# Insert: Identity/Inverses

- the rank $n$ identity matrix $I$ $(= I_n)$ is the $n \times n$ matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

- we have $AI = IA = A$ for all $n \times n$ $A$.

- For $n \times n$ $A$, $A$ is invertible (or nonsingular) iff $\exists$ matrix $A^{-1}$ s.t. $AA^{-1} = A^{-1}A = I$

Thm: $A$ is invertible

$\Leftrightarrow$ columns are independent

$\Leftrightarrow$ nullspace is trivial (i.e. $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$)

$\Leftrightarrow$ $\det(A) \neq 0$

$\vdots$

# (The Zen of) Matrix Multiplication



## Matrix/vector:

- if $A$ $m \times n$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $n \times 1$, usually think of $A\vec{v}$ as "left to right" "rows of $A$ acting on $\vec{v}$":

$$A\vec{v} = \begin{array}{c} \\ m \end{array}\underset{n}{\begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}} \begin{array}{c} \\ n \end{array}\underset{1}{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}} = \begin{array}{c} \\ m \end{array}\underset{1}{\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}} = \vec{u}$$

where $u_i = \vec{r}_i \vec{v}$

$= a_{i1}v_1 + \cdots + a_{in}v_n$

beware bad notation $\vec{r}_i$ is a row vector...

- But observe! Can also view $A\vec{v}$ as "right to left" "$\vec{v}$ acting on columns of $A$"

$$A\vec{v} = \left[\vec{c}_1 \middle| \cdots \middle| \vec{c}_n\right] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{c}_1 + \cdots + v_n\vec{c}_n$$

- gives same result, since $i$th coord of this sum is $v_1 a_{i1} + \cdots + v_n a_{in} = u_i$ as before

- e.g. $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

$$= -1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

- <u>the point</u>: $A\vec{v}$ is a linear combo of columns of $A$!

## Matrix/matrix

- if $A$ is $m \times n$ $= \left[ \vec{c}_1 \middle| \cdots \middle| \vec{c}_n \right]$

  $B$ is $n \times k$ $= \left[ \vec{d}_1 \middle| \cdots \middle| \vec{d}_k \right]$

Can also think of $AB$ "right to left" as "columns of $B$ acting on columns of $A$"

- $AB = \left[ A\vec{d}_1 \middle| \cdots \middle| A\vec{d}_k \right]$

$$= \left[ d_{11}\vec{c}_1 + \cdots + d_{n1}\vec{c}_n \middle| \cdots \middle| d_{1k}\vec{c}_1 + \cdots + d_{nk}\vec{c}_n \right]$$

- e.g. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ -2 & 9 \\ -3 & 12 \end{bmatrix}$

$$= \left[ 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \middle| 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right]$$

<u>exercise</u>: convince yourself this is always same as usual multiplication rule for $AB$

<u>the point</u>: Columns of $AB$ are linear combos of columns of $A$.

- can also "transpose" this view:
do matrix multiplication "left to right" but not in usual way:
- rather, views rows of A acting on rows of B
- if A is $m \times n$ = $\begin{bmatrix} \vec{r}_1^{\,T} \\ \hline \vdots \\ \hline \vec{r}_m^{\,T} \end{bmatrix}$ ← writing T now to emphasize row vectors, i.e. transposes of column vectors.

B is $n \times k$ = $\begin{bmatrix} \vec{s}_1^{\,T} \\ \hline \vdots \\ \hline \vec{s}_n^{\,T} \end{bmatrix}$

then $AB = \begin{bmatrix} \vec{r}_1^{\,T} B \\ \hline \vdots \\ \hline \vec{r}_m^{\,T} B \end{bmatrix}$

$= \begin{bmatrix} r_{11}\vec{s}_1^{\,T} + \cdots + r_{1n}\vec{s}_n^{\,T} \\ \hline \vdots \\ \hline r_{m1}\vec{s}_1^{\,T} + \cdots + r_{mn}\vec{s}_n^{\,T} \end{bmatrix}$

e.g. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ -2 & 9 \\ -3 & 12 \end{bmatrix}$

$= \begin{bmatrix} 1(1,0) + 2(-1,3) \\ \hline 1(1,0) + 3(-1,3) \\ \hline 1(1,0) + 4(-1,3) \end{bmatrix}$

<u>exercise</u>: convince yourself this always works

<u>the point</u>: rows of AB are linear combo of rows of B.

# A = CR

- Given A, an algorithm: create new matrix C by only including those columns of A which are independent over previous columns (reading left to right)

- e.g. if $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

if $A = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 4 & 1 \end{bmatrix}$ then $C = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix}$

if $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 7 \end{bmatrix}$ then $C = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

since $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

- # of columns in C = number of indep columns in A
  = "rank" of A

- <u>column space</u> C(A) is span of columns of A
- columns of C are basis for C(A).

so: dim C(A) = rank of A

- We can get back A from C by mult. on right by matrix R that tells us how to combine columns of C to form missing columns of A:

- remember: if $A = CR$ columns of A linear combos of columns of C; so figure out what R must be to make eq'n hold!

- e.g. if $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $C = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{1} \}2$ $R = \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{2} \}1$

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \checkmark$$

if $A = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 4 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$A = CR \checkmark$$

if $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 7 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$$A = CR \checkmark$$

- in general: if A is $m \times n$ w/ $k$ independent columns, then C is $m \times k$ R is $k \times n$.

\# of columns of C = # of rows of R

**Notice**: rows of R (viewed as vectors) are independent!

**Why**: For every column of A in C we have a column of R w/ one 1 and other entries 0

$\Rightarrow$ ensures no linear combo of other rows of R can equal a given row

- the row space R(A) is the span of the rows of A (viewed as row vectors) = span of columns of $A^T$ (i.e. $R(A) = C(A^T)$)

- from $A = CR$ and our knowledge of matrix mult: knew rows of A linear combos of rows of R

- since rows of R independent, form a basis for R(A)

- but then: dim of row space = # of rows of R
  = # of columns of C

we've proved!

**Theorem** dimension of C(A)
= dimension of R(A) $(= C(A^T))$

In words: # of independent columns in A
= # of independent rows in A.