SVD: round 2

- we give another proof using symmetric matrices
- Observe: A^TA and AA^T are symmetric for any $A \in \mathbb{R}^{m\times n}$.
- if $A = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,V^T$ — Σ diag., U and V ($n\times n$) orthog.

then: $A^TA = (V\Sigma^TU^T)(U\Sigma V^T)$

$= V\underbrace{\Sigma^T\Sigma}_{}V^T$ ~~diag~~ diagonal $n\times n$

$AA^T = (U\Sigma V^T)(V\Sigma^TU^T)$

$= U\underbrace{\Sigma\Sigma^T}_{}V^T$ diag $m\times m$

if $p = \min(n,m)$ then the diagonal entries of $\Sigma^T\Sigma$ and $\Sigma\Sigma^T$ include $\sigma_1^2, \dots, \sigma_p^2$ — and these include all nonzero singular values of A (squared...)

↳ both eq'ns of the form

$A^T A = (\text{orthog})(\text{diag})(\text{orthog})^{-1}$

$AA^T =$ " " "

↳ so there are eigenvalue decompositions of A^TA and AA^T!

So: V contains orthonormal eigenvectors of A^TA (as columns)

U " " " of AA^T } !!

↳ nonzero eigenvalues for each are $\sigma_1^2, \dots, \sigma_p^2$

↳ we can reverse this work to get another proof of the SVD:

- Given A $m \times n$, consider A^TA $(n \times n)$
- by spectral theorem, find orthonormal eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ assoc. to eigenvalues $\lambda_1, \dots, \lambda_n$.

- Observe: $\lambda_i \geq 0$ for all i since:

$$(\vec{v}_i^T (A^T) A \vec{v}_i) = \vec{v}_i^T \lambda_i \vec{v}_i = \lambda_i \|\vec{v}_i\|^2$$

$$= (A\vec{v}_i)^T (A v_i) = \|A\vec{v}_i\|^2$$

equality and $\vec{v}_i \neq 0 \Rightarrow \lambda_i \geq 0$

- Let's assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ (if not: reorder)

- Let $\sigma_k = \sqrt{\lambda_k}$

Want $A\vec{v}_k = \sigma_k \vec{u}_k$ Know: $A^T A \vec{v}_k = \sigma_k^2 \vec{v}_k$

so let $\vec{u}_k = \frac{1}{\sigma_k} A\vec{v}_k$

when this makes sense: when $\sigma_k = 0$ pick $\vec{u}_k$ orthonormal to $\{\vec{u}_1, \ldots, \vec{u}_{k-1}\}$ if this not already a basis for $\mathbb{R}^n$, o.w. define $\vec{u}_k = \vec{0}$ (won't matter: will be deleted from $\hat{U}$ anyway)

Just like before!

then: ① $\vec{u}_k$ is an eigenvector for AA^T:

for $k \leq m$

$$AA^T \vec{u}_k = AA^T \left(\frac{1}{\sigma_k} A\vec{v}_k\right)$$
$$= A \frac{1}{\sigma_k} \underbrace{(A^T A \vec{v}_k)}_{\sigma_k^2 \vec{v}_k}$$
$$= A \sigma_k \vec{v}_k$$
$$= \sigma_k^2 \left(\frac{1}{\sigma_k} A\vec{v}_k\right) = \sigma_k^2 \vec{u}_k$$

② $\vec{u}_k$'s are orthonormal (for $k \leq m$)

$$\vec{u}_k^T \vec{u}_\ell = \left(\frac{1}{\sigma_k} A\vec{v}_k\right)^T \left(\frac{1}{\sigma_\ell} A\vec{v}_\ell\right)$$
$$= \frac{1}{\sigma_k \sigma_\ell} \vec{v}_k^T A^T A \vec{v}_\ell = \frac{1}{\sigma_k \sigma_\ell} \sigma_\ell^2 v_k^T v_\ell$$

$= 0$ if $k \neq \ell$

$= 1$ if $k = \ell$

As before let

$$V = [\vec{v}_1 | \cdots | \vec{v}_n]$$

$$\hat{U} = [\vec{u}_1 | \cdots | \vec{u}_n]$$

$\hat{\Sigma}$ = diagonal $n\times n$ with σ_i's on diag.

Then $\overset{m\times n}{A}\,\overset{n\times n}{V} = \overset{m\times n}{\hat{U}}\,\overset{n\times n}{\hat{\Sigma}}$

modify $\hat{U}\,\hat{\Sigma}$ as before to $\overset{m\times m \text{ orthogonal}}{U}, \overset{m\times n \text{ diagonal}}{\Sigma}$

to get $A\overset{\text{orthog by spectral thm!}}{V} = U\Sigma$ eigenvectors of A^TA for columns

$\Rightarrow A = U\Sigma V^T$ ✓

↳ Complete proof of SVD! Just needed spectral thm applied to A^TA.

<u>Ex</u>: Recall $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

Can check $A^TA = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

and $\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 45\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$

rescale $\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$ orthonormal

eigenvalues $\lambda_1 = 45 \geq 5 = \lambda_2$

let $\sigma_1 = \sqrt{45} = 3\sqrt{5}$, $\sigma_2 = \sqrt{5}$

Now: $A\vec{v}_1 = \underset{\sigma_1}{3\sqrt{5}} \underset{u_1}{\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}}$ (same calculation as before)

u_1 — unit vector

work not shown

$A\vec{v}_2 = \underset{\sigma_2}{\sqrt{5}} \underset{u_2}{\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix}}$

orthog, as expected

Let $V = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$ $\Sigma = \begin{bmatrix}3\sqrt{5} & 0\\0 & \sqrt{5}\end{bmatrix}$

$U = \begin{bmatrix}1/\sqrt{10} & 3/\sqrt{10}\\3/\sqrt{10} & -1/\sqrt{10}\end{bmatrix}$

Then $AV = U\Sigma \longrightarrow A = U\Sigma V^T$

$$\begin{bmatrix}3 & 0\\4 & 5\end{bmatrix} = \begin{bmatrix}1/\sqrt{10} & 3/\sqrt{10}\\3/\sqrt{10} & -1/\sqrt{10}\end{bmatrix}\begin{bmatrix}3\sqrt{5} & 0\\0 & \sqrt{5}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

as before

Worth noting: if $S = S^T$ is symmetric with nonnegative eigenvalues then its eigenvalue decomposition $S = QDQ^T$ $\underline{\text{is}}$ its SVD. (as long as eigenvalues in descending order in D)

A few last facts on SVD:

(1) Let r = # of nonzero singular values of A. Then $r = \text{rank}(A)$

(Why? Follows from orig. pf + rank/nullity)

(2) Let $\sigma_1, \dots, \sigma_r$ denote nonzero singular values of A.

From $A = U \Sigma V^T$

we can write A as a <u>sum</u> <u>of</u> <u>rank 1</u> <u>matrices</u>:

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^{\,T}$$

<u>Why we care</u>: For a given $k \leq r$

the partial sum

$$A_k = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T$$

is the best rank k approximation to A (in a sense that can be made precise)

e.g.

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= 3\sqrt{5} \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \sqrt{5} \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= 3\sqrt{5} \begin{bmatrix} \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} \\ \frac{3}{\sqrt{20}} & \frac{3}{\sqrt{20}} \end{bmatrix} + \sqrt{5} \begin{bmatrix} \frac{3}{\sqrt{20}} & -\frac{3}{\sqrt{20}} \\ -\frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{9}{2} & \frac{9}{2} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

best rank 1 approx to A