Eigenvectors and Eigenvalues

- if A is $n \times n$, a nonzero $\vec{x}$ s.t. for some scalar λ (possibly complex — even if A real) we have $A\vec{x} = \lambda\vec{x}$ is called an eigenvector of A; λ is associated eigenvalue.

- e.g. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

  so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigenvector associated to $\lambda = 3$

  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix}$

  so $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ " " $\lambda = i$

How many eigenthings?

- A always has n eigenvalues if you count multiplicity (possible repeated eigenvalues in char. polynomial)

- usually A has n distinct eigenvalues

- eigenvectors assoc. to distinct eigenvalues always independent (can prove by induction)

- a repeated eigenvalue $\lambda_1 = \lambda_2$ might or might not have 2 independent eigenvectors.

- eigenvectors/values help us understand dynamics of A:
- usually n indep. eigenvectors $\vec{x}_1, \dots, \vec{x}_n$ assoc. to $\lambda_1, \dots, \lambda_n$
- then any $\vec{v} \in \mathbb{R}^n$ is linear combo:

$$\vec{v} = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n$$

- so: $A\vec{v} = c_1\lambda_1\vec{x}_1 + \cdots + c_n\lambda_n\vec{x}_n$

$$A^k\vec{v} = c_1\lambda_1^k\vec{x}_1 + \cdots + c_n\lambda_n^k\vec{x}_n$$

- so: if we knew n (indep.) eigenvectors for A and assoc. eigenvalues, knew all there is to know about A (and powers of A!)

- even more, in this situation can diagonalize A. What's this mean?
- put indep. eigenvectors as columns of a matrix $X = [\vec{x}_1 | \cdots | \vec{x}_n]$
- then $AX = [A\vec{x}_1 | \cdots | A\vec{x}_n] = [\lambda_1\vec{x}_1 | \cdots | \lambda_n\vec{x}_n]$

$$= \left[\vec{x}_1 \middle| \cdots \middle| \vec{x}_n\right] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

zeroes

$= XD$ ← diagonal with eigenvalues on main diagonal.

- X is invertible (since $\vec{x}_1, \dots, \vec{x}_n$ indep. by hypothesis)
- So: $AX = XD \Rightarrow A = XDX^{-1}$

↳ Now: easy to compute powers of A

$$A^2 = XDX^{-1}XDX^{-1} = XD^2X^{-1}$$

$$A^k = XD^kX^{-1}$$

↖ squares of eigenvalues on diag.

- Conversely: if $A = XDX^{-1}$ for some diagonal D, then reversing above we see: entries on diag of D are eigenvalues and columns of X are assoc. (indep.) eigenvectors.

We've proved:

<u>Theorem</u> An $n \times n$ matrix A is diagonalizable, i.e. $A = XDX^{-1}$ for some diagonal D, iff A has n independent eigenvectors.

In this case, eigenvalues of A appear on diagonal of D.