So $AV = U\Sigma$

$$\Rightarrow A = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

<u>If $m \leq n$:</u> in this case, by dimensionality considerations $A\vec{v}_{m+1}, \dots, A\vec{v}_n$ are all $\vec{0}$ — hence so are $\vec{u}_{m+1}, \dots, \vec{u}_n$

so we may delete these $\vec{0}$ columns from $\hat{U}$ $(m\times n)$ to form U $(m \times m)$

$$\hat{U} = \begin{bmatrix} u_1 & \cdots & u_m & \overset{\vec{u}_{m+1}}{\vec{0}} & \cdots & \overset{\vec{u}_n}{\vec{0}} \end{bmatrix}$$

$$\longrightarrow U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$$

we also truncate $\hat{\Sigma}$:

$$\hat{\Sigma} = \left[\begin{array}{ccc|c} \sigma_1 & & & \vec{0} \\ & \ddots & & \\ & & \sigma_n & \\ \hline & & & \ddots \\ & & & \sigma_n \end{array}\right]$$

$$\longrightarrow \Sigma = \left[\begin{array}{ccc|c} \sigma_1 & \cdots & \sigma_n & \vec{0} \end{array}\right]$$

We still have in this case:

$AV = U\Sigma$

$\Rightarrow A = U\Sigma V^T$

We've proved in either case:

<u>Theorem</u> (SVD) Given $A \in \mathbb{R}^{m \times n}$
there are orthogonal matrices
$U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ and
a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$
with diagonal entries

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ ($p = \min(m,n)$)

s.t. $AV = U\Sigma$

$\Rightarrow A = U\Sigma V^T$

Moreover, the σ_i are uniquely determined.
If they are distinct, so are the matrices U, V.

up to signs of each column

An example:

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ 3×2

Want to maximize $\|A\vec{v}\|$ subject to $\|\vec{v}\| = 1$

i.e.

equiv $\|A\vec{v}\|^2$ subject to $\|\vec{v}\|^2$

max length $\left\| \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} x+y \\ 0 \\ 0 \end{bmatrix} \right\|^2 = (x+y)^2$

subject to $x^2 + y^2 = 1$

but $(x+y)^2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

and $\vec{u}^T \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ is maximized for given lengths $\|\vec{u}\|$, $\|\vec{v}\|$ when $\vec{u}$, $\vec{v}$ point in same direction.

so $\begin{bmatrix} x \\ y \end{bmatrix}$ must be multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ but w/ length 1

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \vec{v}_1$

then $A\vec{v} = \begin{bmatrix}\sqrt{2}\\0\\0\end{bmatrix} = \underbrace{\sqrt{2}}_{\sigma_1}\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{u}_1}$ since a unit

to find $\vec{v}_2$ σ_2 need to maximize

$\|A\vec{v}\|$ subject to $\|\vec{v}\| = 1$ and $\vec{v} \perp \vec{v}_1$

only two possibilities $\vec{v}_2 = \begin{bmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{bmatrix}$ or switch negative

in both cases

$A\vec{v}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓

So $A\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}\sqrt{2} & 0\\0 & 0\\0 & 0\end{bmatrix}$

want
$U = 3\times3$
$\Sigma = 3\times2$

$= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & 0\\0 & 0\end{bmatrix}$

↳ chose $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as orthonormal unit vector corresponding to $\sigma_2 = 0$ arbitrarily

then $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ was determined

up to sign

↳ could have e.g. let $\begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ $\begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ be our last 2 columns of U etc.

we have $AV = U\Sigma$

$$\underset{3\times 2}{A}\ \underset{2\times 2}{V} = \underset{3\times 3}{U}\ \underset{3\times 2}{\Sigma}$$

↳ transposing gives

$$A = U\Sigma V^T$$

σ_1, σ_2 ignore sp...

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Pict

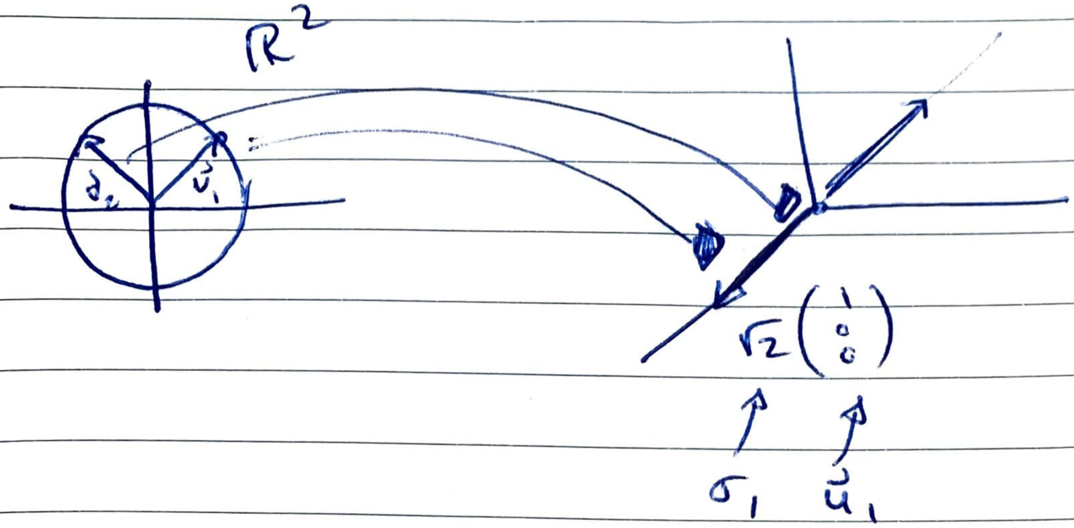

Image is a 1-dim'l "ellipse" with semiaxis length $\sqrt{2}$, i.e. a line.