\[ ax^2 + by^2 = 1 \]

- \( a, b > 0 \): ellipse

- \( a, b \) different sign: hyperbola

- \( ax^2 + cxy + by^2 = 1 \)

- Tilted ellipse

- Tilted hyperbola

- Curved surface of \( P(x,y) = ax^2 + cxy + by^2 \)
Summary

Given a quadratic form

\[ P(x_1, \ldots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + \ldots + c_{1n}x_1x_n + c_{22}x_2^2 + \ldots + c_{nn}x_n^2 \]

Balance cross terms:

\[
\begin{align*}
&= c_{11}x_1^2 + \frac{c_{12}}{2}x_1x_2 + \ldots + \frac{c_{1n}}{2}x_1x_n \\
&\quad + \frac{c_{22}}{2}x_2^2 + \ldots + \frac{c_{nn}}{2}x_n^2 \\
&\quad + \frac{c_{12}}{2}x_1x_2 + \frac{c_{22}}{2}x_1x_2 + \ldots + \frac{c_{nn}}{2}x_1x_n
\end{align*}
\]

\[
= [x_1 \ldots x_n] \begin{bmatrix}
\frac{c_{11}}{2} & \frac{c_{12}}{2} & \cdots & \frac{c_{1n}}{2} \\
\frac{c_{12}}{2} & \frac{c_{22}}{2} & \cdots & \frac{c_{2n}}{2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{c_{1n}}{2} & \frac{c_{2n}}{2} & \cdots & \frac{c_{nn}}{2}
\end{bmatrix} [x_1 \ldots x_n]
\]

\[
= x^T S x
\]

\[ S \text{ symmetric!} \]
A diagonal form looks like

\[ R(x_1, \ldots, x_n) = c_1 x_1^2 + \cdots + c_n x_n^2 \]

\[ = [x_1 \ldots x_n] \begin{bmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_n \end{bmatrix} [x_1 \ldots x_n]^T \]

\[ = X^T DX \]

"Principal Axis Theorem" says (roughly)

c. general diagonal form \( P = X^T S^2 X \)

can be represented as a diagonal form \( X^T DX \)

\[ \text{w.r.t. a new orthonormal basis } c^T P^n \]

The key point:

If \( P = X^T S^2 X \)

\( X \) \text{ orthonormal eigenvectors}

and \( S = Q \Lambda Q^T \)

\[ \text{eigenvalues } \lambda_1, \ldots, \lambda_n \]
Then 
\[ P = x^T Q D Q^T x \]
\[ = (Q^T x)^T D (Q^T x) \]
\[ = \tilde{x}^T D \tilde{x} \]

where \( \tilde{x} \) is column vector \( Q^T x \).

We think of \( \tilde{x} = [x_1, ..., x_n] \) as new co-ords—
these will be w.r.t. the orthonormal basis of
- eigenvectors of \( S \) (columns of \( Q \))

Whereas \( x_1, ..., x_n \) are co-ords w.r.t. standard basis of \( \mathbb{R}^n \). (Not fully explained.)

Expanding
\[ P = \tilde{x}^T D \tilde{x} \]
gives
\[ = \lambda_1 x_1^2 + ... + \lambda_n x_n^2 \]
so \( P \) is diagonal form!
(…w.r.t. co-ords \( x_1, ..., x_n \))
More context:

A generalized form \( p(x_1, \ldots, x_n) = c_1 x_1^2 + \ldots + c_n x_n^2 \) is strictly convex when all \( c_i > 0 \).

So our "tilted form"

\[
P = \mathbf{x}^T \Sigma \mathbf{x}
\]

\[
= \mathbf{x}^T \mathbf{D} \mathbf{x}
\]

is convex when all \( \lambda_i > 0 \)

i.e. when \( \Sigma \) is p.d.!!

In this case, \( P \) is a "hyperparaboloid"

Its level surfaces are

"hyperellipses"
Ex: Consider the form

\[ P(x, y) = 5x^2 + 8xy + 3y^2 \]

to write on \( \mathbb{R}^2 \) we need to balance cross terms:

\[
\begin{align*}
5x^2 + 4xy \\
+ 4yx + 3y^2 \\
= [x y] \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix} [y x]
\end{align*}
\]

Can check: \( \lambda = 9, 1 \) are eigenvalues of \( S \) (so \( S \) is p.d.!!) and \( \mathbf{c} \) to orthonormal eigenvectors \( \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \) and \( \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \) resp.

So: \( S = Q \Lambda Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \)

\[
\Rightarrow P = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \]

\[
= 9x^2 + 1y^2
\]
\[
\begin{bmatrix} x \\ y \end{bmatrix} = P^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{x-y}{\sqrt{2}} \end{bmatrix}
\]

So
\[
P = 9 \left( \frac{x+y}{\sqrt{2}} \right)^2 + 1 \left( \frac{x-y}{\sqrt{2}} \right)^2
\]

But back to:
\[
9x^2 + 1y^2
\]

the curve:
\[
9x^2 + 1y^2 = 1 \quad \text{is an ellipse}
\]

\[
\Rightarrow \quad \frac{x^2}{(\frac{1}{\sqrt{2}})^2} + \frac{y^2}{1^2} = 1
\]

Recall how to graph (in \(xy\) coords):

In \(xy\) coords it is the same ellipse — but with axes along the eigenvectors \( \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \)

\( \lambda_1 = 9 = \frac{1}{(y_3)^2} \)

assoc. to \( \lambda = 1 \frac{1}{(y_3)^2} \)
This is the graph of

\[ P(x, y) = 5x^2 + 6xy + 5y^2 \]

in standard axes.

Summary: \( S = Q \text{D} Q^T \) is positive definite when \( \lambda_i > 0 \) \( \forall i \).

The graph of \( \bar{x}^T S \bar{x} = 1 \) is a (hyper) ellipse with axes pointing along eigenvectors of \( S \).

Length of axes given by \( \frac{1}{\sqrt{\lambda_i}} \).