldots, \vec{q}_k\}$ and $\vec{r}$ orthog. to space spanned by these vectors.

- when $k = n$, $\{\vec{q}_1, \ldots, \vec{q}_n\}$ a basis for $\mathbb{R}^n$, so $\vec{r} = \vec{0}$ and we're written $\vec{u}$ as combo of $\{\vec{q}_1, \ldots, \vec{q}_n\}$.

- an $n \times n$ matrix Q is <u>orthogonal</u> if its columns are orthonormal

  i.e. if $Q = (\vec{q}_1 | \cdots | \vec{q}_n)$ then $\vec{q}_i^T \vec{q}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

- said another way: $Q^T Q = Q Q^T = I$

  i.e. $Q^{-1} = Q^T$

- columns of Q an orthonormal basis for $\mathbb{R}^n$
- another way of viewing work above:

  ↳ given $\vec{u}$ in $\mathbb{R}^n$ if we want to decompose as $c_1 \vec{q}_1 + \cdots + c_n \vec{q}_n = \vec{u}$

write this eq'n as:

$$\vec{u} = Q \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

– so that: $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = Q^{-1}\vec{u} = Q^T\vec{u}$

giving $c_i = \vec{q_i}^T \vec{u}$ as before.