Positive Definite Matrices

* Standing assumption: $S = S^T$ is symmetric and real

**Def** $S$ is **positive definite** (p.d.) if all of its eigenvalues are strictly positive (positive semidefinite (p.s.d.) if all $\lambda \geq 0$)

We know $S = QQ^T$ so given such a factorization can see if p.d. by checking if $D$'s diagonal entries $> 0$.

**e.g.** $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is p.d.

$Q \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} Q^T$ is p.d. for any orthogon Q.

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is p.s.d.

$\begin{bmatrix} 2 & 0 \\ 0 & -6 \end{bmatrix}$ is not p.d.
Vague geometric intuition:

P.D. matrices only "stretch" vectors
do not reflect vectors

Test 1: \( S \) is p.d. iff \( \vec{x}^T S \vec{x} > 0 \) for all \( \vec{x} \in \mathbb{R}^n \).

PF: \((\Rightarrow)\) if \( S \) is p.d. and \( \vec{x} \in \mathbb{R}^n \):

Can write \( \vec{x} \) as linear combo of eigenvectors \( \vec{x}_1, \ldots, \vec{x}_n \) of \( S \) (poss. by Spectral
Theorem --- \( S \) is symmetric!)

orthonormal!

\( \vec{x} = c_1 \vec{x}_1 + \ldots + c_n \vec{x}_n \)

Then \( \vec{x}^T S \vec{x} = (c_1 \vec{x}_1 + \ldots + c_n \vec{x}_n)^T (\lambda_1 c_1 \vec{x}_1 + \ldots + \lambda_n c_n \vec{x}_n) \)

\( = \lambda_1 c_1^2 \| \vec{x}_1 \|^2 + \ldots + \lambda_n c_n^2 \| \vec{x}_n \|^2 \)

All cross terms\( c_i \vec{x}_i \cdot \vec{x}_j = 0 \implies \) \( > 0 \) since \( \lambda_1, \ldots, \lambda_n > 0 \)

by p.d. ness of \( S \).

\((\Leftarrow)\) if \( \vec{x}^T S \vec{x} > 0 \) for all \( \vec{x} \);

Sps \( \vec{x}_0 \) is an eigenvector:

\( S \vec{x}_0 = \lambda_0 \vec{x}_0 \)
then \( \bar{x}_c^T S \bar{x}_c = \lambda_c \|\bar{x}_c\|^2 > 0 \) by hypothesis

\[ \Rightarrow \lambda_c > 0 \checkmark \]

This test gives us an easy proof of the following:

**Fact:** if \( S_1, S_2 \) are p.d. then so is \( S_1 + S_2 \).

**PF:** given \( \bar{x} \in \mathbb{R}^n \):

\[ \bar{x}^T (S_1 + S_2) \bar{x} = \bar{x}^T S_1 \bar{x} + \bar{x}^T S_2 \bar{x} \]

\[ \geq \lambda_1 \geq \lambda_2 \geq 0 \text{ by p.d.ness of } S_1, S_2 \]

\[ > 0. \checkmark \]

---

**Test 2:** \( S \) is p.d. iff \( S = A^T A \) for some \( A \) with independent columns.

**PF:** \((\Rightarrow)\) Suppose \( S \) is p.d. then (by spectral theorem) \( S = QDQ^T \) where

\[ D \text{ diagonal} \]
Entries of \( D \) are positive.

Let \( A = Q \sqrt{D} Q^T \), where \( Q \) is a matrix obtained by taking \( \sqrt{D} \)'s of all entries of \( D \).

Then \( A^T = A = Q \sqrt{D} Q^T \)

and \( A^T A = Q \sqrt{D} Q^T \sqrt{D} Q^T \)

\[ = Q D Q^T = S \]

Of course: \( A \) has independent columns (why?)

\[ \iff \] \( \text{Sp} \quad A^T A = S \) and \( \text{rank} \ A = n \)

Let \( \lambda \) be an eigenvalue, \( \vec{x} \) an associated eigenvector: \( A^T A \vec{x} = \lambda \vec{x} \)

Then: \( \vec{x}^T (A^T A \vec{x}) = \vec{x}^T \lambda \vec{x} = \lambda \| \vec{x} \|^2 \)

\[ (A \vec{x})^T (A \vec{x}) = \| A \vec{x} \|^2 > 0 \]

Since \( \vec{x} \neq 0 \) and \( A \) has trivial nullspace

So all eigenvalues positive
- Given \( S \), consider the "upper left square" matrices \( S_1, \ldots, S_n = S \)

\[
S = \begin{bmatrix}
\end{bmatrix}
\]

- Call the determinants of these matrices \( D_1, \ldots, D_n \)

**Test 3:** \( S \) is p.d. iff \( D_1, \ldots, D_n > 0 \).

It won't work.

One upset: the \( D_i \)'s are connected to the pivots in Gaussian elimination.

Fact: \( k \)th pivot = \( \frac{D_k}{D_{k-1}} \)

So: if \( S \) is p.d. then in particular all pivots are nonzero (in fact > 0).

\[ \rightarrow \text{elimination succeeds with no row exchange} \]
and we can factor $S = LU$.
Moreover: if we factor as usual (1's on diagonal of $L$, pivots on diagonal of $U$) then $L$ and $U$ must be closely related — $U$ is essentially $LT$ up to multiplication by the pivots which we can pull out as a diagonal matrix.

Proof by example:

Given $S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

we don't know $S$ is p.d. — yet.

Let's try to factor: $S = LU$.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & \frac{4}{3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
\[
L = \begin{bmatrix}
1 & 0 & 0 \\
-1/2 & 1 & 0 \\
0 & -2/3 & 1
\end{bmatrix} \begin{bmatrix}
2 & -1 & 0 \\
0 & 3/2 & -1 \\
0 & 0 & 4/3
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 3/2 & 0 \\
0 & 0 & 4/3
\end{bmatrix}
\]

or by magic

\[
L^T \text{ or } L^{-1}
\]

since pivots (diagonal entries in } D) are > 0

Can take } \sqrt{D} = D \rightarrow and absorb left and right:

\[
= (L \sqrt{D}) \sqrt{D}^T L^T = (L \sqrt{D}) (L \sqrt{D})^T
\]
= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}

This writes $S = A^T A$
(with $A = (LD)^T$) where $A$ has indep columns — so $S$ is p.d.

But this factorization has a special form: $A$ is upper triangular!

Comment: This procedure always works when $S$ is p.d.