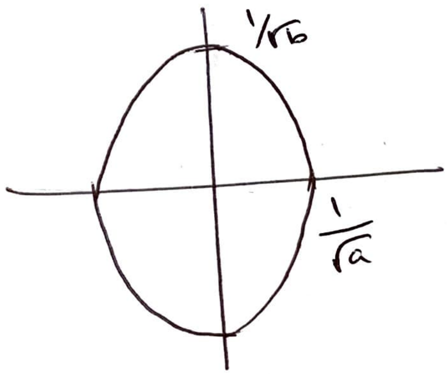
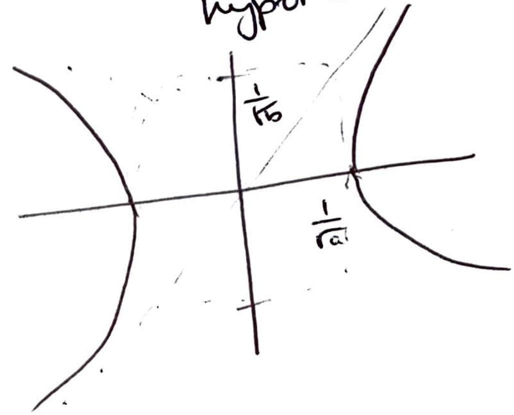
$$ax^2 + by^2 = 1$$

level surface of $P(x,y) = ax^2 + by^2$

$a,b > 0$: ellipse



a,b different sign : hyperbola

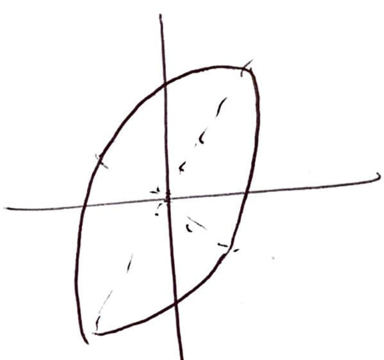


$$ax^2 + cxy + by^2 = 1$$

could be

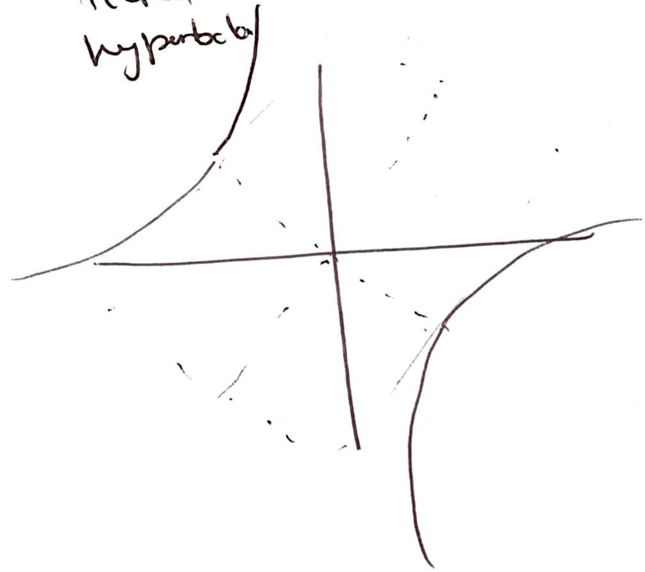
level surface of $P(x,y) = ax^2 + cxy + by^2$

tilted ellipse



How to tell??

tilted hyperbola



Summary

Given a <u>quadratic</u>
<u>form</u>

$$P(x_1, \ldots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + \cdots + c_{1n}x_1x_n$$
$$+ c_{22}x_2^2 + \cdots + c_{2n}x_2x_n$$
$$+ \cdots$$
$$+ c_{nn}x_n^2$$

Balance cross terms:

$$= c_{11}x_1^2 + \frac{c_{12}}{2}x_1x_2 + \cdots + \frac{c_{1n}}{2}x_1x_n$$
$$+ \frac{c_{12}}{2}x_2x_1 + c_{22}x_2^2 + \cdots \frac{c_{2n}}{2}x_2x_n$$
$$\vdots$$
$$+ \frac{c_{1n}}{2}x_nx_1 + \frac{c_{2n}}{2}x_nx_2 + \cdots c_{nn}x_n^2$$

$$= [x_1 \cdots x_n] \begin{bmatrix} c_{11} & \frac{c_{12}}{2} & \cdots & \frac{c_{1n}}{2} \\ \frac{c_{12}}{2} & c_{22} & & \\ \frac{c_{1n}}{2} & \cdots & & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \vec{x}^T S \vec{x}$$

S symmetric!

<u>think</u>:

$P(x,y) = 5x^2 + 8xy + 5y^2$

$\downarrow$

$5x^2 + 4xy + 4yx + 5y^2$

$\downarrow$

$$[x\ y]\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- A <u>diagonal</u> <u>form</u> looks like

$$R(x_1, \ldots, x_n) = c_1 x_1^2 + \cdots + c_n x_n^2$$

$$= [x_1 \cdots x_n] \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \vec{x}^T D \vec{x}$$

diagonal!

<u>think</u>:

$$9x^2 + 1y^2$$

$$= [x\,y] \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- "Principal Axis theorem" says (roughly) a general diagonal form $P = \vec{x}^T S \vec{x}$ can be represented as a diagonal form $\vec{x}^T D \vec{x}$ — w.r.t. a new orthonormal basis of $\mathbb{R}^n$.

the key point:

if $P = \vec{x}^T S \vec{x}$

orthonormal eigenvectors

and $S = Q D Q^T$

- eigenvalues $\lambda_1, \ldots, \lambda_n$

then $P = \vec{x}^T Q D Q^T \vec{x}$

$$= (Q^T\vec{x})^T D (Q^T\vec{x})$$

$$= \vec{X}^T D \vec{X}$$

where $\vec{X}$ is column vector $Q^T\vec{x}$

We think of $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ as new coords —

these will be w.r.t. the orthonormal basis of eigenvectors of S (columns of Q)

not fully explained
think about it!
(see example)

whereas $x_1, \ldots, x_n$ are coords w.r.t. standard basis of $\mathbb{R}^n$.

Expanding

$$P = \vec{X}^T D \vec{X}$$

gives

$$= \lambda_1 X_1^2 + \cdots + \lambda_n X_n^2$$

So: P is diagonal form!
(wrt coords $X_1, \ldots, X_n$)

More context.

a diagonal form $P(x_1, \dots, x_n) = c_1 x_1^2 + \dots + c_n x_n^2$ (whatever that means)
is strictly convex when all $c_i > 0$.

So our "tilted form"

$P = \vec{x}^T S \vec{x}$

$= \vec{x}^T D \vec{x}$

$= \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$

is convex when all $\lambda_i > 0$

↳ i.e. when S is p.d!!

In this case: P is a "hyperparaboloid"

its level surfaces are "hyperellipses"

think
$z = ax^2 + by^2$
if $a, b > 0$ a paraboloid

level surface: ellipse

unique min!

signs disagree:
→ saddle point!

level surface: hyperbola.

→ not convex!

Ex: Consider the form

$$P(x,y) = 5x^2 + 8xy + 5y^2$$

to write as $\vec{x}^T S \vec{x}$ we need to balance cross terms:

$$= 5x^2 + 4xy + 4yx + 5y^2$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Can check: $\lambda = 9, 1$ are eigenvalues of S (so S is p.d!!!) assoc. to orthonormal eigenvectors $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ resp.

So: $S = QDQ^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$\Rightarrow P = \vec{x}^T Q D Q^T \vec{x}$

$= 9X^2 + 1Y^2$

(x)

Where $\begin{bmatrix} X \\ Y \end{bmatrix} = Q^T \vec{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{x-y}{\sqrt{2}} \end{bmatrix}$$

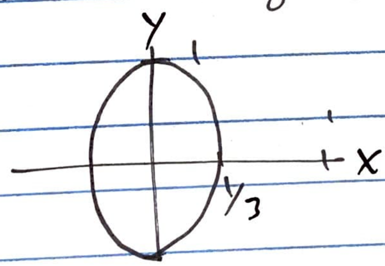
So $P = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2$

But back to: $9X^2 + 1Y^2$

the curve
$9X^2 + 1Y^2 = 1$ is an ellipse

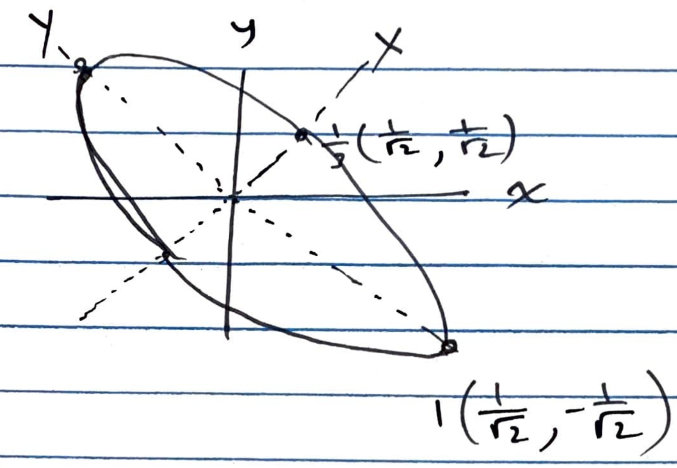
$\Leftrightarrow \quad \frac{X^2}{(1/3)^2} + \frac{Y^2}{1^2} = 1$

Recall how to graph (in X, Y coords):



In x, y coords it is the same ellipse — but with axes along the eigenvectors $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ $\leftarrow \lambda_2 = 1 = \frac{1}{1^2}$

assoc. to $\lambda_1 = 9 = \frac{1}{(1/3)^2}$



this is the graph of

$P(x,y) = 5x^2 + 8xy + 5y^2$ in standard axes!

Summary: $S = QDQ^T$ is positive definite when $\lambda_i > 0 \; \forall i$.

The graph of $\vec{x}^T S \vec{x} = 1$ is a (hyper) ellipse with axes pointing along eigenvectors of S

Length of axes given by $\frac{1}{\sqrt{\lambda_i}}$.